

**VARIOUS ERROR ESTIMATIONS FOR SEVERAL
NEWTON–COTES QUADRATURE FORMULAE IN TERMS OF
AT MOST FIRST DERIVATIVE AND APPLICATIONS IN
NUMERICAL INTEGRATION**

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ABSTRACT. Error estimations for the midpoint, trapezoidal, Simpson's, Milne's, Maclaurin's, 3/8-Simpson's and Boole's rules are obtained. Some related inequalities of Ostrowski's type are pointed out. These results are obtained for mappings of bounded variation, Lipschitzian, and absolutely continuous mappings whose first derivatives are belong to $L_p[a, b]$ ($1 \leq p \leq \infty$). Applications to numerical integration are provided.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following inequality, known as the *Hermite–Hadamard inequality* for convex functions, holds:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the interior of the interval I , such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality,*

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq M(b-a) \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]$$

holds for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In [23], Dragomir, Cerone and Roumeliotis proved the following generalization of Ostrowski's inequality.

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous on $[a, b]$, differentiable on (a, b) and whose derivative f' is bounded on (a, b) . Denote $\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty$.*

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Then,

$$(1.3) \quad \left| (b-a) \left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) f(x) \right] - \int_a^b f(t) dt \right| \\ \leq \left[\frac{(b-a)^2}{4} (\lambda^2 + (1-\lambda)^2) + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty.$$

for all $\lambda \in [0, 1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq b - \lambda \frac{b-a}{2}$.

Using (1.3), the authors obtained estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae. They also gave applications of the mentioned results in numerical integration and for special means. For recent results, generalizations and new inequalities of Hermite–Hadamard, Ostrowski and Simpson's type the reader may refer to [1]–[45] and the references therein.

Motivated by [32], Dragomir in [21] has proved the following companion of the Ostrowski inequality:

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequalities*

$$(1.4) \quad \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \begin{cases} \left[\frac{1}{8} + 2 \left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty, & f' \in L_\infty[a, b] \\ \frac{2^{1/q}}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q} \|f'\|_{[a,b],p}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } f' \in L_p[a, b] \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

In [22], Dragomir established some inequalities for this companion for mappings of bounded variation.

Theorem 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then we have the inequalities:*

$$(1.5) \quad \left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left| \frac{x - \frac{3a+b}{4}}{b-a} \right| \right] \cdot \bigvee_a^b(f),$$

for any $x \in [a, \frac{a+b}{2}]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a, b]$. The constant $1/4$ is best possible.

In [33], Liu introduced some companions of an Ostrowski type inequality for functions whose first derivative are absolutely continuous. In [12], Barnett, Dragomir and Gomma have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality. Recently, Alomari [1] and [8]–[10] proved several inequalities for Dragomir's companion of Ostrowski's inequality for absolutely continuous, differentiable mappings and mappings of bounded variation

Recently, M.W. Alomari [10] proved a companion of (1.3), as follows:

Theorem 5. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable on I° , the interior of the interval I , where $a, b \in I$ with $a < b$. If f' is bounded on $[a, b]$, i.e., $\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty$. Then the inequality holds*

$$(1.6) \quad \left| (b-a) \left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda) \frac{f(x) + f(a+b-x)}{2} \right] - \int_a^b f(t) dt \right| \leq \left[\frac{(b-a)^2}{8} (2\lambda^2 + (1-\lambda)^2) + 2 \left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4} \right)^2 \right] \|f'\|_\infty.$$

holds, for all $\lambda \in [0, 1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

In the recent years, a number of authors have considered an error analysis quadrature rules of Newton-Cotes type. In particular, the midpoint, trapezoidal, Simpson's, Milne's, Maclaurin's, 3/8-Simpson's and Boole's rules have been investigated more recently with the view of obtaining bounds on the quadrature rule in terms of a variety of differentiable mappings, at most first derivative. Simply, they tried to reduce the condition of higher derivatives in these formulae and then to have alternative quadrature formulae with weaker conditions. The reader may refer to a sample of these results in the references. Here, we consider the above mentioned formulae, as follow [15]:

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable mapping on I° , the interior of the interval I , where $a, b \in I$ with $a < b$, such that $f^{(n)} \in L_1[a, b]$. Assume that $f^{(n)}$ is bounded on $[a, b]$, i.e., $\|f^{(n)}\|_\infty := \sup_{t \in [a, b]} |f^{(n)}(t)| < \infty$.

