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ON SOME NEW INEQUALITIES FOR CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, some new inequalities for co-ordinated quasi-convex functions in two variables which are related to the right side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

1. Introduction

The following definition is well known in literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$.

Many important inequalities have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality (see for instance [27]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, a, $b \in I$ with $a < b$. The inequalities in (1.1) hold in reversed order if f a concave function.

In recent years many authors have established several inequalities connected to Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see [12], [13], [17] and [23].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\},$$

for any $x, y \in [a,b]$ and $\lambda \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [16]). For several results concerning inequalties for quasi-convex functions we refer the interested reader to [1]-[5], [16], [24, 25] and [27, 28].

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Let us consider now a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification for convex functions on Δ, which are also known as co-ordinated convex functions, was introduced by S. S. Dragomir [11] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b], y \in [c,d]$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [11]).

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [11]:

Theorem 1. [11] *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
& \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
& \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
& \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.
\end{aligned} \tag{1.2}$$

The above inequalities are sharp.

In a recent paper [22], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1. [22] *A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be quasi-convex on Δ if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \max\{f(x,y), f(z,w)\},$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are quasi-convex where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition of co-ordinated quasi-convex functions may be stated as:

Definition 2. *A function $f : \Delta \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if*

$$f(tx + (1-t)z, sy + (1-s)w) \leq \max\left\{f(x,y), f(x,w), f(z,y), f(z,w)\right\},$$

for all $(x,y),(z,w) \in \Delta$ and $s,t \in [0,1]$.

The class of co-ordinated quasi-convex functions on Δ is denoted by $QC(\Delta)$. It has been also proved in [22] that every quasi-convex functions on Δ is quasi-convex on the co-ordinates of Δ. For further results on several new classes of co-ordinated convex functions and related results we refer the interested reader to [6]-[9], [11], [15], [18]-[22] and [26].

Recently, M. Z. Sarıkaya et al. [26] established some inequalities for differentiable co-ordinated convex functions which are connected to the rightmost terms of the Hermite-Hadamard type inequality (1.2). Motivated by the results proved in [26], the main purpose of the present paper is to establish some new inequalities for co-ordinated quasi-convex functions which are related to the rightmost terms of the Hermite-Hadamard type inequality (1.2) and to deduce some consequent results.

2. Main Results

The following lemma is necessary and plays an important role in establishing our main results:

Lemma 1. [26] *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ with $a < b$, $c < d$. If $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, then the following identity holds:*

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} - A$$

(2.1)

$$= \frac{(b-a)(d-c)}{4} \int_0^1 \int_0^1 (1-2s)(1-2t) \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d)\, dsdt,$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c) + f(x,d)]\, dx + \frac{1}{d-c}\int_c^d [f(a,y) + f(b,y)]\, dy\right].$$

Now we begin with the following result:

Theorem 2. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is quasi-convex on the co-ordinates on Δ, then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} - A\right|$$

(2.2)

$$\leq \frac{(b-a)(d-c)}{16} \max\left\{\left|\frac{\partial^2}{\partial s \partial t} f(a,c)\right|, \left|\frac{\partial^2}{\partial s \partial t} f(a,d)\right|, \left|\frac{\partial^2}{\partial s \partial t} f(b,c)\right|, \left|\frac{\partial^2}{\partial s \partial t} f(b,d)\right|\right\},$$

where is defined in Lemma 1.

Proof. From Lemma 1, we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1|1-2s|\,|1-2t|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|dsdt.$$

Using the fact that f is quasi-convex on Δ, we have from the above inequality that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1|1-2s|\,|1-2t|\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|,\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|,\right.$$
$$\left.\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|,\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|\right\}dsdt$$
$$=\frac{(b-a)(d-c)}{4}\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|,\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|,\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|,\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|\right\},$$

which is the required inequality (2.2), since

$$\int_0^1\int_0^1|2s-1|\,|2t-1|\,dsdt=\left(\int_0^1|2s-1|\,ds\right)\left(\int_0^1|2t-1|\,ds\right)=\frac{1}{4}.$$

This completes the proof of the theorem. □

Corollary 1. *Suppose the conditions of the Theorem 2 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is increasing on the co-ordinates on* Δ*, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|. \tag{2.3}$$

(2) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is decreasing on the co-ordinates on* Δ*, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|. \tag{2.4}$$

