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NEW INEQUALITIES OF HERMITE-HADAMARD-TYPE FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. Several new inequalities of Hermite-Hadmard type for co-ordinated quasi-convex functions in two variables which are related to the left-hand side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

1. Introduction

The following definition is well known in literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all $x,\, y \in I$ and $\lambda \in [0,1]$.

Many important inequalities have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality (see for instance [27]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, $a,\, b \in I$ with $a < b$. The inequalities in (1.1) hold in reversed order if f a concave function.

In recent years many authors have established several inequalities connected to Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see [12], [13], [17] and [23].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\},$$

for any $x, y \in [a,b]$ and $\lambda \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [16]). For several results concerning inequalties for quasi-convex functions we refer the interested reader to [1]-[5], [16], [24, 25] and [27, 28].

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Let us consider now a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by S. S. Dragomir [11] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b], y \in [c,d]$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [11]).

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [11]:

Theorem 1. [11] *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
& \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
& \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
& \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (1.2)
\end{aligned}$$

The above inequalities are sharp.

In a recent paper [22], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1. [22] *A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be quasi-convex on Δ if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \max\{f(x,y), f(z,w)\},$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are quasi-convex where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition of co-ordinated quasi-convex functions may be stated as:

Definition 2. *A function $f:\Delta\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if*

$$f(tx+(1-t)z,sy+(1-s)w)\leq\max\left\{f\left(x,y\right),f\left(x,w\right),f\left(z,y\right),f\left(z,w\right)\right\},$$

for all $(x,y),(z,w)\in\Delta$ and $s,t\in[0,1]$.

The class of co-ordinated quasi-convex functions on Δ is denoted by $QC(\Delta)$. It has been also proved in [22] that every quasi-convex functions on Δ is quasi-convex on the co-ordinates of Δ. For further results on several new classes of co-ordinated convex functions and related results we refer the interested reader to [6]-[9], [11], [15], [18]-[22] and [26]. The main purpose of the present paper is to establish some new inequalities for co-ordinated quasi-convex functions which are related to the leftmost terms of the Hermite-Hadamard type inequality (1.2) and to deduce some consequent results.

2. Main Results

Throughout in this section, for convenience, we will use the following notations:

$$L=\left|\frac{\partial^2}{\partial s\partial t}f\left(a,c\right)\right|,M=\left|\frac{\partial^2}{\partial s\partial t}f\left(a,d\right)\right|,N=\left|\frac{\partial^2}{\partial s\partial t}f\left(b,c\right)\right|,O=\left|\frac{\partial^2}{\partial s\partial t}f\left(b,d\right)\right|,$$
$$P=\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|,Q=\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|,R=\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|,$$
$$S=\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|\text{ and }T=\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|.$$

The following lemma is necessary and plays an important role in establishing our main results:

Lemma 1. [20] *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial s\partial t}\in L\left(\Delta\right)$, then the following identity holds:*

$$\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$
$$=\left(b-a\right)\left(d-c\right)\int_0^1\int_0^1 K\left(t,s\right)\frac{\partial^2}{\partial s\partial t}f\left(ta+\left(1-t\right)b,sc+\left(1-s\right)d\right)dsdt, \tag{2.1}$$

where

$$K\left(t,s\right)=\begin{cases} ts & ,(t,s)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\\ t\left(s-1\right) & ,(t,s)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right]\\ s\left(t-1\right) & ,(t,s)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\\ \left(t-1\right)\left(s-1\right) & ,(t,s)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right]\end{cases}.$$

Now we begin with the following result:

Theorem 2. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is quasi-convex on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right| \\
&\leq \frac{(b-a)(d-c)}{64}\left[\max\{O,Q,S,T\} + \max\{N,Q,R,T\}\right. \\
&\left.+\max\{L,P,R,T\} + \max\{M,P,S,T\}\right], \qquad (2.2)
\end{aligned}$$

where

$$A = \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy.$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A\right| \\
&\leq (b-a)(d-c)\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt\right. \\
&+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 (1-s)t\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} s(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&\left.+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-s)(1-t)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt\right] \qquad (2.3)
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is quasi-convex on the co-ordinates on Δ, we observe that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b, sc+(1-s)d)\right|dsdt \\
&\leq \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} st\left\{\max\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|, \left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|, \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|,\right. \\
&\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\} \\
&= \frac{1}{64}\left\{\max\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|, \left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|, \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|,\right. \\
&\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}. \qquad (2.4)
\end{aligned}$$

Analogously, we also have that the following inequalities:

