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A GENERALIZATION OF COMPANION INEQUALITY OF OSTROWSKI'S TYPE FOR MAPPINGS WHOSE FIRST DERIVATIVES ARE BOUNDED AND APPLICATIONS IN NUMERICAL INTEGRATION

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An inequality for a companion of Ostrowski's integral inequality is proved. Application to a composite quadrature rule is considered.

I. INTRODUCTION

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \le M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{1.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In [11], Dragomir, Cerone and Roumeliotis proved the following generalization of Ostrowski's inequality.

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$, differentiable on (a,b) and whose derivative f' is bounded on (a,b). Denote $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then,*

$$\left| (b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) f(x) \right] - \int_a^b f(t)\,dt \right| \le \left[\frac{(b-a)^2}{4}\left(\lambda^2 + (1-\lambda)^2\right) + \left(x - \frac{a+b}{2}\right)^2 \right] \|f'\|_\infty . \tag{1.2}$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le b - \lambda\frac{b-a}{2}$.

Using (1.2), the authors obtained estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae. They also gave applications of the mentioned results in numerical integration and for special means.

A companion of (1.2) is considered by Alomari in [3], as follows:

Theorem 3 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If f' is bounded on $[a,b]$, i.e., $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then the inequality holds*

$$\left| (b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) \frac{f(x)+f(a+b-x)}{2} \right] - \int_a^b f(t)\,dt \right| \le \left[\frac{(b-a)^2}{8}\left(2\lambda^2 + (1-\lambda)^2\right) + 2\left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right)^2 \right] \|f'\|_\infty . \tag{1.3}$$

holds, for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$.

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Theorem 4 *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If f' is bounded on $[a,b]$, i.e., $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then we have*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty \tag{1.4}$$

for all $x\in[a,\frac{a+b}{2}]$.

In the recent paper [2], Alomari proved the following companion of Ostrowski inequality:

Theorem 5 *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, and let $a,b\in I$ with $a<b$. If $f'\in L^1[a,b]$ and $\gamma\le f'(x)\le\Gamma$, $\forall x\in[a,b]$, then the following inequality holds,*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le (b-a)\left[\frac{1}{16} + \left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right]\cdot(\Gamma-\gamma), \tag{1.5}$$

for all $x\in[a,\frac{a+b}{2}]$.

In [10], Dragomir established some inequalities for this companion for mappings of bounded variation. In [12], Liu introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. Recently, Barnett, Dragomir and Gomma [4], have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality.

The aim of this paper is to study the companion inequality (1.3) which is of Ostrowski's type for differentiable bounded mappings.

II. MAIN RESULTS

Our main result may be stated as follows:

Theorem 6 *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, and let $a,b\in I$ with $a<b$. If $f'\in L^1[a,b]$ and $\gamma\le f'(x)\le\Gamma$, $\forall x\in[a,b]$, then the following inequality holds,*

$$\begin{aligned}&\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)\,dt\right| \\ &\qquad\le \left[\frac{(b-a)^2}{16}\left(2\lambda^2+(1-\lambda)^2\right) + \left(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right)^2\right]\cdot(\Gamma-\gamma),\end{aligned} \tag{2.6}$$

for all $\lambda\in[0,1]$ and $a+\lambda\frac{b-a}{2}\le x\le\frac{a+b}{2}$.

Proof. Defining the mapping

$$K(x,t) = \begin{cases} t-\left(a+\lambda\frac{b-a}{2}\right), & t\in[a,x] \\ t-\frac{a+b}{2}, & t\in(x,a+b-x] \\ t-\left(b-\lambda\frac{b-a}{2}\right), & t\in(a+b-x,b] \end{cases} \tag{2.7}$$

for all $\lambda\in[0,1]$ and $a+\lambda\frac{b-a}{2}\le x\le\frac{a+b}{2}$.

