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Bounding the Čebyšev Functional for Functions of Bounded Variation and Applications

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Abstract. Some sharp bounds for the Čebyšev functional of functions with bounded variation and applications for selfadjoint operators in Hilbert spaces via the spectral representation theorem are given.

1. Introduction

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [11] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $T(f,g)$ was derived in 1882 by Čebyšev [3] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a,b]$.

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In 1970, A.M. Ostrowski [18] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g:[a,b]\to\mathbb{R}$ is absolutely continuous and $g'\in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [12] (see also [15, p. 210]) obtained the following result as well:

$$|C(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{1.5}$$

provided f,g are absolutely continuous and $f',g'\in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [1], P. Cerone and S.S. Dragomir proved the following inequalities:

$$|C(f,g)| \leq \begin{cases} \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_{\infty}\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\ \\ \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_{q}\cdot\frac{1}{b-a}\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} \\ \qquad\qquad\text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma=0$, we get from the first inequality in (1.6)

$$|C(f,g)| \leq \|g\|_{\infty}\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m\leq g\leq M$ for a.e. $x\in[a,b]$, then $\left\|g-\frac{m+M}{2}\right\|_{\infty}\leq\frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [4]

$$|C(f,g)| \leq \frac{1}{2}(M-m)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [2].

2. New Bounds for the Čebyšev Functional

The following result holds.

THEOREM 1. *Let $f:[a,b]\to\mathbb{C}$ be of bounded variation on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|C(f,g)| \leq \frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \tag{2.1}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$.

The constant $\frac{1}{2}$ is best possible in (2.1).

Proof. We start with the following equality of interest that follows from the Sonin's identity

$$C(f,g)=\frac{1}{b-a}\int_a^b\left[f(t)-\frac{f(a)+f(b)}{2}\right]\left[g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right]dt. \tag{2.2}$$

Taking the modulus in (2.2) and utilizing the triangle inequality, we get

$$\begin{aligned}
&|C(f,g)| \\
&\leq \frac{1}{b-a}\int_a^b\left|f(t)-\frac{f(a)+f(b)}{2}\right|\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \\
&\leq \frac{1}{2(b-a)}\int_a^b\left[|f(t)-f(a)|+|f(b)-f(t)|\right]\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \\
&\leq \frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt,
\end{aligned} \tag{2.3}$$

since, f being of bounded variation, we have that

$$|f(t)-f(a)|+|f(b)-f(t)|\leq\bigvee_a^b(f)$$

for all $t\in[a,b]$.

Now, assume that the inequality (2.1) holds with the constant $C>0$ instead of $\frac{1}{2}$, namely

$$|C(f,g)|\leq C\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt \tag{2.4}$$

for any function f and g as in the statement of the theorem.

Consider now the functions $f,g:[a,b]\to\mathbb{R}$ with $f(t)=sgn\left(t-\frac{a+b}{2}\right)$ and $g(t)=t-\frac{a+b}{2}$. Observe that f is of bounded variation and $\bigvee_a^b(f)=2$. The function g is integrable on $[a,b]$ and $\int_a^b g(s)\,ds=0$. Also $\int_a^b|g(t)|\,dt=\frac{1}{4}(b-a)^2$ and

$$\begin{aligned}
\int_a^b f(t)\,g(t)\,dt &= \int_a^b sgn\left(t-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)dt \\
&= \int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{4}(b-a)^2.
\end{aligned}$$

Replacing these values in (2.4) we deduce that $\frac{1}{4}(b-a)\leq\frac{1}{2}C(b-a)$ which implies that $C\geq\frac{1}{2}$ and the proof is complete. □

We denote the variance of the function $f:[a,b]\to\mathbb{C}$ by $D(f)$ and defined as

$$D(f)=\left[C\left(f,\bar{f}\right)\right]^{1/2}=\left[\frac{1}{b-a}\int_a^b|f(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2\right]^{1/2}, \tag{2.5}$$

where $\bar{f}$ denotes the complex conjugate function of f.

Corollary 1. *If the function $f:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$D(f)\leq\frac{1}{2}\bigvee_{a}^{b}(f). \tag{2.6}$$

The constant $\frac{1}{2}$ is best possible in (2.6).

