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A COMPANION OF OSTROWSKI'S INEQUALITY FOR THE RIEMANN-STIELTJES INTEGRAL $\int_a^b f(t)\,du(t)$, WHERE f IS OF r-H-HÖLDER TYPE AND u IS OF BOUNDED VARIATION AND APPLICATIONS

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ABSTRACT. Some companions of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of r-H-Hölder type on $[a,b]$ and u is of bounded variation on $[a,b]$, are proved. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

1. Introduction

In [10], Dragomir has proved an Ostrowski inequality for the Riemann-Stieltjes integral, as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^r, \quad \forall x,y\in[a,b],$$

where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\left|f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\le H\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\cdot\bigvee_a^b(u) \tag{1.1}$$

for all $x\in[a,b]$, where, $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{2}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

In [11], Dragomir has proved the dual case as follows:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then we have the inequality*

(1.2)

$$\left|(u(b)-u(a))f(x)-\int_a^b f(t)\,du(t)\right| \le H\left[(x-a)^r\cdot\bigvee_a^x(f)+(b-x)^r\cdot\bigvee_x^b(f)\right]$$
$$\le H\begin{cases}\left[(x-a)^r+(b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\\ \\ \left[(x-a)^{qr}+(b-x)^{qr}\right]^{1/q}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{1/p}\\ \\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\cdot\bigvee_a^b(f)\end{cases}$$

In [5], Barnett et al. established some Ostrowski and trapezoid type inequalities for the Stieltjes integral $\int_a^b f(t)\,du(t)$ in the case of Lipschitzian integrators for both Hölder continuous and monotonic integrals are obtained. The dual case is also analyzed. In [6], Cerone et al. proved some Ostrowski type inequalities for the Stieltjes integral where the integrand f is absolutely continuous while the integrator u is of bounded variation. For other results concerning inequalities for Stieltjes integrals, see [3, 7, 8, 14, 16, 17, 19, 21, 22].

Motivated by [18], Dragomir in [13], established the following companion of the Ostrowski inequality for mappings of bounded variation.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b(f), \tag{1.3}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

For recent results concerning the above comapnion of Ostrowski's inequality and other related results see [1, 2, 4, 13, 15, 20].

In this paper, we establish a companion of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of r-H-Hölder type on $[a,b]$ and u is of bounded variation on $[a,b]$, are given. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

2. The Results

The following companion of Ostrowski's inequality for Riemann-Stieltjes integral holds.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality*

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \leq H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(u) \tag{2.1}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. where, $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)=f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]-\int_a^{\frac{a+b}{2}}f(t)\,du(t),$$

and

$$\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)=f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_{\frac{a+b}{2}}^b f(t)\,du(t)$$

Adding the above equalities, we have

$$\int_a^{\frac{a+b}{2}}[f(x)-f(t)]\,du(t)+\int_{\frac{a+b}{2}}^b[f(a+b-x)-f(t)]\,du(t)$$
$$=f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t).$$

It is well known that if $p:[c,d]\to\mathbb{R}$ is continuous and $\nu:[c,d]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)d\nu(t)$ exists and the following inequality holds:

$$\left|\int_c^d p(t)\,d\nu(t)\right|\leq\sup_{t\in[c,d]}|p(t)|\bigvee_c^d(\nu). \tag{2.2}$$

Applying the inequality (2.2) for

$$p(t)=\begin{cases} f(x)-f(t), & t\in\left[a,\frac{a+b}{2}\right] \\ f(a+b-x)-f(t), & t\in\left(\frac{a+b}{2},b\right]\end{cases},$$

as above and $\nu(t) = u(t)$, $t \in [a,b]$, we get

$$\left| f(x)\left[u\left(\frac{a+b}{2}\right) - u(a)\right] + f(a+b-x)\left[u(b) - u\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,du(t) \right|$$
$$= \left| \int_a^{\frac{a+b}{2}} [f(x) - f(t)]\,du(t) + \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)]\,du(t) \right|$$
$$\leq \left| \int_a^{\frac{a+b}{2}} [f(x) - f(t)]\,du(t) \right| + \left| \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)]\,du(t) \right|$$
$$\leq \sup_{t\in\left[a,\frac{a+b}{2}\right]} |f(x) - f(t)| \cdot \bigvee_a^{\frac{a+b}{2}}(u) + \sup_{t\in\left[\frac{a+b}{2},b\right]} |f(a+b-x) - f(t)| \cdot \bigvee_{\frac{a+b}{2}}^b (u). \tag{2.3}$$