(1) If $n = 2$, then the midpoint inequality is given by

$$(1.7) \quad \left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{(b-a)^3}{24} \|f''\|_\infty.$$

(2) If $n = 2$, then the trapezoid inequality is given by

$$(1.8) \quad \left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t) dt \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty.$$

(3) If $n = 4$, then the Simpson's inequality is given by

$$(1.9) \quad \left| \frac{(b-a)}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{(b-a)^5}{2880} \|f^{(4)}\|_\infty.$$

(4) If $n = 4$, then the Milne's inequality is given by

$$(1.10) \quad \left| \frac{(b-a)}{3} \left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{7(b-a)^5}{23040} \|f^{(4)}\|_\infty.$$

(5) If $n = 4$, then the Maclaurin's inequality is given by

$$(1.11) \quad \left| \frac{b-a}{8} \left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right) \right] - \int_a^b f(t) dt \right| \leq \frac{7(b-a)^5}{51840} \|f^{(4)}\|_\infty.$$

(6) If $n = 4$, then the 3/8-Simpson's inequality is given by

$$(1.12) \quad \left| \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{(b-a)^5}{6480} \|f^{(4)}\|_\infty.$$

(7) If $n = 6$, then the Boole's inequality is given by

$$(1.13) \quad \left| \frac{b-a}{90} \left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{4}{945} (b-a)^7 \|f^{(6)}\|_\infty.$$

The aim of this paper, is to obtain various error estimations for the midpoint, trapezoidal, Simpson's, Milne's, Maclaurin's, 3/8-Simpson's and Boole's rules and related inequalities of Ostrowski's type for mappings of bounded variation, Lipschitzian, and absolutely continuous mappings whose first derivatives are belong to $L_p[a, b]$ ($1 \leq p \leq \infty$). Applications to numerical integration are provided.

2. THE CASE WHEN f IS OF BOUNDED VARIATION

We begin with the following result:

Theorem 6. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then we have the inequality*

$$(2.1) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{1}{2\delta} \max \{ 2\delta(x-a) - \alpha(b-a), \alpha(b-a), (\alpha + \beta)(b-a) - 2\delta(x-a), (\delta - \alpha - \beta)(b-a) \} \cdot \bigvee_a^b(f),$$

for all $\frac{(2\delta - \alpha)a + \alpha b}{2\delta} \leq x \leq \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta}$, where, $\bigvee_a^b(f)$ denotes to total variation of f over $[a, b]$, and $\alpha, \beta, \gamma, \delta$ are positive constants such that $\alpha + \beta + \gamma = 2\delta$, with $\delta \neq 0$.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\int_a^x K(x, t; \alpha, \beta, \gamma, \delta) df(t) = \left(x - \frac{(2\delta - \alpha)a + \alpha b}{2\delta} \right) f(x) + \frac{\alpha}{2\delta} (b-a) f(a) - \int_a^x f(t) dt,$$

$$\int_x^{\frac{a+b}{2}} K(x, t; \alpha, \beta, \gamma, \delta) df(t) = \left(\frac{a+b}{2} - \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} \right) f\left(\frac{a+b}{2}\right) - \left(x - \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} \right) f(x) - \int_x^{\frac{a+b}{2}} f(t) dt,$$

$$\int_{\frac{a+b}{2}}^{a+b-x} K(x, t; \alpha, \beta, \gamma, \delta) df(t) = \left(a+b-x - \frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} \right) f(a+b-x) - \left(\frac{a+b}{2} - \frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} \right) f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^{a+b-x} f(t) dt,$$

and

$$\int_{a+b-x}^b K(x, t; \alpha, \beta, \gamma, \delta) df(t) = \left(b - \frac{\alpha a + (2\delta - \alpha)b}{2\delta} \right) f(b) - \left(a+b-x - \frac{\alpha a + (2\delta - \alpha)b}{2\delta} \right) f(a+b-x) - \int_{a+b-x}^b f(t) dt.$$

Adding the above equalities, and since $\alpha + \beta + \gamma = 2\delta$, simple calculations yield that

$$\begin{aligned} & \int_a^b K(x, t; \alpha, \beta, \gamma, \delta) df(t) \\ &= \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\delta - \alpha - \beta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \\ &= \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt, \end{aligned}$$

where,

$$K(x, t; \alpha, \beta, \gamma, \delta) = \begin{cases} t - \frac{(2\delta - \alpha)a + \alpha b}{2\delta}, & t \in [a, x] \\ t - \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta}, & t \in \left(x, \frac{a+b}{2}\right] \\ t - \frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta}, & t \in \left(\frac{a+b}{2}, a+b-x\right] \\ t - \frac{\alpha a + (2\delta - \alpha)b}{2\delta}, & t \in (a+b-x, b] \end{cases}$$

Now, we use the fact that for a continuous function $p : [c, d] \rightarrow \mathbb{R}$ and a function $\nu : [c, d] \rightarrow \mathbb{R}$ of bounded variation, one has the inequality

$$(2.2) \quad \left| \int_c^d p(t) d\nu(t) \right| \leq \sup_{t \in [c, d]} |p(t)| \bigvee_a^b(\nu).$$

