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INEQUALITIES FOR THE SUM OF TWO SINES OR COSINES

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Abstract. We prove some inequalities for the expressions $(\sin x + \sin y)/(x+y)$ and $(\cos x + \cos y)/(x+y)$.

1. Introduction and main results

In [1] H. Alzer proved the following intriguing inequality (which holds for all reals x):

$$\alpha|\sin(\cos x)+\sin(\sin x)| \le |\cos x+\sin x| \le \beta\left(\cos(\cos x)+\cos(\sin x)\right), \tag{1}$$

where the constant factors

$$\alpha := \frac{1}{\sqrt{2}\sin(1/\sqrt{2})} \approx 1.0884 \quad \text{and} \quad \beta := \frac{1}{\sqrt{2}\cos(1/\sqrt{2})} \approx 0.9301$$

are best possible. Part of his main motivation was to expand on an older and just as nice set of inequalities proved by R.J. Webster (see [6] and also [2] and [4]):

$$|\sin(\cos x)| \le |\cos x| \le \cos(\sin x), \qquad \forall\, x \in \mathbb{R}.$$

In this note I will focus less on the iteration aspect of these results, and rather on the estimate of the sum of sines, thereby obtaining a broader scope within which Alzer's result will follow as a special case. To summarize, here is our main result in the form of a chain of inequalities:

Main Theorem. *For (x,y) in the disk $\sqrt{x^2+y^2} \le \pi/2$, we have*

$$\begin{aligned}\cos\sqrt{\frac{x^2+y^2}{2}} \quad &\le \quad \frac{\sin x+\sin y}{x+y} \le \frac{\sin\sqrt{\frac{x^2+y^2}{2}}}{\sqrt{\frac{x^2+y^2}{2}}} \\ &\le \quad \frac{1}{2}\left(\frac{\sin x}{x}+\frac{\sin y}{y}\right) \le \frac{1}{2}\left(1+\frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right).\end{aligned}$$

Identity in the first inequality holds if and only if $x=-y$ (in which case the central term is redefined as $\cos x$ by continuity), while identity in the second inequality holds if and only if $x=y$. Identity holds in the third inequality if and only if $x=\pm y$. Finally, in the fourth we have identity if and only if $x=0$ or $y=0$.

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The proofs are to be found in section 2. Specifically, the first two inequalities are proved in Theorem 2.6, while the third and fourth are proved in Theorem 2.11.

A common pattern in these inequalities is the function $(\sin x)/x$, familiar to every calculus student thanks to its appealing analytic continuation at $x = 0$. To avoid cumbersome notation and needless technicalities, we will liberally assume that such a continuation is in place, and such a remark includes the situations in the plane where, for example, the denominator of $\frac{\sin x+\sin y}{x+y}$ should vanish: in such cases we would naturally redefine the function as the appropriate limit.

Back to the Main Theorem, if we specialize the first two inequalities to the unit circle we obtain a result where we recognize the "sine half" of Alzer's result (1), albeit without the need for absolute values:

Corollary 1.1. *Let $x^2 + y^2 = 1$. Then*

$$0.760245 \approx \cos\frac{1}{\sqrt{2}} \le \frac{\sin x + \sin y}{x + y} \le \sqrt{2}\sin\frac{1}{\sqrt{2}} \approx 0.918725$$

Though it's not the focus of this paper, a cosine counterpart to the above results is possible and to illustrate this we give the next theorem, whose proof is an immediate consequence of Theorem 2.9 in the next section.

Theorem 1.2. *For any x, y with $\sqrt{x^2 + y^2} \le \pi/\sqrt{2}$ we have*

$$\frac{\cos\sqrt{\frac{x^2+y^2}{2}}}{\sqrt{\frac{x^2+y^2}{2}}} \le \left|\frac{\cos x + \cos y}{x + y}\right|.$$

with identity if and only if $x = y$.