Proof. If we apply Theorem 1 for $g=\bar{f}$ we get

$$\begin{aligned} D^{2}(f) &\leq \frac{1}{2}\bigvee_{a}^{b}(f)\cdot\frac{1}{b-a}\int_{a}^{b}\left|\bar{f}(t)-\frac{1}{b-a}\int_{a}^{b}\bar{f}(s)\,ds\right|dt \\ &= \frac{1}{2}\bigvee_{a}^{b}(f)\cdot\frac{1}{b-a}\int_{a}^{b}\left|f(t)-\frac{1}{b-a}\int_{a}^{b}f(s)\,ds\right|dt. \end{aligned} \tag{2.7}$$

By Cauchy-Bunyakovsky-Schwarz's integral inequality we have

$$\begin{aligned} &\frac{1}{b-a}\int_{a}^{b}\left|f(t)-\frac{1}{b-a}\int_{a}^{b}f(s)\,ds\right|dt \\ &\leq\left[\frac{1}{b-a}\int_{a}^{b}\left|f(t)-\frac{1}{b-a}\int_{a}^{b}f(s)\,ds\right|^{2}dt\right]^{1/2} \\ &=\left[\frac{1}{b-a}\int_{a}^{b}|f(t)|^{2}\,dt-\left|\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|^{2}\right]^{1/2} \\ &=D(f). \end{aligned} \tag{2.8}$$

On making use of (2.7) and (2.8) we deduce the desired inequality (2.6).

Now, if we choose $f:[a,b]\to\mathbb{R}$ with $f(t)=sgn\left(t-\frac{a+b}{2}\right)$, then we obtain in both sides of (2.6) the same quantity 1, which proves the sharpness of the constant $\frac{1}{2}$. □

Now we can state the following result when both functions are of bounded variation:

Corollary 2. *If $f,g:[a,b]\to\mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|C(f,g)|\leq\frac{1}{4}\bigvee_{a}^{b}(f)\bigvee_{a}^{b}(g). \tag{2.9}$$

The constant $\frac{1}{4}$ is best possible in (2.9).

Proof. On making use of Theorem 1 and Corollary 1 we have successively

$$\begin{aligned} |C(f,g)| &\leq\frac{1}{2}\bigvee_{a}^{b}(f)\cdot\frac{1}{b-a}\int_{a}^{b}\left|g(t)-\frac{1}{b-a}\int_{a}^{b}g(s)\,ds\right|dt \\ &\leq\frac{1}{2}\bigvee_{a}^{b}(f)\,D(g)\leq\frac{1}{4}\bigvee_{a}^{b}(f)\bigvee_{a}^{b}(g). \end{aligned}$$

The case of equality is obtained in (2.9) for $f(t)=g(t)=sgn\left(t-\frac{a+b}{2}\right), t\in[a,b]$. □

Remark 1. *We can consider the following quantity associated with a complex valued function* $f:[a,b]\to\mathbb{C}$,

$$E(f):=|C(f,f)|^{1/2}=\left|\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2\right|^{1/2}.$$

Utilising the above results we can state that

$$\begin{aligned}E^2(f)&\le\frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt\\&\le\frac{1}{2}\bigvee_a^b(f)\,D(f)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2.\end{aligned}\tag{2.10}$$

If we consider

$$\begin{aligned}G(f)&:=|C(f,|f|)|^{1/2}\\&=\left|\frac{1}{b-a}\int_a^b f(t)\,|f(t)|\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b|f(t)|\,dt\right|^{1/2},\end{aligned}$$

then we also have

$$\begin{aligned}G^2(f)&\le\frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left||f(t)|-\frac{1}{b-a}\int_a^b|f(s)|\,ds\right|dt\\&\le\frac{1}{2}\bigvee_a^b(f)\,D(|f|)\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(|f|)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}G^2(f)&\le\frac{1}{2}\bigvee_a^b(|f|)\cdot\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt\\&\le\frac{1}{2}\bigvee_a^b(|f|)\,D(f)\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(|f|)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2.\end{aligned}\tag{2.12}$$

The following representation is of interest in itself. The result was firstly obtained in [**5**] (see also [**6**]). For the sake a completeness we give here a short proof as well.