Applying the inequality (2.2) for $p(t) = K(x, t; \alpha, \beta, \gamma, \delta)$, as above and $\nu(t) = f(t)$, $t \in [a, b]$, we get

$$\begin{aligned} & \left| \int_a^b K(x, t; \alpha, \beta, \gamma, \delta) df(t) \right| \\ & \leq \sup_{t \in [a, b]} |K(x, t; \alpha, \beta, \gamma, \delta)| \cdot \bigvee_a^b(f) \\ & = \frac{1}{2\delta} \max \{ 2\delta(x-a) - \alpha(b-a), \alpha(b-a), (\alpha + \beta)(b-a) - 2\delta(x-a), (\delta - \alpha - \beta)(b-a) \} \cdot \bigvee_a^b(f), \end{aligned}$$

which completes the proof. \square

Remark 1. Let f as in Theorem 6. Choose $\gamma = \delta = 1$, $\alpha = \lambda$ and $\beta = 1 - \lambda$, then we refer to (1.6).

Corollary 1. Let f as in Theorem 6. Choose $\alpha = 0$, then we get the following three-point inequality

$$\begin{aligned} (2.3) \quad & \left| \frac{(b-a)}{2\delta} \left[\beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) \right] - \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2\delta} \max \{ 2\delta(x-a), \beta(b-a) - 2\delta(x-a), (\delta - \beta)(b-a) \} \cdot \bigvee_a^b(f). \end{aligned}$$

Remark 2. In Corollary 1, if we choose $\beta = \delta \neq 0$, then we refer to (1.5).

Corollary 2. Let f as in Corollary 1. Choose $\beta = \lambda$, $\gamma = 2 - \lambda$ and $\delta = 1$. Then we have

$$\begin{aligned} (2.4) \quad & \left| (b-a) \left[\lambda \frac{f(x) + f(a+b-x)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2} \max \{ 2(x-a), \lambda(b-a) - 2(x-a), (1-\lambda)(b-a) \} \cdot \bigvee_a^b(f). \end{aligned}$$

for all $\lambda \in [0, 1]$ and $a \leq x \leq \frac{a+b}{2}$. For instance, let $x = a$, we get

$$\begin{aligned} (2.5) \quad & \left| (b-a) \left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)}{2} \left[\frac{1}{2} + \left| \frac{1}{2} - \lambda \right| \right] \cdot \bigvee_a^b(f). \end{aligned}$$

Corollary 3. Let f as above, if we choose

(1) $\alpha = \beta = 0$, $\gamma = \frac{3}{2}$ and $\delta = 1$, then we get the following midpoint inequality

$$(2.6) \quad \left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b-a) \cdot \bigvee_a^b(f).$$

(2) $\alpha = 1, \beta = 0$ and $\delta = \gamma = 1$, then we get the following trapezoid inequality

$$(2.7) \quad \left| (b-a) \frac{f(a)+f(b)}{2} - \int_a^b f(t) dt \right| \leq \frac{1}{2} (b-a) \cdot \bigvee_a^b(f).$$

(3) $\alpha = 1, \beta = 0, \gamma = 5$ and $\delta = 3$, then we get the following Simpson's inequality

$$(2.8) \quad \left| \frac{(b-a)}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{1}{3} (b-a) \cdot \bigvee_a^b(f).$$

(4) $\alpha = 1, \beta = 0, \gamma = 1$ and $\delta = \frac{3}{2}$, then we get the following Milne's inequality

$$(2.9) \quad \left| \frac{(b-a)}{3} \left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{2}{3} (b-a) \cdot \bigvee_a^b(f).$$

(5) if we choose $\alpha = 0, \beta = 3, \gamma = 5$ and $\delta = 4$ with $x = \frac{5a+b}{6}$, then we get the following Maclaurin's inequality

$$(2.10) \quad \left| \frac{b-a}{8} \left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right) \right] - \int_a^b f(t) dt \right| \leq \frac{5}{24} (b-a) \cdot \bigvee_a^b(f).$$

(6) $\alpha = 1, \beta = 3$ and $\gamma = \delta = 4$ with $x = \frac{2a+b}{3}$, then we get the following 3/8-Simpson's inequality

$$(2.11) \quad \left| \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{5}{24} (b-a) \cdot \bigvee_a^b(f).$$

(7) $\alpha = 7, \beta = 32, \gamma = 51$ and $\delta = 45$ with $x = \frac{3a+b}{4}$, then we get the following Boole's inequality

$$(2.12) \quad \left| \frac{b-a}{90} \left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{11}{60} (b-a) \cdot \bigvee_a^b(f).$$

We may write the following result regarding monotonic mappings:

Corollary 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonous mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality*

$$(2.13) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \frac{1}{2\delta} \max \{ 2\delta(x-a) - \alpha(b-a), \alpha(b-a), (\alpha + \beta)(b-a) - 2\delta(x-a), (\delta - \alpha - \beta)(b-a) \} \\ \times |f(b) - f(a)|.$$