Of course the behavior of cosine is quite different from sine's near the origin, and thus it would be pointless to pursue the parallels too far. For example, while the level curves of $(\sin x + \sin y)/(x + y)$ have a roughly ellipsoidal shape (up to periodicity) and thus are all closed curves, the level curves of $(\cos x+\cos y)/(x+y)$ can be roughly sinusoidal (and periodic). This being said, by specializing Theorem 1.2 to the case when $2k^2 = 1$, that is, when $k = \sqrt{2}\cos(1/\sqrt{2})$, we obtain the "cosine half" of Alzer's result:

Corollary 1.3 (Alzer [1])**.** *For any $x \in \mathbb{R}$ we have*

$$1.07515 \approx \sqrt{2}\cos\frac{1}{\sqrt{2}} \le \frac{\cos(\cos x) + \cos(\sin x)}{|\cos x + \sin x|}.$$

2. Proofs

At least the first part of the following lemma is well known. Note that the inequality stated there (in part (1)) would be true in a broader interval, but for our purposes we only need it to hold in $(0, \pi/2)$.

Lemma 2.1. *(1) For $x \in (0, \pi/2)$ we have*

$$0 < \frac{1}{x} - \cot x < x.$$

(2) For $x \in (0,1)$ we have

$$\arccos x < \frac{\sqrt{1-x^2}}{x}.$$

Proof. Set $s(x) := \frac{1}{x} - \cot x$. Because of the limits $\lim_{x\downarrow 0} s(x) = 0$, $\lim_{x\downarrow 0} s'(x) = \frac{1}{3}$, we know that $s(x) < x$ when x is positive and small. Since $s(x)$ clearly is increasing, and $s(\pi/2) = 2/\pi < \pi/2$, part (1) follows. Note that $s(x)$ is convex for $x \in (0,\pi)$ (this is equivalent to the well-known inequality $\cos x < \left(\frac{\sin x}{x}\right)^3$ — see [5, 3.4.18]) and thus $s(x) = x$ exactly once in $(\pi/2, \pi)$.

The inequality in part (2) is easier to study if we apply the change of variable $x \to 1/\sqrt{1+y^2}$, thus obtaining the equivalent

$$\arccos \frac{1}{1+y^2} < y \qquad \forall\, y \in (0,\infty). \tag{2}$$

However, this is immediately verified by the calculation

$$\left(\arccos \frac{1}{1+y^2}\right)'' = -\frac{2y}{(1+y^2)^2},$$

from which it follows that the left hand side of (2) is concave for all positive y, and thus must be less than y given that both functions agree at $y = 0$. □

Lemma 2.2. *(1) When $k \in [0,1]$ and $x \in (0, \arccos k]$ we have*

$$\sqrt{\arccos^2 k - x^2} \le \arccos\left(\frac{kx}{\sin x}\right).$$

(2) Fix $k \in [\frac{2}{\pi}, 1]$, and let $r := \min\{t \in (0,\pi] : k = \frac{\sin t}{t}\}$ and $x \in (0, r]$. Then

$$\arccos\left(\frac{kx}{\sin x}\right) \le \sqrt{r^2 - x^2}$$

Proof. Both parts of the lemma can be seen as statements about the function

$$f(x) := \arccos^2\left(\frac{kx}{\sin x}\right) + x^2.$$

A calculation gives that

$$f'(x) = 2x - 2\frac{\arccos(h(x))}{\sqrt{1-h(x)^2}/h(x)}\left(\frac{1}{x} - \cot x\right), \tag{3}$$

where we put $h(x) := kx/\sin x$ to simplify the expression. Now, by construction, since $x \in (0,r]$ we have $h(x) \in (0,1)$ and accordingly the second part of Lemma 2.1 implies

$$\frac{\arccos(h(x))}{\sqrt{1-h(x)^2}/h(x)} < 1,$$

and thus

$$f'(x) > 2\left(x - \frac{1}{x} + \cot x\right) \qquad \forall\, x \in (0,r].$$

The first part of Lemma 2.2 only needs looking at $x \in (0, \arccos k]$, and thus $x \in (0, \pi/2]$ in all cases, meaning that by the first part of Lemma 2.1 we know that $f(x)$ is increasing on $(0,\pi/2)$. Hence, $f(x)$ achieves its infimum over $(0, \arccos k]$ at $x = 0$, where it equals $\arccos k$ — and thus the first part of Lemma 2.2 is proved.