Lemma 1. *If* $v:[a,b]\to\mathbb{C}$ *is continuous (of bounded variation) on* $[a,b]$ *and* $h:[a,b]\to\mathbb{C}$ *is of bounded variation (continuous) on* $[a,b]$, *then we have the identity*

$$\begin{aligned}&\frac{v(b)\int_a^b(t-a)\,dh(t)+v(a)\int_a^b(b-t)\,dh(t)}{b-a}-\int_a^b v(t)\,dh(t)\\&=\int_a^b h(t)\,dv(t)-\frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt.\end{aligned}\tag{2.13}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \\
&= \int_a^b \left[\frac{v(b)(t-a)+v(a)(b-t)}{b-a} - v(t)\right] dh(t) \\
&= \left[\frac{(t-a)v(b)+(b-t)v(a)}{b-a} - v(t)\right] h(t)\Big|_a^b \\
&\qquad - \int_a^b h(t)\,d\left[\frac{(t-a)v(b)+(b-t)v(a)}{b-a} - v(t)\right] \\
&= [v(b)-v(b)]\,h(b) - [v(a)-v(a)]\,h(a) \\
&\qquad - \int_a^b h(t)\left[\frac{v(b)-v(a)}{b-a}dt - dv(t)\right] \\
&= \int_a^b h(t)\,dv(t) - \frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt
\end{aligned} \tag{2.14}$$

and the identity is proven. □

We can provide now the following corollary of Theorem 1:

Corollary 3. *If $v : I \to \mathbb{C}$ is differentiable on the interior of the interval I denoted $\mathring{I}$ and $[a,b] \subset \mathring{I}$, v' is of bounded variation on $[a,b]$ and $h : [a,b] \to \mathbb{C}$ is integrable on $[a,b]$, then we have the inequality*

$$\begin{aligned}
&\left|\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t)\right| \\
&\leq \frac{1}{2}\bigvee_a^b (v') \int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right| dt.
\end{aligned} \tag{2.15}$$

Proof. From (2.13) we have

$$\begin{aligned}
&\frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \\
&= \int_a^b h(t)\,v'(t)\,dt - \frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt.
\end{aligned} \tag{2.16}$$

Since v' is of bounded variation on $[a,b]$, then by (2.1) we have

$$\begin{aligned}
&\left|\int_a^b h(t)\,v'(t)\,dt - \frac{v(b)-v(a)}{b-a}\int_a^b h(t)\,dt\right| \\
&\leq \frac{1}{2}\bigvee_a^b (v') \cdot \int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right| dt
\end{aligned}$$

and the inequality (2.15) is proved. □

REMARK 2. *If in Corollary 3 we assume that $h : [a,b] \to \mathbb{C}$ is of bounded variation, then we have*

$$\left| \frac{v(b) \int_a^b (t-a)\, dh(t) + v(a) \int_a^b (b-t)\, dh(t)}{b-a} - \int_a^b v(t)\, dh(t) \right| \leq \frac{1}{4}(b-a) \bigvee_a^b (v') \bigvee_a^b (h). \tag{2.17}$$

COROLLARY 4. *If $v : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$ and $h : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then we have the inequality*

$$\left| \frac{v(b) \int_a^b (t-a)\, dh(t) + v(a) \int_a^b (b-t)\, dh(t)}{b-a} - \int_a^b v(t)\, dh(t) \right| \leq \frac{1}{2} \bigvee_a^b (h) \int_a^b \left| v'(t) - \frac{v(b) - v(a)}{b-a} \right| dt. \tag{2.18}$$

The constant $\frac{1}{2}$ is best possible in (2.18).

PROOF. By applying the inequality (2.1) for h of bounded variation we have

$$\left| \int_a^b h(t)\, v'(t)\, dt - \frac{v(b) - v(a)}{b-a} \int_a^b h(t)\, dt \right| \leq \frac{1}{2} \bigvee_a^b (h) \cdot \int_a^b \left| v'(t) - \frac{1}{b-a} \int_a^b v'(s)\, ds \right| dt$$

which together with (2.16) produces the desired inequality (2.18).

The case of equality holds for $v : [a,b] \to \mathbb{R}$ with $v(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$ and $h : [a,b] \to \mathbb{R}$ with $h(t) = sgn\left(t - \frac{a+b}{2}\right)$. □

Lemma 1 provides also the possibility to obtain other error bounds in approximating the Riemann-Stieltjes integral $\int_a^b v(t)\, dh(t)$ by the rule

$$\frac{v(b) \int_a^b (t-a)\, dh(t) + v(a) \int_a^b (b-t)\, dh(t)}{b-a}$$

when bounds for the Čebyšev functional $C(v', h)$ are provided.

