

**REFINEMENTS OF HADAMARD-TYPE INEQUALITIES FOR
CO-ORDINATED QUASI-CONVEX FUNCTIONS**

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ABSTRACT. In this paper, some inequalities of Hermite-Hadamard type for co-ordinated quasi-convex functions in two variables are given. The obtained results give refinements of the Hermite-Hadamard type inequalities for co-ordinated quasi convex functions proved in [21].

1. INTRODUCTION

Let $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ be a convex on I , $a, b \in I$ with $a < b$. Then the inequalities:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2},$$

hold. The inequalities in (1.1) is known as the Hermite-Hadamard's inequalities for convex mappings. The inequalities in (1.1) hold in reversed order if f is a concave function.

In recent years, many authors have established several inequalities connected to Hermite-Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see [12], [13], [17] and [24].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \rightarrow \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [16]). For several results concerning inequalities for quasi-convex functions we refer the interested reader to [1]-[5], [16], [25, 26] and [28, 29].

Let us consider now a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in \mathbb{R}^2 with $a < b$ and $c < d$, a mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w),$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A modification for convex functions on Δ , which are also known as co-ordinated convex functions, was introduced by S. S. Dragomir [11] as follows:

Date: May 24, 2011.

2000 Mathematics Subject Classification. 26A51, 26D15.

Key words and phrases. convex function, co-ordinated quasi-convex functions, Hermite-Hadamard-type inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

Clearly, every convex mapping $f : \Delta \rightarrow \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [11]).

The following Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane \mathbb{R}^2 was also proved in [11]:

Theorem 1. [11] *Suppose that $f : \Delta \rightarrow \mathbb{R}$ is co-ordinated convex on Δ . Then one has the inequalities:*

$$\begin{aligned}
 & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
 & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
 & \leq \frac{1}{4} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\
 & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
 (1.2) \quad & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
 \end{aligned}$$

The above inequalities are sharp.

In a recent paper [23], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 1. [23] *A function $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be quasi-convex on Δ if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \max\{f(x, y), f(z, w)\},$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are quasi-convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

A formal definition of co-ordinated quasi-convex functions may be stated as:

Definition 2. *A function $f : \Delta \rightarrow \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if*

$$f(tx + (1-t)z, sy + (1-s)w) \leq \max\{f(x, y), f(x, w), f(z, y), f(z, w)\},$$

for all $(x, y), (z, w) \in \Delta$ and $s, t \in [0, 1]$.

The class of co-ordinated quasi-convex functions on Δ is denoted by $QC(\Delta)$. It has been also proved in [23] that every quasi-convex functions on Δ is quasi-convex on the co-ordinates on Δ . For further results on several new classes of co-ordinated convex functions and related results we refer the interested reader to [6]-[9], [11], [15], [18]-[23] and [27]. Motivated by the results proved in [21, 27], the main purpose of the present paper is to establish some new inequalities for co-ordinated quasi-convex functions which are related to the rightmost terms of the Hermite-Hadamard type inequality (1.2) and to get refinements of the results for co-ordinated quasi-convex functions proved in [21].

2. MAIN RESULTS

Throughout in this section, for convenience, we will use the notations:

$$\begin{aligned} L &= \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|, M = \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|, N = \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|, O = \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|, \\ P &= \left| \frac{\partial^2}{\partial s \partial t} f\left(a, \frac{c+d}{2}\right) \right|, Q = \left| \frac{\partial^2}{\partial s \partial t} f\left(b, \frac{c+d}{2}\right) \right|, R = \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, c\right) \right|, \\ S &= \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, d\right) \right| \text{ and } T = \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \end{aligned}$$

The following lemma is necessary and plays an important role in establishing our main results:

Lemma 1. *Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ with $a < b$, $c < d$. If $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, then the following identity holds:*