In the second part of Lemma 2.2 we only consider $k \in [\frac{2}{\pi}, 1]$, and in this range $r \in [0, \pi/2]$ and thus the same argument as above shows that $f(x)$ is increasing for $x \in (0, r]$: therefore, $f(x)$ reaches its maximum at $f(r) = r^2$, as claimed. □

Theorem 2.3. *Fix $k \in [0,1]$, let $r := \min\{t \in (0,\pi] : k = \frac{\sin t}{t}\}$, and define $g(x,y) := \frac{1}{x}\sin x \cos y$ for (x,y) in the rectangle with vertices $(\pm\pi, \pm\frac{\pi}{2})$ (with the natural continuation when $x = 0$). The circle centered at the origin and with radius $\arccos k$ lies in the interior and is tangent to the k-level curve of $g(x,y)$.*

Further, if $k \in [\frac{2}{\pi}, 1]$ we also know that the circle centered at the origin and with radius r lies outside of and is tangent to the k-level curve of $g(x,y)$.

Proof. $g(x,y)$ (defined as in the statement) clearly is invariant if we flip the signs of x or y, and thus it is enough to study the behavior of the k-level curve of $g(x,y)$ in the first quadrant, where we can solve its equation as

$$y = \arccos\left(\frac{kx}{\sin x}\right).$$

Consequently, the rest of the statement follows straight from Lemma 2.2. □

The k-level curves of $\frac{1}{x}\sin x \cos y$ look a lot like less and less eccentrical ellipses when $k \uparrow 1$, while in the limit $k \downarrow 0$ the level curves uniformly approximate the rectangle with vertices $(\pm\pi, \pm\frac{\pi}{2})$ from the inside.

Corollary 2.4.
1. *If $k \in [0,1]$ and $\sqrt{\xi^2 + \eta^2} = \arccos k$ then*
$$k \le \frac{1}{\xi}\sin\xi\cos\eta$$
with identity if and only if $\xi = 0$ and $\eta = \pm\arccos k$.
2. *Let $k \in [\frac{2}{\pi}, 1]$ and let $r := \min\{t \in (0,\pi] : k = \frac{\sin t}{t}\}$. If $\sqrt{\xi^2 + \eta^2} = r$ then*
$$\frac{1}{\xi}\sin\xi\cos\eta \le k$$
with identity if and only if $\xi = \pm r$ and $\eta = 0$.

If we apply the change of variables $\xi := \frac{x+y}{2}$, $\eta := \frac{x-y}{2}$ to the setting of Theorem 2.3, we can summarize our work so far into

Theorem 2.5.
1. *If $k \in [0,1]$, the circle centered at the origin and with radius $\sqrt{2}\arccos k$ lies in the interior and is tangent to the k-level curve of $h(x,y) := \frac{\sin x + \sin y}{x+y}$. Accordingly, if $\sqrt{x^2+y^2} = \sqrt{2}\arccos k$ then*
$$k \le \frac{\sin x + \sin y}{x+y}$$
with identity if and only if $x = -y = \pm\arccos k$.
2. *Let $k \in [\frac{2}{\pi}, 1]$ and $r := \min\{t \in (0,\pi] : k = \frac{\sin t}{t}\}$. Then the circle centered at the origin and with radius $\sqrt{2}r$ lies outside of and is tangent to the k-level curve of $h(x,y) := \frac{\sin x + \sin y}{x+y}$. Accordingly, if $\sqrt{x^2+y^2} = \sqrt{2}r$ then*
$$\frac{\sin x + \sin y}{x+y} \le k$$
with identity if and only if $x = y = \pm r$.

Theorem 2.6. *For x, y with $\sqrt{x^2+y^2} \le \frac{\pi}{2}$ we have*

$$\cos\sqrt{\frac{x^2+y^2}{2}} \le \frac{\sin x + \sin y}{x+y} \le \frac{\sin\sqrt{\frac{x^2+y^2}{2}}}{\sqrt{\frac{x^2+y^2}{2}}}$$

Identity in the first inequality holds if and only if $x = -y$ (in which case the central term is redefined as $\cos x$ by continuity), while identity in the second inequality holds if and only if $x = y$.

Proof. In the notation of Theorem 2.5, if $x^2 + y^2 = 2(\arccos k)^2$ then we have

$$\cos\sqrt{\frac{x^2+y^2}{2}} = k,$$

and the first inequality follows. Note that as $k \in [0,1]$ the inequality holds for x, y such that $\sqrt{x^2+y^2} \le \pi/\sqrt{2}$. Similarly, the second inequality follows from the second part of Theorem 2.5. Since there we restrict ourselves to $k \in [\frac{2}{\pi}, 1]$, we obtain an overall limitation $\sqrt{x^2+y^2} \le \frac{\pi}{2}$ for the validity of our inequalities, even though they remain true for a broader (but harder to define) region of the (x, y)-plane. □