For instance, we can state the following result:

PROPOSITION 1. *Assume that $v : I \to \mathbb{C}$ is differentiable on the interior of the interval I denoted $\mathring{I}$ and $[a,b] \subset \mathring{I}$.*

a) *If there exists the real constants γ and Γ so that $\gamma \leq v'(t) \leq \Gamma$ for a.e. $t \in [a,b]$ and h is Lebesgue integrable on $[a,b]$, then*

$$\left| \frac{v(b) \int_a^b (t-a)\, dh(t) + v(a) \int_a^b (b-t)\, dh(t)}{b-a} - \int_a^b v(t)\, dh(t) \right| \leq \frac{1}{2}(\Gamma - \gamma) \cdot \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s)\, ds \right| dt. \tag{2.19}$$

b) *If there exists the constants* n, N *such that* $n \leq h(t) \leq N$ *for almost* $t \in [a,b]$ *and* v *is twice differentiable with* $v'' \in L_\infty [a,b]$, *then we have the inequality*

$$\left| \frac{v(b)\int_a^b (t-a)\,dh(t) + v(a)\int_a^b (b-t)\,dh(t)}{b-a} - \int_a^b v(t)\,dh(t) \right| \leq \frac{1}{8}(b-a)^2 (N-n)\|v''\|_\infty . \tag{2.20}$$

3. Applications for Functions of Selfadjoint Operators

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [**10**, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [**10**] and the references therein.

For other recent results see [**7**], [**8**], [**9**], [**13**], [**16**], [**17**] and [**19**].

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle), \tag{3.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

THEOREM 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is absolutely continuous on $[m, M]$, then we have the inequality*

$$\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \tag{3.2}$$
$$\leq \frac{1}{2} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \int_m^M \left| f'(t) - \frac{f(M) - f(m)}{M - m} \right| dt$$
$$\leq \frac{1}{2} \|x\| \|y\| \int_m^M \left| f'(t) - \frac{f(M) - f(m)}{M - m} \right| dt$$

for any $x, y \in H$.

PROOF. If we apply the inequality (2.18) for $v(t) = f(t)$ and $h(t) = \langle E_t x, y \rangle$ where $t \in [m, M]$ and $x, y \in H$ we get

$$\left| \frac{v(M) \int_{m-0}^M (t - m)\, d\langle E_t x, y \rangle + f(m) \int_{m-0}^M (M - t)\, d\langle E_t x, y \rangle}{M - m} \right. \tag{3.3}$$
$$\left. - \int_{m-0}^M f(t)\, d\langle E_t x, y \rangle \right|$$
$$\leq \frac{1}{2} \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \int_m^M \left| f'(t) - \frac{f(M) - f(m)}{M - m} \right| dt.$$

Since

$$\int_{m-0}^M (t - m)\, d\langle E_t x, y \rangle = \langle (A - 1_H m) x, y \rangle, \int_{m-0}^M (M - t)\, d\langle E_t x, y \rangle = \langle (M1_H - A) x, y \rangle$$

and

$$\int_{m-0}^M f(t)\, d\langle E_t x, y \rangle = \langle f(A) x, y \rangle,$$

then we get from (3.3) the first part of (3.2).

If P is a nonnegative operator on H, i.e., $\langle Px, x \rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle$$

for any $x, y \in H$.

Further, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators that

$$\bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right)$$
$$= \sup_d \left\{ \sum_{i=0}^{n-1} \left| \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, y \right\rangle \right| \right\}$$
$$\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \right\rangle^{1/2} \right] \right\} := I.$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$
\begin{aligned}
I &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \right\rangle \right]^{1/2} \right\} \\
&\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) x, x \right\rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i} \right) y, y \right\rangle \right]^{1/2} \right\} \\
&= \left[\bigvee_m^M \left(\left\langle E_{(\cdot)} x, x \right\rangle \right) \right]^{1/2} \left[\bigvee_m^M \left(\left\langle E_{(\cdot)} y, y \right\rangle \right) \right]^{1/2} = \|x\| \|y\|
\end{aligned}
$$

for any $x, y \in H$. These prove the last part of (3.2). □

Since the integral

$$\int_m^M \left| f'(t) - \frac{f(M) - f(m)}{M - m} \right| dt$$

is sometimes difficult to calculate we can provide simpler, however coarser, bounds as in the following corollary.