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}(1-s)\,t\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)\,b,sc+(1-s)\,d\right)\right|dsdt$$
$$\leq\frac{1}{64}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f\left(b,c\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\tag{2.5}$$

$$\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}s\,(1-t)\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)\,b,sc+(1-s)\,d\right)\right|dsdt$$
$$\leq\frac{1}{64}\left\{\max\left|\frac{\partial^2}{\partial s\partial t}f\left(a,d\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\tag{2.6}$$

and

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(1-s)\,(1-t)\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)\,b,sc+(1-s)\,d\right)\right|dsdt$$
$$\leq\frac{1}{64}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f\left(a,c\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\tag{2.7}$$

Substitution of (2.4)-(2.7) in (2.3) gives the desired inequality (2.2). This completes the proof. □

Corollary 1. *Suppose the conditions of the Theorem 2 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is increasing on the co-ordinates on* Δ*, then*

$$\left|\frac{1}{(b-a)\,(d-c)}\int_a^b\int_c^d f\left(x,y\right)dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq\frac{(b-a)\,(d-c)}{64}\left[O+Q+S+T\right].\tag{2.8}$$

(2) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is decreasing on the co-ordinates on* Δ*, then*

$$\left|\frac{1}{(b-a)\,(d-c)}\int_a^b\int_c^d f\left(x,y\right)dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq\frac{(b-a)\,(d-c)}{64}\left[L+R+P+T\right].\tag{2.9}$$

(3) $\frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, c\right) = \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) = \frac{\partial^2}{\partial s \partial t} f\left(a, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) = 0$

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right| \leq \frac{(b-a)(d-c)}{64} [L + M + N + O]. \tag{2.10}$$

(4) $\frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, c\right) = \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) = \frac{\partial^2}{\partial s \partial t} f\left(a, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f(a, c) = \frac{\partial^2}{\partial s \partial t} f(a, d) = \frac{\partial^2}{\partial s \partial t} f(b, c) = \frac{\partial^2}{\partial s \partial t} f(b, d) = 0$

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right| \leq \frac{(b-a)(d-c)}{16} T. \tag{2.11}$$

Proof. Follows directly from Theorem 2. □

Theorem 3. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is quasi-convex on the co-ordinates on Δ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\right| \\ &\leq \frac{(b-a)(d-c)}{4[2(p+1)]^{\frac{2}{p}}} \left[\max\{N^q, Q^q, R^q T^q\}^{\frac{1}{q}} + \max\{N^q, Q^q, R^q, T^q\}^{\frac{1}{q}}\right. \\ &\left. + \max\{M^q, T^q, P^q, S^q\}^{\frac{1}{q}} + \max\{L^q, P^q, R^q, T^q\}^{\frac{1}{q}}\right] \end{aligned} \tag{2.12}$$

where A is as given in Theorem 2.

Proof. Suppose $p > 1$. From Lemma 1 and well-known Hölder inequality for double integrals, we obtain

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \leq (b-a)(d-c) \\
&\times \left[\left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^p s^p \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \right. \\
&+ \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t^p (1-s)^p \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (1-t)^p s^p \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \\
&\left. + \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t)^p (1-s)^p \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \right]. \qquad (2.13)
\end{aligned}$$

Now by the quasi-convexity of $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ on Δ, we have that the following inequalities hold:

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \\
&\leq \max\left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) \right|^q, \right. \\
&\left. \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q \right\}, \qquad (2.14)
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \\
&\leq \max\left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, c\right) \right|^q, \right. \\
&\left. \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q \right\}, \qquad (2.15)
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q dsdt \\
&\leq \max\left\{ \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f\left(a, \frac{c+d}{2}\right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) \right|^q, \right. \\
&\left. \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^q \right\} \qquad (2.16)
\end{aligned}$$

and

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b,sc+(1-s)d\right)\right|^q dsdt$$
$$\leq \max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}. \tag{2.17}$$

Also, we notice that

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}t^p s^p dsdt=\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}t^p(1-s)^p dsdt=\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}s^p(1-t)^p dsdt$$
$$=\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(1-t)^p(1-s)^p dsdt=\frac{1}{(p+1)^2}\left(\frac{1}{2}\right)^{2(p+1)}. \tag{2.18}$$

Utilizing the inequalities (2.14)-(2.3) in (2.13), we get the required inequality (2.12), which completes the proof of the theorem. □

Corollary 2. *Suppose the conditions of the Theorem 3 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is increasing on the co-ordinates on Δ, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4\left[2(p+1)\right]^{\frac{2}{p}}}\left[O+Q+S+T\right]. \tag{2.19}$$

(2) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is decreasing on the co-ordinates on Δ, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4\left[2(p+1)\right]^{\frac{2}{p}}}\left[L+R+P+T\right]. \tag{2.20}$$