Lemma 2.7. *Fix $k > 0$. The function $\frac{\cos x}{x}$ is invertible when defined on $(0, \pi/2)$, and we define $r : \mathbb{R}^+ \to (0, \pi/2)$ as its inverse. Then, for $x \in (0, r(k))$ we have*

$$\sqrt{r(k)^2 - x^2} \le r\left(\frac{k}{\cos x}\right).$$

Lemma 2.8. *Fix $k \in (0, \pi/2)$. Since $f(x) := \frac{\cos x}{x}$ is invertible on $(0, \pi/2)$ (with inverse f^{-1}), the implicit equation*

$$\cos(x)\cos(y) = y\frac{\cos k}{k}$$

uniquely defines a function

$$g(x) := f^{-1}\left(\frac{f(k)}{\cos x}\right)$$

over the same interval $(0, \pi/2)$. Then, the circle centered at the origin and with radius k lies below and is tangent to the graph of $g(x)$ (the actual tangent point being $(0, k)$, at the left boundary of the interval). Similarly, the same circle lies above and is tangent to the graph of $-g(x)$ at the point of coordinates $(0, -k)$.

Proof. Fix $k \in (0, \pi/2)$ and let $f(x), g(x)$ be as in the theorem's statement. We need to prove that $g(x) \ge \sqrt{k^2 - x^2}$ for all $x \in (0, k)$. Since $f(x)$ is strictly decreasing, this is equivalent to proving

$$\frac{f(k)}{\cos x} \le f\left(\sqrt{k^2 - x^2}\right)$$

or, in other words,

$$\cos\left(\sqrt{k^2 - x^2}\right)\cos x \ge \frac{\cos k}{k}\sqrt{k^2 - x^2}, \qquad \forall\, x \in (0, k).$$

The change of variables $x \to \sqrt{k^2-\xi^2}$ simplifies this inequality to

$$\cos\left(\sqrt{k^2-\xi^2}\right)\cos\xi \geq \frac{\cos k}{k}\xi, \qquad \forall\,\xi\in(0,k). \tag{4}$$

Now, the function $t(\xi) := \cos\left(\sqrt{k^2-\xi^2}\right)\cos\xi$ has the derivative

$$\xi\cos\left(\sqrt{k^2-\xi^2}\right)\cos\xi\left(\frac{\tan\left(\sqrt{k^2-\xi^2}\right)}{\sqrt{k^2-\xi^2}} - \frac{\tan\xi}{\xi}\right),$$

from which we see (since $\frac{\tan x}{x}$ is strictly increasing) that $t(\xi)$ only has one critical point in $(0,k)$, and that's where $\xi = \sqrt{k^2-\xi^2} = k/\sqrt{2}$ (another calculation shows that $t(\xi)$ has a maximum there – we omit the details). We conclude that $t(\xi)$'s minimum is achieved at the endpoints, thus being $t(0) = t(k) = \cos k$. Since the straight line with equation $\ell(\xi) := \frac{\cos k}{k}\xi$ increases from zero to $\cos k$ on the interval $(0,k)$ (see the right hand side of inequality (4)), the inequality (4) is proved. □

Theorem 2.9. *Fix $k\in(0,\pi/2)$. If $x^2+y^2\leq 2k^2$ then*

$$\left|\frac{\cos x+\cos y}{x+y}\right| \geq \frac{\cos k}{k}$$

with identity if and only if $x = y = \pm k$.

Proof. Let $k\in(0,\pi/2)$, and consider the exact situation described in Lemma 2.8, but written in terms of variables A, B instead of x, y. That is, we know from the Lemma that the circle of equation $A^2+B^2=k^2$ is tangent to and lies below the curve defined implicitly by

$$\cos A\cos B = B\frac{\cos k}{k} \tag{5}$$

(where we by symmetry may assume $A\in(-\pi/2,\pi/2)$). Consider now the change of variables $A\to(x-y)/2$, $B\to(x+y)/2$. In terms of x and y the circle of equation $A^2+B^2=k^2$ remains centered at the origin but has now the equation $x^2+y^2=2k^2$, while the implicit equation (5) becomes $\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) = \frac{x+y}{2}\frac{\cos k}{k}$, i.e., by a well-known trigonometric identity,

$$\frac{\cos x+\cos y}{x+y} = \frac{\cos k}{k}. \tag{6}$$

The theorem now easily follows. □

Lemma 2.10. *(1) For $x\in(0,\pi/2)$ we have*

$$\tan x > \frac{3x}{3-x^2}.$$

(2) The function

$$x\mapsto -\frac{\cos x}{x^3}(\tan x - x)$$

is strictly increasing on $(0,\pi/2)$.