As f is of r-H-Hölder type, we have

$$\begin{aligned}\sup_{t\in\left[a,\frac{a+b}{2}\right]} |f(x) - f(t)| &\leq \sup_{t\in\left[a,\frac{a+b}{2}\right]} \left[H|x-t|^r\right] \\ &= H\max\left\{(x-a)^r, \left(\frac{a+b}{2} - x\right)^r\right\} \\ &= H\left[\max\left\{(x-a), \left(\frac{a+b}{2} - x\right)\right\}\right]^r \\ &= H\left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4}\right|\right]^r,\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in\left[\frac{a+b}{2},b\right]} |f(a+b-x) - f(t)| &\leq \sup_{t\in\left[\frac{a+b}{2},b\right]} \left[H|a+b-x-t|^r\right] \\ &= H\max\left\{\left(a+b-x-\frac{a+b}{2}\right)^r, (b-a-b+x)^r\right\} \\ &= H\left[\max\left\{(x-a), \left(\frac{a+b}{2} - x\right)\right\}\right]^r \\ &= H\left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4}\right|\right]^r.\end{aligned}$$

Therefore, by (2.3), we have

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right|$$
$$\leq H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^{\frac{a+b}{2}}(u)+H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_{\frac{a+b}{2}}^b(u)$$
$$=H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(u).$$

To prove the sharpness of the constant $\frac{1}{4}$ for any $r\in(0,1]$, assume that (2.1) holds with a constant $C>0$, that is,

(2.4)
$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right|$$
$$\leq H\left[C(b-a)+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(u).$$

Choose $f(t)=t^r$, $r\in(0,1]$, $t\in[0,1]$ and $u:[0,1]\to[0,\infty)$ given by

$$u(t)=\begin{cases}0, & t\in(0,1]\\ -1, & t=0\end{cases}$$

As

$$|f(x)-f(y)|=|x^r-y^r|\leq|x-y|^r,\ \forall x\in[0,1],\ r\in(0,1],$$

it follows that f is r-H-Hölder type with the constant $H=1$.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:

(2.5)
$$\int_0^1 f(t)\,du(t)=f(1)\,u(1)-f(0)\,u(0)-\int_0^1 u(t)\,df(t)=0,\quad \textit{and}\quad \bigvee_0^1(u)=1.$$

Consequently, by (2.5), we get

$$|x^r|\leq\left[C+\left|x-\frac{1}{4}\right|\right]^r,\ \forall x\in\left[0,\frac{1}{2}\right].$$

For $x=\frac{1}{2}$, we get $\frac{1}{2^r}\leq\left(C+\frac{1}{4}\right)^r$, which implies that $C\geq\frac{1}{4}$, and the theorem is completely proved.

□

The following inequalities are hold:

Corollary 1. *Let f and u as in Theorem 4. In (2.1) choose*

(1) $x=a$, *then we get the following trapezoid type inequality*

$$\left|f(a)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(b)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \le H\left(\frac{b-a}{2}\right)^r\cdot\bigvee_a^b(u). \tag{2.6}$$

(2) $x=\frac{a+b}{2}$, *then we get the following mid-point type inequality*

$$\left|(u(b)-u(a))\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right| \le H\left(\frac{b-a}{2}\right)^r\cdot\bigvee_a^b(u). \tag{2.7}$$

We may state the following Ostrowski type inequality:

Corollary 2. *Let f and u as in Theorem 4. Additionally, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, then we have*

$$\left|(u(b)-u(a))\,f(x)-\int_a^b f(t)\,dt\right| \le H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(u), \tag{2.8}$$

for all $x\in[a,\frac{a+b}{2}]$.

Corollary 3. *Let u as in Theorem 4, and $f:[a,b]\to\mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, that is,*

$$|f(x)-f(y)|\le L\,|x-y|\,,\quad \forall x,y\in[a,b],$$

where, $L>0$ is fixed. Then, for all $x\in[a,\frac{a+b}{2}]$, we have the inequality

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \le L\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]\cdot\bigvee_a^b(u). \tag{2.9}$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Corollary 4. *In Theorem 4, if u is monotonic on $[a,b]$, and f is of r-H-Hölder type. Then, for all $x\in[a,\frac{a+b}{2}]$, we have the inequality*

$$\left|f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right| \le H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot|u(b)-u(a)|. \tag{2.10}$$

Corollary 5. *Let f be of r-H-Hölder type and $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then we have the inequality*

$$\left|f(x)\int_a^{\frac{a+b}{2}} g(s)ds+f(a+b-x)\int_{\frac{a+b}{2}}^b g(s)\,ds-\int_a^b f(t)\,g(t)\,dt\right| \le H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r \|g\|_1, \tag{2.11}$$

for all $x\in[a,\frac{a+b}{2}]$, where $\|g\|_1=\int_a^b |g(t)|\,dt$.