Proof. Follows directly from Theorem 2. □

Theorem 3. *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is quasi-convex on the co-ordinates on* Δ

and $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}$$
$$\times\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right\} \tag{2.5}$$

where A *is as given in Theorem 1.*

Proof. Suppose $p>1$. From Lemma 1 and well-known Hölder inequality for double integrals, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1|2t-1|^p|2s-1|^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}. \tag{2.6}$$

Since f is quasi-convex, we obtain

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt$$
$$=\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right\}. \tag{2.7}$$

We also notice that

$$\int_0^1\int_0^1|2t-1|^p|2s-1|^p\,dsdt=\left(\int_0^1|2t-1|^p\,dt\right)\left(\int_0^1|2s-1|^p\,ds\right)$$
$$=\frac{1}{(p+1)^2}. \tag{2.8}$$

A combination of (2.6), (2.7) and (2.8), gives the desired inequality (2.5). Hence the proof of the Theorem is complete. □

Corollary 2. *Suppose the conditions of the Theorem 3 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is increasing on the co-ordinates on* Δ*, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right| \tag{2.9}$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|.$$

(2) $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is decreasing on the co-ordinates on* Δ*, then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right| \tag{2.10}$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|.$$

Proof. It is direct consequence of Theorem 3. □

Our next result gives an improvement of the constant of the result given in Theorem 3.

Theorem 4. *Let* $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is quasi-convex on the co-ordinates on* Δ *and* $q\geq 1$*, then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq \frac{(b-a)(d-c)}{16} \tag{2.11}$$
$$\times\max\left\{\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q,\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q\right\},$$

where A *is as given in Theorem 1.*

Proof. Suppose $q\geq 1$. From using Lemma 1 and the power mean inequality, we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}|2t-1|\,|2s-1|\right)^{1-\frac{1}{q}} \tag{2.12}$$
$$\times\left(\int_0^1\int_0^1|2t-1|\,|2s-1|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}$$

Now by the quasi-convexity, we have

$$\begin{aligned}
&\int_0^1\int_0^1 |2t-1|\,|2s-1| \left|\frac{\partial^2}{\partial s\partial t} f\left(ta+(1-t)\,b, sc+(1-s)\,d\right)\right|^q dsdt \\
&\leq \max\left\{\left|\frac{\partial^2}{\partial s\partial t} f\left(a,c\right)\right|, \left|\frac{\partial^2}{\partial s\partial t} f\left(a,d\right)\right|, \left|\frac{\partial^2}{\partial s\partial t} f\left(b,c\right)\right|, \left|\frac{\partial^2}{\partial s\partial t} f\left(b,d\right)\right|\right\} \\
&\times \int_0^1\int_0^1 |2t-1|\,|2s-1|\, dsdt \\
&= \frac{1}{4}\max\left\{\left|\frac{\partial^2}{\partial s\partial t} f\left(a,c\right)\right|, \left|\frac{\partial^2}{\partial s\partial t} f\left(a,d\right)\right|, \left|\frac{\partial^2}{\partial s\partial t} f\left(b,c\right)\right|, \left|\frac{\partial^2}{\partial s\partial t} f\left(b,d\right)\right|\right\} \qquad (2.13)
\end{aligned}$$

Making use of the inequality (2.13) and the fact

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} |2t-1|\,|2s-1| = \frac{1}{4},$$

in (2.12), we obtain the required inequality (2.11). This completes the proof. □

Remark 1. *Since* $2^p > p+1$*if* $p>1$ *and accordingly*

$$\frac{1}{4} < \frac{1}{2\left(p+1\right)^{\frac{1}{p}}}$$

and hence we have that the following inequality:

$$\frac{1}{16} < \frac{1}{4}\cdot\frac{1}{4} < \frac{1}{2\left(p+1\right)^{\frac{1}{p}}}\cdot\frac{1}{2\left(p+1\right)^{\frac{1}{p}}} = \frac{1}{4\left(p+1\right)^{\frac{2}{p}}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Improvements of the inequalities of Corollary 2 are given in the following result:

Corollary 3. *Suppose the conditions of the Theorem 4 are satisfied. Additionally, if*

(1) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is increasing on the co-ordinates on* Δ*, then (2.3) holds.*
(2) $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is decreasing on the co-ordinates on* Δ*, then (2.4) holds.*

Proof. It follows from Theorem 4. □

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