Integrating by parts

$$\int_a^b K(x,t)f'(t)\,dt = \frac{b-a}{2}\left[\lambda\left(f(a)+f(b)\right)+(1-\lambda)\left(f(x)+f(a+b-x)\right)\right] - \int_a^b f(t)\,dt. \tag{2.8}$$

We also have

$$\int_a^b K(x,t)\,dt = 0. \tag{2.9}$$

Let $C = \frac{\Gamma - \gamma}{2}$. From (2.8) and (2.9), it follows that

$$\int_a^b K(x,t)\left[f'(t) - C\right] dt = \frac{b-a}{2}\left[\lambda\left(f(a)+f(b)\right) + (1-\lambda)\left(f(x)+f(a+b-x)\right)\right] - \int_a^b f(t)\,dt.$$

On the other hand, we have

$$\left|\int_a^b p(x,t)\left[f'(t) - C\right] dt\right| \le \cdot \max_{t\in[a,b]} \left|f'(t) - C\right| \int_a^b \left|K(x,t)\right| dt. \tag{2.10}$$

Now, since

$$\begin{aligned}\int_p^r |t-q|\,dt = \int_p^q (q-t)\,dt + \int_q^r (t-q)\,dt &= \frac{(q-p)^2 + (r-q)^2}{2}\\ &= \frac{1}{4}(p-r)^2 + \left(q - \frac{r+p}{2}\right)^2, \end{aligned} \tag{2.11}$$

for all r,p,q such that $p \le q \le r$. Then, we observe that

$$\int_a^x \left|t - \left(a + \lambda\frac{b-a}{2}\right)\right| dt = \frac{1}{4}(x-a)^2 + \left(\lambda\frac{b-a}{2} - \frac{x-a}{2}\right)^2,$$

$$\int_x^{a+b-x} \left|t - \frac{a+b}{2}\right| dt = \left(x - \frac{a+b}{2}\right)^2,$$

and

$$\int_{a+b-x}^b \left|t - \left(b - \lambda\frac{b-a}{2}\right)\right| dt = \frac{1}{4}(x-a)^2 + \left(\frac{x-a}{2} - \lambda\frac{b-a}{2}\right)^2.$$

Then, we have

$$\begin{aligned}\int_a^b |K(x,t)|\,dt &= \frac{(x-a)^2 + \left((x-a) - \lambda(b-a)\right)^2}{2} + \left(x - \frac{a+b}{2}\right)^2\\ &= \underbrace{\frac{1}{4}\lambda^2(b-a)^2 + \left(x - \frac{(2-\lambda)a + \lambda b}{2}\right)^2}_{by\ (2.11)} + \left(x - \frac{a+b}{2}\right)^2\\ &= \frac{\lambda^2}{4}(b-a)^2 + \underbrace{\frac{(1-\lambda)^2}{8}(b-a)^2 + 2\left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right)^2}_{by\ (2.11)},\\ &= \frac{(b-a)^2}{8}\left(2\lambda^2 + (1-\lambda)^2\right) + 2\left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right)^2, \end{aligned} \tag{2.12}$$

and

$$\max_{t\in[a,b]} \left|f'(t) - C\right| \le \frac{\Gamma - \gamma}{2} \tag{2.13}$$

therefore from (2.10), (2.12) and (2.13), it follows that

$$\begin{aligned}&\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)\,dt\right|\\ &\qquad \le \left[\frac{(b-a)^2}{16}\left(2\lambda^2 + (1-\lambda)^2\right) + \left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right)^2\right]\cdot(\Gamma - \gamma),\end{aligned}$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$.

Remark 1 *In Theorem 6, if one chooses* $\lambda = 0$*, then (2.6) reduces to (1.5).*

Corollary 1 *In Theorem 6, choose* $x = \frac{a+b}{2}$*, we get*

$$\left|(b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt\right| \le \left(\lambda^2 + (1-\lambda)^2\right) \frac{(b-a)^2}{8} \cdot (\Gamma - \gamma).$$

Remark 2 *In Corollary 1, if we choose*

1. $\lambda = 0$*, then we get*

$$\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)^2 \cdot (\Gamma - \gamma).$$

2. $\lambda = \frac{1}{3}$*, then we get*

$$\left|\frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b f(t)\,dt\right| \le \frac{5}{72}(b-a)^2 \cdot (\Gamma - \gamma).$$

3. $\lambda = \frac{1}{2}$*, then we get*

$$\left|\frac{(b-a)}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt\right| \le \frac{1}{16}(b-a)^2 \cdot (\Gamma - \gamma). \tag{2.14}$$

4. $\lambda = 1$*, then we get*

$$\left|(b-a) \frac{f(a)+f(b)}{2} - \int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)^2 \cdot (\Gamma - \gamma).$$