(3) $\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)=\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)=\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)=\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)=0$

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4\left[2(p+1)\right]^{\frac{2}{p}}}\left[L+M+N+O\right]. \tag{2.21}$$

(4) $\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)=\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)=\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)=\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)=\frac{\partial^2}{\partial s\partial t}f(a,c)=\frac{\partial^2}{\partial s\partial t}f(a,d)=\frac{\partial^2}{\partial s\partial t}f(b,c)=\frac{\partial^2}{\partial s\partial t}f(b,d)=0$

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\leq\frac{(b-a)(d-c)}{4\left[2(p+1)\right]^{\frac{2}{p}}}T. \tag{2.22}$$

Proof. It is direct consequence of Theorem 3. □

Our next result gives an improvement of the constant of the result given in Theorem 3.

Theorem 4. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is quasi-convex on the co-ordinates on Δ and $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\&\leq\frac{(b-a)(d-c)}{64}\Big[\max\{N^q,Q^q,R^q,T^q\}^{\frac{1}{q}}+\max\{N^q,Q^q,R^q,T^q\}^{\frac{1}{q}}\\&+\max\{M^q,P^q,S^q,T^q\}^{\frac{1}{q}}\max\{L^q,P^q,R^q,T^q\}^{\frac{1}{q}}\Big],\end{aligned}\tag{2.23}$$

where A is as given in Theorem 2.

Proof. Suppose $q\geq 1$. From using Lemma 1 and the power mean inequality, we have

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\leq(b-a)(d-c)\\&\times\left[\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}ts\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}ts\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\right.\\&+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(1-t)s\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(1-t)s\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1(1-t)(1-s)\right)^{1-\frac{1}{q}}\\&\left.\times\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1(1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.24}$$

Now by the quasi-convexity, we have that the following inequalities hold:

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} ts\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b, sc+(1-s)d\right)\right|^q dsdt$$
$$\leq \frac{1}{64}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{2.25}$$

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t(1-s)\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b, sc+(1-s)d\right)\right|^q dsdt$$
$$\leq \frac{1}{64}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(b,\frac{c+d}{2}\right)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{2.26}$$

$$\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t)s\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b, sc+(1-s)d\right)\right|^q dsdt$$
$$\leq \frac{1}{64}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},d\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\} \tag{2.27}$$

and

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t)(1-s)\left|\frac{\partial^2}{\partial s\partial t}f\left(ta+(1-t)b, sc+(1-s)d\right)\right|^q dsdt$$
$$\leq \frac{1}{64}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(a,\frac{c+d}{2}\right)\right|^q, \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},c\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}. \tag{2.28}$$

Also, we notice that

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} tsdsdt = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t(1-s)dsdt = \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} s(1-t)dsdt$$
$$= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t)(1-s)dsdt = \frac{1}{64}. \tag{2.29}$$

Making use of the inequalities (2.25)-(2.29) in (2.24), we obtain the required inequality (2.23). This completes the proof. □

Remark 1. *Since $4^p > 2(p+1)$ if $p > 1$ and accordingly*

$$\frac{1}{8} < \frac{1}{2\left[2(p+1)\right]^{\frac{1}{p}}}$$

and hence we have that the following inequality:

$$\frac{1}{64} < \frac{1}{8} \cdot \frac{1}{8} < \frac{1}{2\left[2\left(p+1\right)\right]^{\frac{1}{p}}} \cdot \frac{1}{2\left[2\left(p+1\right)\right]^{\frac{1}{p}}} = \frac{1}{4\left[2\left(p+1\right)\right]^{\frac{2}{p}}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Improvements of the inequalities of Corollary 2 are given in the following result:

Corollary 3. *Suppose the conditions of the Theorem 4 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ *is increasing on the co-ordinates on* Δ, *then (2.8) holds.*

(2) $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ *is decreasing on the co-ordinates on* Δ, *then (2.9) holds.*

(3) $\frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, c\right) = \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) = \frac{\partial^2}{\partial s \partial t} f\left(a, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) = 0$, *then (2.10) holds.*

(4) $\frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, c\right) = \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) = \frac{\partial^2}{\partial s \partial t} f\left(a, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) = \frac{\partial^2}{\partial s \partial t} f\left(a, c\right) = \frac{\partial^2}{\partial s \partial t} f\left(a, d\right) = \frac{\partial^2}{\partial s \partial t} f\left(b, c\right) = \frac{\partial^2}{\partial s \partial t} f\left(b, d\right) = 0$, *then (2.11) holds.*

Proof. It follows from Theorem 4. □

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