The following result holds for L -lipschitz mappings:

Corollary 5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a L -lipschitz mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality*

$$(2.14) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \frac{L(b-a)}{2\delta} \max \{ 2\delta(x-a) - \alpha(b-a), \alpha(b-a), (\alpha + \beta)(b-a) - 2\delta(x-a), (\delta - \alpha - \beta)(b-a) \}.$$

Remark 3. *If we assume that f is continuous differentiable on (a, b) and f' is integrable on (a, b) , then we have*

$$(2.15) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \frac{\|f'\|_1}{2\delta} \max \{ 2\delta(x-a) - \alpha(b-a), \alpha(b-a), (\alpha + \beta)(b-a) - 2\delta(x-a), (\delta - \alpha - \beta)(b-a) \}.$$

3. THE CASE WHEN f IS LIPSCHITZIAN

Theorem 7. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an L -Lipschitzian mapping on $[a, b]$. Then we have the following inequality*

$$(3.1) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \\ \leq L \left[\frac{(b-a)^2}{16} + \left(x - \frac{3a+b}{4} \right)^2 + \left(\frac{\alpha(b-a) - \delta(x-a)}{2\delta} \right)^2 \right. \\ \left. + \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - \frac{1}{2} \left(\frac{a+b}{2} + x \right) \right)^2 \right. \\ \left. + \left(\frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} - \frac{a+b}{2} - \frac{1}{2} \left(\frac{a+b}{2} - x \right) \right)^2 \right],$$

for all $\frac{(2\delta - \alpha)a + \alpha b}{2\delta} \leq x \leq \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta}$, where, $\alpha, \beta, \gamma, \delta$ are positive constants such that $\alpha + \beta + \gamma = 2\delta$, with $\delta \neq 0$.

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned} & \int_a^b K(x, t; \alpha, \beta, \gamma, \delta) df(t) \\ &= \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \end{aligned}$$

Now, we use the fact that for a Riemann integrable function $p : [c, d] \rightarrow \mathbb{R}$ and L -Lipschitzian function $\nu : [c, d] \rightarrow \mathbb{R}$, one has the inequality

$$(3.2) \quad \left| \int_c^d p(t) d\nu(t) \right| \leq L \int_c^d |p(t)| dt.$$

Applying the inequality (3.2) for $p(t) = K(x, t; \alpha, \beta, \gamma, \delta)$, as above and $\nu(t) = f(t)$, $t \in [a, b]$, we get

$$(3.3) \quad \left| \int_a^b K(x, t; \alpha, \beta, \gamma, \delta) df(t) \right| \leq L \int_a^b |K(x, t; \alpha, \beta, \gamma, \delta)| dt.$$

Now, since

$$\begin{aligned} \int_p^r |t-q| dt &= \int_p^q (q-t) dt + \int_q^r (t-q) dt = \frac{(q-p)^2 + (r-q)^2}{2} \\ (3.4) \quad &= \frac{1}{4} (p-r)^2 + \left(q - \frac{r+p}{2} \right)^2, \end{aligned}$$

for all r, p, q such that $p \leq q \leq r$. Then, we observe that

$$\int_a^x \left| t - \frac{(2\delta - \alpha)a + \alpha b}{2\delta} \right| dt = \frac{1}{4} (x-a)^2 + \left(\frac{\alpha(b-a) - \delta(x-a)}{2\delta} \right)^2,$$

$$\begin{aligned} & \int_x^{\frac{a+b}{2}} \left| t - \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} \right| dt \\ &= \frac{1}{4} \left(\frac{a+b}{2} - x \right)^2 + \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - \frac{1}{2} \left(\frac{a+b}{2} + x \right) \right)^2, \end{aligned}$$

$$\begin{aligned} & \int_{\frac{a+b}{2}}^{a+b-x} \left| t - \frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} \right| dt \\ &= \frac{1}{4} \left(\frac{a+b}{2} - x \right)^2 + \left(\frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} - \frac{a+b}{2} - \frac{1}{2} \left(\frac{a+b}{2} - x \right) \right)^2, \end{aligned}$$

and

$$\int_{a+b-x}^b \left| t - \frac{\alpha a + (2\delta - \alpha)b}{2\delta} \right| dt = \frac{1}{4} (x-a)^2 + \left(\frac{\alpha(b-a) - \delta(x-a)}{2\delta} \right)^2.$$