Corollary 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

a) *If there exists the constants γ, Γ such that $\gamma \leq f'(t) \leq \Gamma$ for almost every $t \in [m, M]$, then*

$$
\begin{aligned}
&\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \\
&\leq \frac{1}{4} (\Gamma - \gamma)(M - m) \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \\
&\leq \frac{1}{4} (\Gamma - \gamma)(M - m) \|x\| \|y\|,
\end{aligned} \tag{3.4}
$$

for all $x, y \in H$;

b) *If f is convex on $[m, M]$ and $f'_-(M), f'_+(m)$ are finite, then*

$$
\begin{aligned}
&\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \\
&\leq \frac{1}{4} \left(f'_-(M) - f'_+(m) \right)(M - m) \bigvee_m^M \left(\left\langle E_{(\cdot)} x, y \right\rangle \right) \\
&\leq \frac{1}{4} \left(f'_-(M) - f'_+(m) \right)(M - m) \|x\| \|y\|,
\end{aligned} \tag{3.5}
$$

for all $x, y \in H$;

c) *If f' is absolutely continuous on $[m,M]$ and $f'' \in L_2[m,M]$, then*

$$\left|\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m}\right]x,y\right\rangle - \langle f(A)x,y\rangle\right| \tag{3.6}$$
$$\leq \frac{1}{2\pi}\|f''\|_2 (M-m)^{3/2} \bigvee_m^M (\langle E_{(\cdot)}x,y\rangle)$$
$$\leq \frac{1}{2\pi}\|f''\|_2 (M-m)^{3/2}\|x\|\|y\|,$$

for all $x,y \in H$, where $\|f''\|_2 = \left(\int_m^M |f''(t)|^2 dt\right)^{1/2}$.

Proof. By the Cauchy-Bunyakovsky-Schwarz's integral inequality we have that

$$\frac{1}{M-m}\int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt \tag{3.7}$$
$$= \frac{1}{M-m}\int_m^M \left|f'(t) - \frac{1}{M-m}\int_m^M f'(s)\,ds\right| dt$$
$$\leq \left[\frac{1}{M-m}\int_m^M |f'(t)|^2 dt - \left|\frac{1}{M-m}\int_m^M f(t)\,dt\right|^2\right]^{1/2}$$
$$= B(m,M).$$

a) By the Grüss inequality for f' we have that

$$B(m,M) \leq \frac{1}{2}(\Gamma - \gamma)$$

which together with the first part of (3.2) and (3.7) produces the desired result (3.4).

b) Follows from a).

c) Follows from the Lupaş inequality (1.5). □

In the following we give only two examples that are particular cases of the inequality (3.2). We invite the interested reader to apply the above results for other functions of interest such as the exponential function or power functions.

Example 1. *We consider the concave function $f : [m,M] \subset (0,\infty) \to \mathbb{R}$, $f(t) = \ln t$. If we define by*

$$L(m,M) := \begin{cases} \frac{M-m}{\ln M - \ln m} & \text{if } m \neq M \\ m & \text{if } m = M \end{cases}$$

the logarithmic mean of the positive numbers m, M, then

$$\int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt$$
$$= \int_m^M \left|\frac{1}{t} - \frac{1}{L(m,M)}\right| dt$$
$$= 2\left\{\ln\left[\frac{L(m,M)}{G(m,M)}\right] + \frac{A(m,M) - L(m,M)}{L(m,M)}\right\}$$

where $G(m,M) := \sqrt{mM}$ *is the geometric mean and* $A(m,M) := \frac{m+M}{2}$ *is the arithmetic mean of the positive numbers* m, M.

Utilising the inequality (3.2) we can state the inequality

$$\left|\left\langle\left[\frac{(M1_H - A)\ln m + (A - m1_H)\ln M}{M-m}\right]x,y\right\rangle - \langle \ln Ax, y\rangle\right| \le \left\{\ln\left[\frac{L(m,M)}{G(m,M)}\right] + \frac{A(m,M) - L(m,M)}{L(m,M)}\right\}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \le \left\{\ln\left[\frac{L(m,M)}{G(m,M)}\right] + \frac{A(m,M) - L(m,M)}{L(m,M)}\right\}\|x\|\,\|y\| \tag{3.8}$$

for any A *a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m,M] \subset (0,\infty)$ *and for any* $x, y \in H$.