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \\ &= \frac{(b-a)(d-c)}{16} \left[\int_0^1 \int_0^1 ts \frac{\partial^2}{\partial s \partial t} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d\right) ds dt \right. \\ &+ \int_0^1 \int_0^1 (-t)s \frac{\partial^2}{\partial s \partial t} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d\right) ds dt \\ &+ \int_0^1 \int_0^1 t(-s) \frac{\partial^2}{\partial s \partial t} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d\right) ds dt \\ (2.1) \quad & \left. + \int_0^1 \int_0^1 (-t)(-s) \frac{\partial^2}{\partial s \partial t} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d\right) ds dt \right\}, \end{aligned}$$

where

$$A = \frac{1}{2} \left[\frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{d-c} \int_a^d [f(a, y) + f(b, y)] dy \right].$$

Proof. By integration by parts, we have

$$\begin{aligned}
& \int_0^1 \int_0^1 (-t)(-s) \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) ds dt \\
&= \frac{4}{(b-a)(d-c)} f(a, c) - \frac{4}{(b-a)(d-c)} \int_0^1 f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, c \right) dt \\
&\quad - \frac{4}{(b-a)(d-c)} f \left(a, \frac{1+s}{2}c + \frac{1-s}{2}d \right) ds \\
(2.2) \quad &+ \frac{4}{(b-a)(d-c)} \int_0^1 \int_0^1 f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) ds dt.
\end{aligned}$$

Setting $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ and $y = \frac{1+s}{2}c + \frac{1-s}{2}d$, we get from (2.2) the following inequality:

$$\begin{aligned}
& \int_0^1 \int_0^1 (-t)(-s) \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) ds dt \\
&\leq \frac{4}{(b-a)(d-c)} f(a, c) - \frac{8}{(b-a)^2(d-c)} \int_a^{\frac{a+b}{2}} f(x, c) dx \\
(2.3) \quad &- \frac{8}{(b-a)(d-c)^2} \int_c^{\frac{c+d}{2}} f(a, y) dy + \frac{16}{(b-a)^2(d-c)^2} \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x, y) dy dx.
\end{aligned}$$

In a similar way, we can have the following inequalities:

$$\begin{aligned}
& \int_0^1 \int_0^1 (-t)s \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) ds dt \\
&\leq \frac{4}{(b-a)(d-c)} f(a, d) - \frac{8}{(b-a)^2(d-c)} \int_a^{\frac{a+b}{2}} f(x, d) dx \\
(2.4) \quad &- \frac{8}{(b-a)(d-c)^2} \int_{\frac{c+d}{2}}^d f(a, y) dy + \frac{16}{(b-a)^2(d-c)^2} \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x, y) dy dx,
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 t(-s) \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) ds dt \\
&\leq \frac{4}{(b-a)(d-c)} f(b, c) - \frac{8}{(b-a)^2(d-c)} \int_{\frac{a+b}{2}}^b f(x, c) dx \\
(2.5) \quad &- \frac{8}{(b-a)(d-c)^2} \int_c^{\frac{c+d}{2}} f(b, y) dy + \frac{16}{(b-a)^2(d-c)^2} \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(x, y) dy dx
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 ts \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) ds dt \\
& \leq \frac{4}{(b-a)(d-c)} f(b, d) - \frac{8}{(b-a)^2(d-c)} \int_{\frac{a+b}{2}}^b f(x, dd) dx \\
(2.6) \quad & - \frac{8}{(b-a)(d-c)^2} \int_{\frac{c+d}{2}}^d f(b, y) dy + \frac{16}{(b-a)^2(d-c)^2} \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x, y) dy dx.
\end{aligned}$$

Substituting (2.3)-(2.6) in (2.2), simplifying and multiplying the resulting inequality by $\frac{(b-a)(d-c)}{16}$, we get (2.1). Hence the proof of the Lemma is complete. \square

Now we begin with the following result:

Theorem 2. *Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ with $a < b, c < d$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is quasi-convex on the co-ordinates on Δ , then the following inequality holds:*

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
& \leq \frac{(b-a)(d-c)}{64} [\sup \{O, Q, S, T\} + \sup \{N, Q, R, T\} + \sup \{L, P, R, T\} \\
(2.7) \quad & + \sup \{M, P, S, T\}],
\end{aligned}$$

where A is defined as in Lemma 1.

