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A COMPANION OF OSTROWSKI'S INEQUALITY FOR MAPPINGS WHOSE FIRST DERIVATIVES ARE BOUNDED AND APPLICATIONS IN NUMERICAL INTEGRATION

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Abstract. A companion of Ostrowski's integral inequality for differentiable mappings whose first derivatives are bounded is proved. Applications to a composite quadrature rule and to probability density functions are considered.

1. Introduction

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows [4]:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss' inequality [8].

Theorem 2. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\phi \le f(t) \le \Phi$ and $\gamma \le g(t) \le \Gamma$ for all $t \in [a,b]$, ϕ, Φ, γ and Γ are constants. Then we have*

$$\left| \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \right| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{1.2}$$

Motivated by [3], Dragomir in [5] has proved the following companion of the Ostrowski inequality:

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Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequalities*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$
$$\leq \begin{cases} \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty, & f' \in L_\infty[a,b] \\ \frac{2^{1/q}}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \text{ and } f' \in L_p[a,b] \\ \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left\|f'\right\|_{[a,b],1} & \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Recently, Alomari [1] proved a companion inequality for differentiable mappings whose derivatives are bounded.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, and let $a, b \in I$ with $a < b$. If $f' \in L^1[a, b]$ and $\gamma \leq f'(x) \leq \Gamma$, $\forall x \in [a, b]$, then the following inequality holds,*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq (b-a)\left[\frac{1}{16}+\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma-\gamma), \tag{1.4}$$

for all $x \in [a, \frac{a+b}{2}]$.

In [6], Dragomir established some inequalities for this companion for mappings of bounded variation. In [7], Liu introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. Recently, Barnett, Dragomir and Gomma [2], have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality.

In the present paper we shall derive a companion inequality of Ostrowski's type using Grüss' result and then discuss its applications for a composite quadrature rule and for probability density functions.

2. The Results

The following companion of Ostrowski's inequality holds.

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b). If $f' \in L^1[a, b]$ and $\gamma \leq f'(t) \leq \Gamma$, for all $t \in [a, b]$, then the inequality holds*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)(\Gamma-\gamma), \tag{2.1}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Let us define the mapping

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ t-\frac{a+b}{2}, & t \in (x, a+b-x] \\ t-b, & t \in (a+b-x, b] \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt = \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{2.2}$$

It is clear that for all $t \in [a,b]$ and $x \in [a, \frac{a+b}{2}]$, we have

$$x - \frac{a+b}{2} \le p(x,t) \le x-a.$$

Applying Theorem 2 to the mappings $p(x,\cdot)$ and $f'(\cdot)$, we obtain

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt - \frac{1}{b-a}\int_a^b p(x,t)\,dt \cdot \frac{1}{b-a}\int_a^b f'(t)\,dt\right| \\ &\le \frac{1}{4}\left\{x-a-\left(x-\frac{a+b}{2}\right)\right\}(\Gamma-\gamma) = \frac{1}{8}(b-a)(\Gamma-\gamma), \end{aligned} \tag{2.3}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. By a simple calculation we get

$$\int_a^b p(x,t)\,dt = 0, \quad \textit{and} \quad \int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a}. \tag{2.4}$$

Finally, combining (2.2)–(2.4), we obtain (2.1) as required. □

Corollary 1. *In the inequality (2.1), choose*

(1) $x = a$, *we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{2.5}$$

(2) $x = \frac{3a+b}{4}$, *we get*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{2.6}$$

(3) $x = \frac{a+b}{2}$

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{2.7}$$

An inequality of Ostrowski's type may be stated as follows:

Corollary 2. *Let f as in Theorem 5. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x) = f(x)$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)(\Gamma-\gamma), \tag{2.8}$$

for all $x \in [a, \frac{a+b}{2}]$. For instance, choose $x = a$, we have

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{2.9}$$

3. A Composite Quadrature Formula

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a,b]$ and $h_i = x_{i+1} - x_i$, $(i = 0, 1, 2, \cdots, n-1)$.

Consider the general quadrature formula

$$Q_n(I_n, f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i. \tag{3.1}$$

The following result holds.

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If $f' \in L^1[a,b]$ and $\gamma \le f'(x) \le \Gamma$, $\forall x \in [a,b]$. Then, we have*

$$\int_a^b f(t)\,dt = Q_n(I_n, f) + R_n(I_n, f). \tag{3.2}$$

where, $Q_n(I_n, f)$ is defined by formula (3.1), and the remainder satisfies the estimates

$$|R_n(I_n, f)| \le \frac{(\Gamma-\gamma)}{8}\cdot\sum_{i=0}^{n-1} h_i. \tag{3.3}$$

Proof. Applying inequality (2.1) on the intervals $[x_i, x_{i+1}]$, we may state that

$$R_i(I_i, f) = \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i.$$

Summing the above inequality over i from 0 to $n-1$, we get

$$\begin{aligned} R_n(I_n, f) &= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i \\ &= \int_a^b f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i, \end{aligned}$$

which follows form (2.1), that

$$|R_n(I_n, f)| = \left|\int_a^b f(t)\,dt - \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i\right|$$
$$\leq \frac{(\Gamma - \gamma)}{8} \cdot \sum_{i=0}^{n-1} h_i.$$

which completes the proof. □

4. APPLICATIONS TO PROBABILITY DENSITY FUNCTIONS

Let X be a random variable taking values in the finite interval $[a, b]$, with the probability density function $f : [a, b] \to [0, 1]$ with the cumulative distribution function $F(x) = Pr(X \leq x) = \int_a^b f(t)dt$.

Theorem 7. *With the assumptions of Theorem 4, we have the inequality*

$$\left|\frac{1}{2}[F(x) + F(a + b - x)] - \frac{b - E(X)}{b - a}\right| \leq \frac{1}{8}(b - a)(\Gamma - \gamma),$$

for all $x \in [a, \frac{a+b}{2}]$, where $E(X)$ is the expectation of X.

Proof. In the proof of Theorem 4, let $f = F$, and taking into account that

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)\,dt.$$

We left the details to the interested reader. □

Corollary 3. *In Theorem 7, choose $x = \frac{3a+b}{4}$, we get*

$$\left|\frac{1}{2}\left[F\left(\frac{3a + b}{4}\right) + F\left(\frac{a + 3b}{4}\right)\right] - \frac{b - E(X)}{b - a}\right| \leq \frac{1}{8}(b - a)(\Gamma - \gamma).$$

Corollary 4. *In Theorem 7, if F is symmetric about the x-axis, i.e., $F(a + b - x) = F(x)$, we have*

$$\left|F(x) - \frac{b - E(X)}{b - a}\right| \leq \frac{1}{8}(b - a)(\Gamma - \gamma),$$

for all $x \in [a, \frac{a+b}{2}]$.

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