Then, we have

$$\begin{aligned} \int_a^b |K(x, t; \alpha, \beta, \gamma, \delta)| dt &= \frac{1}{2} (x-a)^2 + \frac{1}{2} \left(\frac{a+b}{2} - x \right)^2 + \left(\frac{\alpha(b-a) - \delta(x-a)}{2\delta} \right) \\ &+ \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - \frac{1}{2} \left(\frac{a+b}{2} + x \right) \right)^2 \\ &+ \left(\frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} - \frac{a+b}{2} - \frac{1}{2} \left(\frac{a+b}{2} - x \right) \right)^2 \end{aligned}$$

Using (3.4), to simplify the above equality, we get

$$\begin{aligned} (3.5) \quad \int_a^b |K(x, t; \alpha, \beta, \gamma, \delta)| dt &= \frac{(b-a)^2}{16} + \left(x - \frac{3a+b}{4} \right)^2 + \left(\frac{\alpha(b-a) - \delta(x-a)}{2\delta} \right)^2 \\ &+ \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - \frac{1}{2} \left(\frac{a+b}{2} + x \right) \right)^2 \\ &+ \left(\frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} - \frac{a+b}{2} - \frac{1}{2} \left(\frac{a+b}{2} - x \right) \right)^2. \end{aligned}$$

Thus, by (3.3), we have the required result, which completes the proof. \square

Corollary 6. *Let f as in Theorem 7. Choose $\alpha = 0$, then we get the following three-point inequality*

$$\begin{aligned} (3.6) \quad &\left| \frac{(b-a)}{2\delta} \left[\beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) \right] - \int_a^b f(t) dt \right| \\ &\leq L \left[\frac{(b-a)^2}{16} + \left(x - \frac{3a+b}{4} \right)^2 + \frac{(x-a)^2}{4} + \left(\frac{(2\delta - \beta)a + \beta b}{2\delta} - \frac{1}{2} \left(\frac{a+b}{2} + x \right) \right)^2 \right. \\ &\quad \left. + \left(\frac{\beta a + (2\delta - \beta)b}{2\delta} - \frac{a+b}{2} - \frac{1}{2} \left(\frac{a+b}{2} - x \right) \right)^2 \right], \end{aligned}$$

for all $a \leq x \leq \frac{(2\delta - \beta)a + \beta b}{2\delta}$.

Corollary 7. *Let f as in Corollary 1. Choose $\beta = \lambda$, $\gamma = 2 - \lambda$ and $\delta = 1$. Then we have*

$$\begin{aligned} (3.7) \quad &\left| (b-a) \left[\lambda \frac{f(x) + f(a+b-x)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(t) dt \right| \\ &\leq L \left[\frac{(b-a)^2}{16} + \left(x - \frac{3a+b}{4} \right)^2 + \frac{(x-a)^2}{4} + \left(\frac{(2-\lambda)a + \lambda b}{2} - \frac{1}{2} \left(\frac{a+b}{2} + x \right) \right)^2 \right. \\ &\quad \left. + \left(\frac{\lambda a + (2-\lambda)b}{2} - \frac{a+b}{2} - \frac{1}{2} \left(\frac{a+b}{2} - x \right) \right)^2 \right], \end{aligned}$$

for all $\lambda \in [0, 1]$ and $a \leq x \leq \frac{a+b}{2}$.

Corollary 8. *Let f as above, if we choose*

- (1) $\alpha = \beta = 0$, $\gamma = \frac{3}{2}$ and $\delta = 1$ with $x = a$, then we get the following midpoint inequality

$$(3.8) \quad \left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{1}{4} L (b-a)^2.$$

- (2) $\alpha = 1$, $\beta = 0$ and $\delta = \gamma = 1$ with $x = \frac{a+b}{2}$, then we get the following trapezoid inequality

$$(3.9) \quad \left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t) dt \right| \leq \frac{1}{4} L (b-a)^2.$$

- (3) $\alpha = 1$, $\beta = 0$, $\gamma = 5$ and $\delta = 3$ with $x = \frac{5a+b}{6}$, then we get the following Simpson's inequality

$$(3.10) \quad \left| \frac{(b-a)}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{19}{144} L (b-a)^2.$$

- (4) $\alpha = 0$, $\beta = 1$, and $\gamma = \delta = 1$, then we get the following two-point inequality

$$(3.11) \quad \left| (b-a) \frac{f(x) + f(a+b-x)}{2} - \int_a^b f(t) dt \right| \leq L \left[\frac{(b-a)^2}{16} + \left(x - \frac{3a+b}{4}\right)^2 + \frac{(x-a)^2}{4} + \frac{1}{2} \left(\frac{a+b}{2} - x\right)^2 \right].$$

for all $a \leq x \leq \frac{a+b}{2}$.