Note that since the MacLaurin series of $\frac{3x}{3-x^2}$ consists of positive terms (for $x>0$) and its first two are $x+x^3/3$, part (1) of Lemma 2.10 clearly improves on the classical result quoted in [5, 3.4.27].

Proof. (1): This inequality is a special case of a stronger result proved by Chen and Qi [3, remark after Theorem 1]. To keep this paper self-contained we present the following alternate proof. Define the function

$$f(x) := \tan x - \frac{3x}{3-x^2}.$$

Since

$$f''(x) = \frac{2\tan x}{\cos^2 x} + \frac{6x(x^2+9)}{(x^2-3)^3},$$

we know that $f(x)$ is strictly convex on $(0, \pi/2)$. This implies that the first derivative

$$f'(x) = \frac{1}{\cos^2 x} - \frac{3(x^2+3)}{(x^2-3)^2}$$

is strictly increasing on $(0, \pi/2)$. Due to $\lim_{x\to 0} f'(x) = 0$, we conclude that $f'(x) \geq 0$ for $x \in (0, \pi/2)$. Consequently, $f(x)$ itself is strictly increasing for $x \in (0, \pi/2)$ and, since $\lim_{x\to 0} f(x) = 0$, the statement in part (1) follows.

(2): A calculation shows that the derivative of this function can be written as

$$(3-x^2)\cos x\left(\tan x - \frac{3x}{3-x^2}\right),$$

and thus part (1) yields the claim. □

Theorem 2.11. *For all x, y such that $\sqrt{x^2+y^2} \leq \pi/2)$ we have*

$$\frac{\sin\sqrt{\frac{x^2+y^2}{2}}}{\sqrt{\frac{x^2+y^2}{2}}} \leq \frac{1}{2}\left(\frac{\sin x}{x} + \frac{\sin y}{y}\right) \leq \frac{1}{2}\left(1 + \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right).$$

Identity holds in the first inequality if and only if $x = \pm y$, while in the second we have identity if and only if $x = 0$ or $y = 0$.

Proof. Let $r := \sqrt{x^2+y^2}$ and consider all (x, y) with fixed $r \in (0, \pi/2)$. We need to study the function

$$f(\alpha) := \frac{\sin(r\cos\alpha)}{r\cos\alpha} + \frac{\sin(r\sin\alpha)}{r\sin\alpha}$$

where $\alpha \in (0, \pi/2)$ (indeed, it's easy to see that $f(\alpha)$ is periodic of period $\pi/2$). Setting $g(x) := \frac{\sin x}{x}$, we can write

$$f(\alpha) = g(r\cos\alpha) + g\,(r\sin\alpha)$$

and thus a critical point α of $f(\alpha)$ in $(0, \pi)$ must satisfy

$$g'(r\cos\alpha)\sin\alpha = g'\,(r\sin\alpha)\cos\alpha,$$

or, rearranging,

$$\frac{g'(r\cos\alpha)}{r\cos\alpha} = \frac{g'\,(r\sin\alpha)}{r\sin\alpha}. \tag{7}$$

However, from differentiating $g(x)$ we know that

$$\frac{g'(x)}{x} = -\frac{\cos x}{x^3}(\tan x - x),$$

and thus part (2) of Lemma 2.10 tells us that $g'(x)/x$ is strictly increasing on $(0, \pi/2)$. Therefore, condition (7) can only be satisfied when $\cos\alpha = \sin\alpha$, i.e., when $\alpha = \pi/4$. This value easily turns out to be the location of the minimum of $f(\alpha)$ over $(0, \pi/2)$ (while $f(\alpha)$ attains its maximum at $\alpha = 0 = \pi/2$). We thus have proved that

$$2\frac{\sin\left(r/\sqrt{2}\right)}{r/\sqrt{2}} \le \frac{\sin(r\cos\alpha)}{r\cos\alpha} + \frac{\sin(r\sin\alpha)}{r\sin\alpha} \le 1 + \frac{\sin r}{r},$$

and this immediately translates into what the theorem required. □

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