Proof. From Lemma 1, we have

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
& \leq \frac{(b-a)(d-c)}{16} \left[\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right| ds dt \right. \\
& + \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right| ds dt \\
& + \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right| ds dt \\
(2.8) \quad & \left. + \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right| ds dt \right]
\end{aligned}$$

By the quasi-convexity of $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ on $\Delta := [a, b] \times [c, d]$, we observe that the following inequality holds:

$$\begin{aligned}
& \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right| ds dt \\
& \leq \int_0^1 \int_0^1 st \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(a, \frac{c+d}{2} \right) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, c \right) \right|, \right. \\
& \quad \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\} \\
& = \frac{1}{4} \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|, \right. \\
(2.9) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\}.
\end{aligned}$$

Analogously, we also have that the following inequalities:

$$\begin{aligned}
& \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right| ds dt \\
& \leq \frac{1}{4} \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(a, \frac{c+d}{2} \right) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|, \right. \\
(2.10) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\},
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right| ds dt \\
& \leq \frac{1}{4} \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, c \right) \right|, \right. \\
(2.11) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right| ds dt \\
& \leq \frac{1}{4} \max \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|, \right. \\
(2.12) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\},
\end{aligned}$$

Substitution of (2.9)-(2.12) in (2.8) gives the desired inequality (2.7). This completes the proof. \square

Corollary 1. *Suppose the conditions of the Theorem 2 are satisfied. Additionally, if*

(1) $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is increasing on the co-ordinates on Δ , then

$$(2.13) \quad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\ \leq \frac{(b-a)(d-c)}{64} [O + Q + S + T].$$

(2) $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is decreasing on the co-ordinates on Δ , then

$$(2.14) \quad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\ \leq \frac{(b-a)(d-c)}{64} [L + R + P + T].$$

Proof. It follows directly from Theorem 2. □

Theorem 3. Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ with $a < b$, $c < d$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is quasi-convex on the co-ordinates on Δ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$(2.15) \quad \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\ \leq \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}} [\sup \{L^q, P^q, R^q, T^q\} + \sup \{M^q, P^q, S^q, T^q\} \\ + \sup \{N^q, Q^q, R^q, T^q\} + \sup \{O^q, Q^q, S^q, T^q\}].$$

where A is as given in Lemma 1.

Proof. Suppose $p > 1$. From Lemma 1 and well-known Hölder inequality for double integrals, we obtain

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
& \leq \frac{(b-a)(d-c)}{16} \left(\int_0^1 \int_0^1 t^p s^p \right)^{\frac{1}{p}} \\
& \times \left[\left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \right. \\
& + \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \\
(2.16) \quad & \left. + \left(\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Now by the quasi-convexity of $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ on Δ , we have that the following inequalities hold:

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \\
& \leq \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(a, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, c \right) \right|^q, \right. \\
(2.17) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\},
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \\
& \leq \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(a, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|^q, \right. \\
(2.18) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\},
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \\
& \leq \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, c \right) \right|^q, \right. \\
(2.19) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \\
& \leq \max \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|^q, \right. \\
(2.20) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}.
\end{aligned}$$

Also, we notice that

$$(2.21) \quad \int_0^1 \int_0^1 t^p s^p ds dt = \frac{1}{(p+1)^2}.$$

Utilizing the inequalities (2.17)-(2.21) in (2.16), we get the required inequality (2.15), which completes the proof of the theorem. \square