Proof. Define the mapping $u:[a,b]\to\mathbb{R}$, $u(t)=\int_a^t g(s)ds$. Then u is differentiable on (a,b) and $u'(t)=g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t)=\int_a^b f(t)\,g(t)\,dt,$$

and

$$\bigvee_a^b(u)=\int_a^b |u'(t)|\,dt=\int_a^b |g(t)|\,dt$$

□

Remark 1. *In Corollary 5, if f is symmetric about the x-axis, i.e., $f(a+b-x)=f(x)$, then we have*

$$\left|f(x)\int_a^b g(s)ds-\int_a^b f(t)\,g(t)\,dt\right|\le H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\|g\|_1, \tag{2.12}$$

for all $x\in[a,\frac{a+b}{2}]$. For instance, choose $x=\frac{a+b}{2}$, then we get

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(s)ds-\int_a^b f(t)\,g(t)\,dt\right|\le H\left(\frac{b-a}{2}\right)^r\|g\|_1. \tag{2.13}$$

In the folowing, some examples of weighted Ostrowski inequalities for some of the most popular weights.

Example 1. *(Legendre) If $g(t)=1$, and $t\in[a,b]$ then we get the following companion of Ostrowski inequality for Hölder type mappings $f:[a,b]\to\mathbb{R}$*

$$\left|(b-a)\frac{f(x)+f(a+b-x)}{2}-\int_a^b f(t)\,dt\right|\le H(b-a)\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r, \tag{2.14}$$

for all $x\in[a,\frac{a+b}{2}]$. In particular, the mid-point inequality

$$\left|(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|\le\frac{H}{2^r}(b-a)^{r+1}. \tag{2.15}$$

and the trapezoid inequality

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|\le\frac{H}{2^r}(b-a)^{r+1}. \tag{2.16}$$

Example 2. *(Logarithm) If $g(t) = \ln\left(\frac{1}{t}\right)$, and $t \in (0,1]$ then we get the following companion of Ostrowski inequality for Hölder type mappings $f : [a,b] \to \mathbb{R}$*

$$\left| f(x) \int_0^{1/2} \ln\left(\frac{1}{t}\right) dt + f(1-x) \int_{1/2}^1 \ln\left(\frac{1}{t}\right) dt - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt \right| \tag{2.17}$$

$$= \left| \frac{(1+\ln(2)) f(x) + (1-\ln(2)) f(1-x)}{2} - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt \right|$$

$$\leq H \left[\frac{1}{4} + \left| x - \frac{1}{4} \right| \right]^r,$$

for all $x \in [0, \frac{1}{2}]$. In particular,

$$\left| f\left(\frac{1}{2}\right) - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt \right| \leq \frac{H}{2^r} \tag{2.18}$$

and

$$\left| \frac{(1+\ln(2)) f(0) + (1-\ln(2)) f(1)}{2} - \int_0^1 f(t) \ln\left(\frac{1}{t}\right) dt \right| \leq \frac{H}{2^r} \tag{2.19}$$

Example 3. *(Jacobi) If $g(t) = \frac{1}{\sqrt{t}}$, and $t \in (0,1]$ then we get the following companion of Ostrowski inequality for Hölder type mappings $f : [a,b] \to \mathbb{R}$*

$$\left| f(x) \int_0^{1/2} \frac{1}{\sqrt{t}} dt + f(1-x) \int_{1/2}^1 \frac{1}{\sqrt{t}} dt - \frac{1}{2} \int_0^1 \frac{f(t)}{\sqrt{t}} dt \right| \tag{2.20}$$

$$= \left| \frac{\sqrt{2} f(x) + \left(2-\sqrt{2}\right) f(1-x)}{2} - \frac{1}{2} \int_0^1 \frac{f(t)}{\sqrt{t}} dt \right| \leq H \left[\frac{1}{4} + \left| x - \frac{1}{4} \right| \right]^r,$$

for all $x \in [0, \frac{1}{2}]$. In particular,

$$\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \int_0^1 \frac{f(t)}{\sqrt{t}} dt \right| \leq \frac{H}{2^r} \tag{2.21}$$

and

$$\left| \frac{\sqrt{2} f(0) + \left(2-\sqrt{2}\right) f(1)}{2} - \frac{1}{2} \int_0^1 \frac{f(t)}{\sqrt{t}} dt \right| \leq \frac{H}{2^r} \tag{2.22}$$