- (5) $\alpha = 1$, $\beta = 0$, $\gamma = 1$ and $\delta = \frac{3}{2}$ with $x = \frac{2a+b}{3}$, then we get the following Milne's inequality

$$(3.12) \quad \left| \frac{(b-a)}{3} \left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{1}{9} L (b-a)^2.$$

- (6) if we choose $\alpha = 0$, $\beta = 3$, $\gamma = 5$ and $\delta = 4$ with $x = \frac{5a+b}{6}$, then we get the following Maclaurin's inequality

$$(3.13) \quad \left| \frac{b-a}{8} \left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right) \right] - \int_a^b f(t) dt \right| \leq \frac{19}{144} L (b-a)^2.$$

- (7) $\alpha = 1$, $\beta = 3$ and $\gamma = \delta = 4$ with $x = \frac{2a+b}{3}$, then we get the following 3/8-Simpson's inequality

$$(3.14) \quad \left| \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{19}{144} L (b-a)^2.$$

(8) $\alpha = 7, \beta = 32, \gamma = 51$ and $\delta = 45$ with $x = \frac{3a+b}{4}$, then we get the following Boole's inequality

$$(3.15) \quad \left| \frac{b-a}{90} \left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b) \right] - \int_a^b f(t) dt \right| \leq \frac{777887}{10951200} L(b-a)^2.$$

4. THE CASE WHEN $f' \in L_\infty[a, b]$

Theorem 8. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° , the interior of the interval I , where $a, b \in I$ with $a < b$, such that $f' \in L_1[a, b]$. If f' is bounded on $[a, b]$, i.e., $\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty$, then we have the following inequality:

$$(4.1) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \leq \|f'\|_\infty \left[\frac{(b-a)^2}{16} + \left(x - \frac{3a+b}{4}\right)^2 + \left(\frac{\alpha(b-a) - \delta(x-a)}{2\delta}\right)^2 + \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - \frac{1}{2}\left(\frac{a+b}{2} + x\right)\right)^2 + \left(\frac{(\alpha + \beta)a + (2\delta - \alpha - \beta)b}{2\delta} - \frac{a+b}{2} - \frac{1}{2}\left(\frac{a+b}{2} - x\right)\right)^2 \right],$$

for all $\frac{(2\delta - \alpha)a + \alpha b}{2\delta} \leq x \leq \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta}$, where, $\alpha, \beta, \gamma, \delta$ are positive constants such that $\alpha + \beta + \gamma = 2\delta$, with $\delta \neq 0$.

Proof. Integrating by parts, we obtain

$$\int_a^b K(x, t; \alpha, \beta, \gamma, \delta) f'(t) dt = \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt.$$

Since, f' is bounded, we can state that

$$\begin{aligned} & \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \\ & \leq \int_a^b |K(x, t; \alpha, \beta, \gamma, \delta)| |f'(t)| dt \\ & \leq \|f'\|_\infty \int_a^b |K(x, t)| dt. \end{aligned}$$

By (3.5), we get the required result. \square

Remark 4. In the inequalities (3.6)–(3.15), replace L by $\|f'\|_\infty$ and assume f as in Theorem 8, then we get new inequalities for absolutely continuous mappings whose first derivatives are bounded.

5. THE CASE WHEN $f' \in L_p[a, b]$

Theorem 9. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° , the interior of the interval I , where $a, b \in I$ with $a < b$. If f' is belong to $L_p[a, b]$, $p > 1$, Then we have the following inequality

$$(5.1) \quad \left| \frac{(b-a)}{2\delta} \left[\alpha f(a) + \beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) + \alpha f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \left(\frac{2}{q+1} \right)^{1/q} \left[\left(x - \frac{(2\delta - \alpha)a + \alpha b}{2\delta} \right)^{q+1} + \left(\frac{\alpha(b-a)}{2\delta} \right)^{q+1} + \left(\frac{(\delta - \alpha - \beta)(b-a)}{2\delta} \right)^{q+1} \right. \\ \left. + \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - x \right)^{q+1} \right]^{1/q} \|f'\|_p,$$

for all $\frac{(2\delta - \alpha)a + \alpha b}{2\delta} \leq x \leq \frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta}$, where, $\alpha, \beta, \gamma, \delta$ are positive constants such that $\alpha + \beta + \gamma = 2\delta$, with $\delta \neq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Proof. Using the well known Hölder inequality, we have

$$\left| \frac{b-a}{90} \left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \left(\int_a^b |K(x, t; \alpha, \beta, \gamma, \delta)|^q dt \right)^{1/q} \left(\int_a^b |f(t)|^p dt \right)^{1/p} \\ = \left(\frac{2}{q+1} \right)^{1/q} \left[\left(x - \frac{(2\delta - \alpha)a + \alpha b}{2\delta} \right)^{q+1} + \left(\frac{\alpha(b-a)}{2\delta} \right)^{q+1} + \left(\frac{(\delta - \alpha - \beta)(b-a)}{2\delta} \right)^{q+1} \right. \\ \left. + \left(\frac{(2\delta - \alpha - \beta)a + (\alpha + \beta)b}{2\delta} - x \right)^{q+1} \right]^{1/q} \|f'\|_p,$$