Corollary 2. *Suppose the conditions of the Theorem 3 are satisfied. Additionally, if*

$$(1) \quad \left| \frac{\partial^2 f}{\partial s \partial t} \right|^q \text{ is increasing on the co-ordinates on } \Delta, \text{ then}$$

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
(2.22) \quad & \leq \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}} [O + Q + S + T].
\end{aligned}$$

$$(2) \quad \left| \frac{\partial^2 f}{\partial s \partial t} \right|^q \text{ is decreasing on the co-ordinates on } \Delta, \text{ then}$$

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
(2.23) \quad & \leq \frac{(b-a)(d-c)}{16(p+1)^{\frac{2}{p}}} [L + R + P + T].
\end{aligned}$$

Proof. It is a direct consequence of Theorem 3. \square

Our next result gives an improvement of the constant of the result given in Theorem 3.

Theorem 4. *Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ with $a < b, c < d$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ is quasi-convex on the co-ordinates on Δ*

and $q \geq 1$, then the following inequality holds:

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
& \leq \frac{(b-a)(d-c)}{64} \left[\sup \{L^q, P^q, R^q, T^q\}^{\frac{1}{q}} + \sup \{M^q, P^q, S^q, T^q\}^{\frac{1}{q}} \right. \\
(2.24) \quad & \left. + \sup \{N^q, Q^q, R^q, T^q\}^{\frac{1}{q}} + \sup \{O^q, Q^q, S^q, T^q\}^{\frac{1}{q}} \right].
\end{aligned}$$

where A is as given in Theorem 1.

Proof. Suppose $q \geq 1$. From using Lemma 1 and the power mean inequality, we have

$$\begin{aligned}
& \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - A \right| \\
& \leq \frac{(b-a)(d-c)}{16} \left(\int_0^1 \int_0^1 ts \right)^{1-\frac{1}{q}} \\
& \times \left[\left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \right. \\
& + \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \\
(2.25) \quad & \left. + \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Now by the quasi-convexity of $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$ on Δ , we have that the following inequalities hold:

$$\begin{aligned}
& \int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \\
& \leq \frac{1}{4} \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(a, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, c \right) \right|^q, \right. \\
(2.26) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\},
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \\
& \leq \frac{1}{4} \sup \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(a, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|^q, \right. \\
(2.27) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\},
\end{aligned}$$

$$\begin{aligned}
 & \int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d \right) \right|^q ds dt \\
 & \leq \frac{1}{4} \max \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, c \right) \right|^q, \right. \\
 (2.28) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d \right) \right|^q ds dt \\
 & \leq \frac{1}{4} \max \left\{ \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(b, \frac{c+d}{2} \right) \right|^q, \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, d \right) \right|^q, \right. \\
 (2.29) \quad & \left. \left| \frac{\partial^2}{\partial s \partial t} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}.
 \end{aligned}$$

Also, we notice that

$$\int_0^1 \int_0^1 ts ds dt = \frac{1}{4}.$$

Making use of the inequalities (2.26)-(2.29) in (2.25), we obtain the required inequality (2.24). This completes the proof. \square

Remark 1. Since $2^p > p + 1$ if $p > 1$ and accordingly

$$\frac{1}{8} < \frac{1}{4(p+1)^{\frac{1}{p}}}$$

and hence we have that the following inequality:

$$\frac{1}{64} < \frac{1}{8} \cdot \frac{1}{8} < \frac{1}{4(p+1)^{\frac{1}{p}}} \cdot \frac{1}{4(p+1)^{\frac{1}{p}}} = \frac{1}{16(p+1)^{\frac{2}{p}}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Improvements of the inequalities of Corollary 2 are given in the following result:

Corollary 3. Suppose the conditions of the Theorem 4 are satisfied. Additionally, if

- (1) $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is increasing on the co-ordinates on Δ , then (2.13) holds.
- (2) $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is decreasing on the co-ordinates on Δ , then (2.14) holds.

Proof. It follows from Theorem 4. \square

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