Example 4. *(Chebychev) If $g(t) = \frac{1}{\sqrt{1-t^2}}$, and $t \in (-1,1)$ then we get the following companion of Ostrowski inequality for Hölder type mappings $f : [a,b] \to \mathbb{R}$*

$$\left| f(x) \int_0^{1/2} \frac{1}{\sqrt{1-t^2}} dt + f(1-x) \int_{1/2}^1 \frac{1}{\sqrt{1-t^2}} dt - \frac{1}{\pi} \int_0^1 \frac{f(t)}{\sqrt{1-t^2}} dt \right| \tag{2.23}$$

$$= \left| \pi \frac{f(x) + f(1-x)}{2} - \frac{1}{\pi} \int_0^1 \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq H \left[\frac{1}{4} + \left| x - \frac{1}{4} \right| \right]^r,$$

for all $x \in [0, \frac{1}{2}]$. In particular,

$$\left| \pi f\left(\frac{1}{2}\right) - \frac{1}{\pi} \int_0^1 \frac{f(t)}{\sqrt{1-t^2}} dt \right| \leq \frac{H}{2^r} \tag{2.24}$$

and

$$\left|\pi\frac{f(-1)+f(1)}{2}-\frac{1}{\pi}\int_0^1\frac{f(t)}{\sqrt{1-t^2}}dt\right|\leq\frac{H}{2^r} \tag{2.25}$$

Comment 1. *There are a new studies for this companion of Ostrowski's inequality and other related results for Stieltjes integral in a preparation. Such as, the dual case of the obtained inequalities, and other interesting results will see the light soon.*

3. An Approximation for the Riemann-Stieltjes Integral

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $(i = 0,1,2,\cdots,n-1)$ and $\nu(h) := \max\{h_i | i = 0,1,2,\ldots,n-1\}$. Define the general Riemann-Stieltjes sum

(3.1)

$$S(f,u,I_n,\xi) = \sum_{i=0}^{n-1} f(\xi_i)\left[u\left(\frac{x_i+x_{i+1}}{2}\right)-u(x_i)\right]+f(x_i+x_{i+1}-\xi_i)\left[u(x_{i+1})-u\left(\frac{x_i+x_{i+1}}{2}\right)\right]$$

In the following, we establish some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by its Riemann-Stieltjes sum $S(f,u,I_n,\xi)$.

Theorem 5. *Let $u : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then*

$$\int_a^b f(t)\,du(t) = S(f,u,I_n,\xi) + R(f,u,I_n,\xi)$$

where, $S(f,u,I_n,\xi)$ is given in (3.1) and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$\begin{aligned} |R(f,u,I_n,\xi)| &\leq H\left[\frac{1}{4}\nu(h)+\max_{i=0,1,\ldots,n-1}\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\cdot\bigvee_a^b(u) \\ &\leq H\left[\frac{1}{2}\nu(h)\right]^r\cdot\bigvee_a^b(u) \end{aligned} \tag{3.2}$$

Proof. Applying Theorem 4 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\begin{aligned} \Bigg|f(\xi_i)\left[u\left(\frac{x_i+x_{i+1}}{2}\right)-u(x_i)\right]+f(x_i+x_{i+1}-\xi_i)\left[u(x_{i+1})-u\left(\frac{x_i+x_{i+1}}{2}\right)\right] \\ -\int_{x_i}^{x_{i+1}} f(t)\,du(t)\Bigg| \\ \leq H\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\cdot\bigvee_{x_i}^{x_{i+1}}(u), \end{aligned}$$

for all $i \in \{0,1,2,\cdots,n-1\}$.

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned}
&|R(f,u,I_n,\xi)| \\
&= \sum_{i=0}^{n-1}\left| f(\xi_i)\left[u\left(\frac{x_i+x_{i+1}}{2}\right)-u(x_i)\right]+f(x_i+x_{i+1}-\xi_i)\left[u(x_{i+1})-u\left(\frac{x_i+x_{i+1}}{2}\right)\right]\right. \\
&\qquad \left. -\int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \\
&\le H\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\cdot\bigvee_{x_i}^{x_{i+1}}(u) \\
&\le H\sup_{i=0,1,\dots,n-1}\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\cdot\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u).
\end{aligned}$$

However,

$$\sup_{i=0,1,\dots,n-1}\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\le\left[\frac{1}{4}\nu(h)+\sup\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r,$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u)=\bigvee_a^b(u).$$

which completely proves the first inequality in (3.2).

For the second inequality, we observe that

$$\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\le\frac{1}{4}h_i$$

for all $i\in\{0,1,2,\cdots,n-1\}$. which completes the proof. □

Corollary 6. *In Theorem 5, additionally, if f is symmetric about the x-axis, then we have $S(f,u,I_n,\xi)$ reduced to be*

$$S(f,u,I_n,\xi)=\sum_{i=0}^{n-1}f(\xi_i)\,[u(x_{i+1})-u(x_i)]. \tag{3.3}$$

Then

$$\int_a^b f(t)\,du(t)=S(f,u,I_n,\xi)+R(f,u,I_n,\xi)$$

where, $S(f,u,I_n,\xi)$ is given in (3.3) and the remainder $R(f,u,I_n,\xi)$ satisfies the bound in (3.2).

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