Since f *is a concave function, then we get from (3.8) the following inequality*

$$0 \le \ln A - \frac{(M1_H - A)\ln m + (A - m1_H)\ln M}{M-m} \le \left\{\ln\left[\frac{L(m,M)}{G(m,M)}\right] + \frac{A(m,M) - L(m,M)}{L(m,M)}\right\}1_H \tag{3.9}$$

in the operator order of $B(H)$.

Example 2. *We consider the convex function* $f : [m,M] \subset (0,\infty) \to \mathbb{R}$, $f(t) = \frac{1}{t}$. *If we define by*

$$H(m,M) := \frac{2}{\frac{1}{m} + \frac{1}{M}}$$

the harmonic mean of the positive numbers m, M, *then*

$$\int_m^M \left| f'(t) - \frac{f(M) - f(m)}{M-m}\right| dt = \int_m^M \left|\frac{1}{t^2} - \frac{1}{G^2(m,M)}\right| dt = \frac{2}{G(m,M)}\left(\frac{G(m,M) - H(m,M)}{H(m,M)} + \frac{A(m,M) - G(m,M)}{G(m,M)}\right).$$

Utilising the inequality (3.2) we can state the inequality

$$\left|\left\langle\left[\frac{m^{-1}(M1_H - A) + M^{-1}(A - m1_H)}{M-m}\right]x,y\right\rangle - \langle A^{-1}x, y\rangle\right| \le \frac{1}{G(m,M)}\left[\frac{G(m,M) - H(m,M)}{H(m,M)} + \frac{A(m,M) - G(m,M)}{G(m,M)}\right] \times \bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \le \frac{1}{G(m,M)}\left[\frac{G(m,M) - H(m,M)}{H(m,M)} + \frac{A(m,M) - G(m,M)}{G(m,M)}\right] \times \|x\|\,\|y\| \tag{3.10}$$

for any A *a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m,M]$ *and for any* $x, y \in H$.

Since f is convex, then by (3.10) we also have

$$0 \le \frac{m^{-1}(M1_H - A) + M^{-1}(A - m1_H)}{M - m} - A^{-1} \le \left\{ \frac{1}{G(m,M)} \left[\frac{G(m,M) - H(m,M)}{H(m,M)} + \frac{A(m,M) - G(m,M)}{G(m,M)} \right] \right\} 1_H \tag{3.11}$$

in the operator order of $B(H)$.

References

[1] P. CERONE and S.S. DRAGOMIR, New bounds for the Čebyšev functional, *Appl. Math. Lett.,* **18** (2005), 603-611.

[2] P. CERONE and S.S. DRAGOMIR, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint available at *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE `http://rgmia.vu.edu.au/v5n2.html`].

[3] P.L. CHEBYSHEV, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[4] X.-L. CHENG and J. SUN, Note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.,* **3**(2) (2002), Art. 29. [ONLINE `http://jipam.vu.edu.au/article.php?sid=181`].

[5] S.S. DRAGOMIR, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.*, **26** (2004), 89-112.

[6] S.S. DRAGOMIR, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.,* **20** (2007), 123-130.

[7] S.S. DRAGOMIR, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[8] S.S. DRAGOMIR, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[9] S.S. DRAGOMIR, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, Prerpint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp]

[10] T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ and Y. SEO, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[11] G. GRÜSS, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$, *Math. Z.,* **39** (1934), 215-226.

[12] A. LUPAŞ, The best constant in an integral inequality, *Mathematica* (Cluj), **15** (**38**) (1973), No. 2, 219-222.

[13] A. MATKOVIĆ, J. PEČARIĆ and I. PERIĆ, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), No. 2-3, 551–564.

[14] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, Dordrecht, 1991.

[15] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[16] B. MOND and J. PEČARIĆ, Convex inequalities in Hilbert spaces, *Houston J. Math.*, **19**(1993), 405-420.

[17] B. MOND and J. PEČARIĆ, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.

[18] A.M. OSTROWSKI, On an integral inequality, *Aequationes Math.,* **4** (1970), 358-373.

[19] J. PEČARIĆ, J. MIĆIĆ and Y. SEO, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207.

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