which is required. \square

Corollary 9. Let f as above, choose $\alpha = 0$, then we get the following three-point inequality

$$(5.2) \quad \left| \frac{(b-a)}{2\delta} \left[\beta f(x) + 2(\gamma - \delta) f\left(\frac{a+b}{2}\right) + \beta f(a+b-x) \right] - \int_a^b f(t) dt \right| \\ \leq \left(\frac{2}{q+1} \right)^{1/q} \left[(x-a)^{q+1} + \left(\frac{\delta - \beta}{2\delta} (b-a) \right)^{q+1} + \left(a - x + \beta \frac{b-a}{2\delta} \right)^{q+1} \right]^{1/q} \|f'\|_p.$$

for all $a \leq x \leq \frac{(2\delta - \beta)a + \beta b}{2\delta}$.

Corollary 10. Let f as above, if we choose

- (1) $\alpha = \beta = 0$, $\gamma = \frac{3}{2}$ and $\delta = 1$ with $x = a$, then we get the following midpoint inequality

$$(5.3) \quad \left| (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \frac{(b-a)^{(q+1)/q}}{2(q+1)^{1/q}} \|f'\|_p.$$

- (2) $\alpha = 1$, $\beta = 0$ and $\delta = \gamma = 1$ with $x = \frac{a+b}{2}$, then we get the following trapezoid inequality

$$(5.4) \quad \left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t) dt \right| \leq \frac{(b-a)^{(q+1)/q}}{2(q+1)^{1/q}} \|f'\|_p.$$

- (3) $\alpha = 1$, $\beta = 0$, $\gamma = 5$ and $\delta = 3$ with $x = \frac{5a+b}{6}$, then we get the following Simpson's inequality

$$(5.5) \quad \left| \frac{(b-a)}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t) dt \right| \leq \left(\frac{2(2^{q+1} + 1)}{6^{q+1}(q+1)} \right)^{1/q} (b-a)^{(q+1)/q} \|f'\|_p.$$

- (4) $\alpha = 0$, $\beta = 1$, and $\gamma = \delta = 1$, then we get the following two-point inequality

$$(5.6) \quad \left| (b-a) \frac{f(x) + f(a+b-x)}{2} - \int_a^b f(t) dt \right| \leq \left(\frac{2}{q+1} \right)^{1/q} \left[(x-a)^{q+1} + \left(\frac{a+b}{2} - x \right)^{q+1} \right] \|f'\|_p.$$

for all $a \leq x \leq \frac{a+b}{2}$.

- (5) $\alpha = 1$, $\beta = 0$, $\gamma = 1$ and $\delta = \frac{3}{2}$ with $x = \frac{2a+b}{3}$, then we get the following Milne's inequality

$$(5.7) \quad \left| \frac{(b-a)}{3} \left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b) \right] - \int_a^b f(t) dt \right| \leq \left(\frac{2(2^{q+1} + 1)}{6^{q+1}(q+1)} \right)^{1/q} (b-a)^{(q+1)/q} \|f'\|_p.$$

- (6) if we choose $\alpha = 0$, $\beta = 3$, $\gamma = 5$ and $\delta = 4$ with $x = \frac{5a+b}{6}$, then we get the following Maclaurin's inequality

$$(5.8) \quad \left| \frac{b-a}{8} \left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right) \right] - \int_a^b f(t) dt \right| \leq \left(\frac{2}{q+1} \right)^{1/q} \left[\left(\frac{1}{6} \right)^{q+1} + \left(\frac{1}{8} \right)^{q+1} + \left(\frac{5}{24} \right)^{q+1} \right]^{1/q} (b-a)^{(q+1)/q} \|f'\|_p.$$

(7) $\alpha = 1$, $\beta = 3$ and $\gamma = \delta = 4$ with $x = \frac{2a+b}{3}$, then we get the following 3/8-Simpson's inequality

$$(5.9) \quad \left| \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \left(\frac{2}{q+1} \right)^{1/q} \left[\left(\frac{5}{24} \right)^{q+1} + \left(\frac{1}{8} \right)^{q+1} + \left(\frac{1}{3} \right)^{q+1} \right]^{1/q} (b-a)^{(q+1)/q} \|f'\|_p.$$

(8) $\alpha = 7$, $\beta = 32$, $\gamma = 51$ and $\delta = 45$ with $x = \frac{3a+b}{4}$, then we get the following Boole's inequality

$$(5.10) \quad \left| \frac{b-a}{90} \left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b) \right] - \int_a^b f(t) dt \right| \\ \leq \left(\frac{2}{q+1} \right)^{1/q} \left[\left(\frac{31}{180} \right)^{q+1} + \left(\frac{7}{90} \right)^{q+1} + \left(\frac{1}{15} \right)^{q+1} + \left(\frac{11}{60} \right)^{q+1} \right]^{1/q} (b-a)^{(q+1)/q} \|f'\|_p.$$

Remark 5. One may generalize Theorem 8 and give different approaches for Theorem 9, by applying the Hölder inequality in a different way and we shall leave the details to the interested reader.