Corollary 2 *In Theorem 6, Setting*

1. $\lambda = \frac{1}{n}$*, for* $n = 1, 2, 3, ...$*, then we get*

$$\left|\frac{b-a}{2n}\left[f(a) + (n-1)(f(x) + f(a+b-x)) + f(b)\right] - \int_a^b f(t)\,dt\right| \le \left[\frac{(b-a)^2}{16n^2}\left(2 + (n-1)^2\right) + \left(x - \frac{(3n-1)a + (n+1)b}{4n}\right)^2\right] \cdot (\Gamma - \gamma). \tag{2.15}$$

2. $\lambda = \frac{n-1}{n}$*, for* $n = 1, 2, 3, ...$*, then we get*

$$\left|\frac{b-a}{2n}\left[(n-1) f(a) + f(x) + f(a+b-x) + (n-1) f(b)\right] - \int_a^b f(t)\,dt\right| \le \left[\frac{(b-a)^2}{16n^2}\left(2(n-1)^2 + 1\right) + \left(x - \frac{(2n+1)a + (2n-1)b}{4n}\right)^2\right] \cdot (\Gamma - \gamma). \tag{2.16}$$

Corollary 3 *In (2.15), choose* $n = 4$ *and* $x = \frac{2a+b}{3}$*, then we get the following* $3/8$*-Simpson's inequality*

$$\left|\frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \int_a^b f(t)\,dt\right| \le \frac{25}{576}(b-a)^2 \cdot (\Gamma - \gamma). \tag{2.17}$$

III. A COMPOSITE QUADRATURE FORMULA

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a,b]$ and $h_i = x_{i+1} - x_i$, $(i = 0, 1, 2, \cdots, n-1)$.

Consider the general quadrature formula

$$Q_n(I_n, f) := \sum_{i=0}^{n-1} \frac{h_i}{2} [\lambda (f(x_i) + f(x_{i+1})) + (1-\lambda)(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i))]. \tag{3.18}$$

for all $\lambda \in [0,1]$ and $x_i + \lambda \frac{x_{i+1}-x_i}{2} \le \alpha_i \le \frac{x_i + x_{i+1}}{2}$.

The following result holds.

Theorem 7 *Let f as in Theorem 6, then we have*

$$\int_a^b f(t)\,dt = Q_n(I_n, f) + R_n(I_n, f).$$

where, $Q_n(I_n, f)$ is defined by formula (3.18), and the remainder satisfies the estimates

$$|R_n(I_n, f)| \le (\Gamma - \gamma) \sum_{i=0}^{n-1} \left[\frac{h_i^2}{16} \left(2\lambda^2 + (1-\lambda)^2\right) + \left(\alpha_i - \frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4}\right)^2 \right],$$

for all $\lambda \in [0,1]$ and $x_i + \lambda \frac{x_{i+1}-x_i}{2} \le \alpha_i \le \frac{x_i + x_{i+1}}{2}$.

Proof. Applying inequality (2.6) on the intervals $[x_i, x_{i+1}]$, we may state that

$$R_i(I_i, f) = \int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h_i}{2} [\lambda (f(x_i) + f(x_{i+1})) + (1-\lambda)(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i))].$$

Summing the above inequality over i from 0 to $n-1$, we get

$$\begin{aligned} R_n(I_n, f) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t)\,dt - \sum_{i=0}^{n-1} \frac{h_i}{2} [\lambda (f(x_i) + f(x_{i+1})) + (1-\lambda)(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i))] \\ &= \int_a^b f(t)\,dt - \sum_{i=0}^{n-1} \frac{h_i}{2} [\lambda (f(x_i) + f(x_{i+1})) + (1-\lambda)(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i))], \end{aligned}$$

which follows form (2.6), that

$$\begin{aligned} |R_n(I_n, f)| &= \left| \int_a^b f(t)\,dt - \sum_{i=0}^{n-1} \frac{h_i}{2} [\lambda (f(x_i) + f(x_{i+1})) + (1-\lambda)(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i))] \right| \\ &\le (\Gamma - \gamma) \sum_{i=0}^{n-1} \left[\frac{h_i^2}{16} \left(2\lambda^2 + (1-\lambda)^2\right) + \left(\alpha_i - \frac{(3-\lambda)x_i + (1+\lambda)x_{i+1}}{4}\right)^2 \right], \end{aligned}$$

which completes the proof.

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