Remark 6. One may write new inequalities for mappings whose $|f'|$ is convex on $[a, b]$, using the inequality

$$|f'(t)| \leq \frac{t-a}{b-a} |f'(b)| + \frac{b-t}{b-a} |f'(a)|,$$

for any $t \in [a, b]$. Also, the corresponding version for powers $|f'|^q$ ($q > 1$) may be considered by applying the well-known Hölder inequality in two different ways. We leave the details to the interested reader.

6. APPLICATIONS IN NUMERICAL INTEGRATION

Consider $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of $[a, b]$ and let $h_i = x_{i+1} - x_i$. In what follows, we point out some upper bounds for the error approximation of the general five-point formula.

$$(6.1) \quad S(f, I_n) := \sum_{i=0}^{n-1} \frac{h_i}{2\delta} \left[\alpha f(x_i) + \beta f(t_i) + 2(\gamma - \delta) f\left(\frac{x_i + x_{i+1}}{2}\right) + \beta f(x_i + x_{i+1} - t_i) + \alpha f(x_{i+1}) \right]$$

for all $\frac{(2\delta-\alpha)x_i + \alpha x_{i+1}}{2\delta} \leq t_i \leq \frac{(2\delta-\alpha-\beta)x_i + (\alpha+\beta)x_{i+1}}{2\delta}$, where, $\alpha, \beta, \gamma, \delta$ are positive constants such that $\alpha + \beta + \gamma = 2\delta$, with $\delta \neq 0$.

Theorem 10. Assume that the assumptions of Theorem 6 hold. Then, we have

$$\int_a^b f(t) dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (6.1) and the remainder $R(f, I_n)$ satisfies the bound

$$(6.2) \quad |R(f, I_n)| \leq \frac{1}{2\delta} \sum_{i=1}^{n-1} [\max \{2\delta(t_i - x_i) - \alpha(x_{i+1} - x_i), \alpha(x_{i+1} - x_i), (\alpha + \beta)(x_{i+1} - x_i) - 2\delta(t_i - x_i), (\delta - \alpha - \beta)(x_{i+1} - x_i)\} \cdot \bigvee_{x_i}^{x_{i+1}}(f)].$$

Proof. Applying Theorem 6 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \dots, n-1$, we get the required result. We shall omit the details. \square

Theorem 11. Assume that the assumptions of Theorem 7 hold. Then, we have

$$\int_a^b f(t) dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (6.1) and the remainder $R(f, I_n)$ satisfies the bound

$$(6.3) \quad |R(f, I_n)| \leq L \sum_{i=0}^{n-1} \left[\frac{(x_{i+1} - x_i)^2}{16} + \left(t_i - \frac{3x_i + x_{i+1}}{4} \right)^2 + \left(\frac{\alpha(x_{i+1} - x_i) - \delta(t_i - x_i)}{2\delta} \right)^2 + \left(\frac{(2\delta - \alpha - \beta)x_i + (\alpha + \beta)x_{i+1}}{2\delta} - \frac{1}{2} \left(\frac{x_i + x_{i+1}}{2} + t_i \right) \right)^2 + \left(\frac{(\alpha + \beta)x_i + (2\delta - \alpha - \beta)x_{i+1}}{2\delta} - \frac{x_i + x_{i+1}}{2} - \frac{1}{2} \left(\frac{x_i + x_{i+1}}{2} - t_i \right) \right)^2 \right],$$

Proof. Applying Theorem 7 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \dots, n-1$, we get the required result. We shall omit the details. \square

Theorem 12. Assume that the assumptions of Theorem 9 hold. Then, we have

$$\int_a^b f(t) dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (6.1) and the remainder $R(f, I_n)$ satisfies the bound

$$(6.4) \quad |R(f, I_n)| \leq \sum_{i=0}^{n-1} \left(\frac{2}{q+1} \right)^{1/q} \left[\left(t_i - \frac{(2\delta - \alpha)x_i + \alpha x_{i+1}}{2\delta} \right)^{q+1} + \left(\frac{\alpha(x_{i+1} - x_i)}{2\delta} \right)^{q+1} + \left(\frac{(\delta - \alpha - \beta)(x_{i+1} - x_i)}{2\delta} \right)^{q+1} + \left(\frac{(2\delta - \alpha - \beta)x_i + (\alpha + \beta)x_{i+1}}{2\delta} - t_i \right)^{q+1} \right]^{1/q} \|f'\|_p.$$

Proof. Applying Theorem 9 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \dots, n-1$, we get the required result. We shall omit the details. \square

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