Received 03/07/11

SOME GRÜSS TYPE INEQUALITIES IN $\ell_p^2(X)$

A. G. GHAZANFARI[1,*], S. S. DRAGOMIR[2,3], B. GHAZANFARI[4]

[1,4] *Department of Mathematics, Lorestan University*
P. O. Box 465, Khoramabad, Iran
[2] *School of Engineering and Science, Victoria University*
PO Box 14428 Melbourne City, MC 8001, Australia.
[3] *School of Computational and Applied Mathematics*
University of the Witwatersrand, Private Bag 3
Wits 2050, Johannesburg, South Africa.

Abstract. Let $\mathcal{A}$ be a C^*-algebra and X be an inner product $\mathcal{A}$-module. Some Grüss type inequalities in $\ell_p^2(X)$ are established.
Mathematics Subject Classification (2000). Primary 46L08, 46H25; Secondary 26D99, 46C99.
Keywords: Inner product C^*-modules, Grüss inequality.

1. Introduction

We start by recalling some of the most important Grüss type discrete inequalities for inner product spaces that are available in [1].

Theorem 1. *Let $(H;\langle,\rangle)$ be an inner product space over $\mathbb{K}$; $(\mathbb{K}=\mathbb{C},\mathbb{R})$, $x_i,\ y_i\in H,\ p_i\geq 0\ (i=1,...,n)\ (n\geq 2)$ with $\sum_{i-1}^n p_i=1$. If $x,X,y,Y\in H$ are such that*

$$\operatorname{Re}\langle X-x_i,x_i-x\rangle\geq 0 \quad and \quad \operatorname{Re}\langle Y-y_i,y_i-y\rangle\geq 0$$

for all $i\in\{1,...,n\}$, or, equivalently,

$$\left\|x_i-\frac{x+X}{2}\right\|\leq\frac{1}{2}\|X-x\| \quad and \quad \left\|y_i-\frac{y+Y}{2}\right\|\leq\frac{1}{2}\|Y-y\|$$

for all $i\in\{1,...,n\}$, then we have the inequality

$$\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^n p_ix_i,\sum_{i=1}^n p_iy_i\right\rangle\right|\leq\frac{1}{4}\|X-x\|\|Y-y\|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Theorem 2. *Let $(H;\langle,\rangle)$ and $\mathbb{K}$ be as above and $\overline{x}=(x_1,...,x_n)\in H^n$, $\overline{\alpha}=(\alpha_1,...,\alpha_n)\in\mathbb{K}^n$ and $\overline{p}=(p_1,...,p_n)$ a probability vector. If $x,X\in H$ are such that*

$$\operatorname{Re}\langle X-x_i,x_i-x\rangle\geq 0 \ for\ all\ i\in\{1,...,n\},$$

*Corresponding author.
E-mail addresses:
ghazanfari.amir@gmail.com, sever.dragomir@vu.edu.au, Bahman_ghazanfari@yahoo.com.

or, equivalently,

$$\left\| x_i - \frac{x+X}{2} \right\| \leq \frac{1}{2}\|X - x\| \ for\ all\ i \in \{1, ..., n\},$$

holds, then we have the inequality

$$\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2}\|X - x\| \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{j=1}^{n} p_j \alpha_j \right|$$
$$\leq \frac{1}{2}\|X - x\| \left[\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ *in the first and second inequalities is best possible.*

Let $\mathcal{A}$ be a C^*-algebra. A semi-inner product module over $\mathcal{A}$ is a right module X over $\mathcal{A}$ together with a generalized semi-inner product, that is with a mapping $\langle .,. \rangle$ on $X \times X$, which is $\mathcal{A}$-valued and having the following properties:

(i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in X, a \in \mathcal{A}$,
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in X$,
(iv) $\langle x, x \rangle \geq 0$ for $x \in X$.

We will say that X is a semi-inner product C^*-module. If, in addition,

(v) $\langle x, x \rangle = 0$ implies $x = 0$,

then X is called an inner product C^*-module.

It is well known that if X is a semi-inner product C^*-module, then the following Schwarz inequality holds:

$$\|\langle x, y \rangle\|^2 \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\| \quad (x, y \in X). \tag{1.1}$$

(e.g. [8, Lemma 15.1.3]).

Now let $\mathcal{A}$ be a C^*-algebra and X be an inner product $\mathcal{A}$-module, $p_i \in \mathbb{R}, p_i \geq 0, i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$ we define

$$\ell_p^2(X) = \{(x_n) | x_n \in X, n \in \mathbb{N}, \sum_{i=1}^{\infty} p_i \|\langle x_i . x_i \rangle\|^2 < \infty\}.$$

In this paper we obtain inequalities of Grüss type in $\ell_p^2(X)$.

2. THE MAIN RESULT

If $x = (x_n) \in \ell_p^2(X)$ then the series $\sum_{i=1}^{\infty} p_i \langle x_i . x_i \rangle$ is convergent in $\mathcal{A}$, since it's absolutely convergent. We put $a = \sum_{i=1}^{\infty} p_i \langle x_i . x_i \rangle, \quad a_n = \sum_{i=1}^{n} p_i \langle x_i . x_i \rangle \quad (n \in \mathbb{N})$ and $t = \|a\|$, then $a_n \geq 0$ and by [7, Lemma 2.2.2] $\|a_n - t\| \leq t$. This implies that $\|a - t\| \leq t$ and again by [7, Lemma 2.2.2] we conclude that $a \geq 0$.
Therefore if X is an inner product $\mathcal{A}$-module then $\ell_p^2(X)$ is an inner product $\mathcal{A}$-module with point-wise vector operations and the inner product is defined by

$$\langle x, y \rangle_p = \sum_{i=1}^{\infty} p_i \langle x_i . y_i \rangle \quad x = (x_i), y = (y_i) \in \ell_p^2(X).$$

and module operation $(x_i)a = (x_i a) \quad (a \in \mathcal{A})$. Therefore for the norm $\|.\|_p$ of $\ell^2_p(X)$ we have

$$\|x\|_p = \|\langle x, x\rangle_p\|^{\frac{1}{2}} = \left\|\sum_{i=1}^{\infty} p_i\langle x_i, x_i\rangle\right\|^{\frac{1}{2}} \leq \left(\sum_{i=1}^{\infty} p_i\|\langle x_i, x_i\rangle\|^2\right)^{\frac{1}{2}}.$$

It is obvious that for every $z \in X$ if we put $\overline{z} := (z, z, z, ...)$ then $\overline{z} \in \ell^2_p(X)$. Furthermore if X is a Hilbert C^*-module then $\ell^2_p(X)$ is a Hilbert C^*-module.

Let $\mathcal{A}$ be a C^*-algebra and $p = (p_i)$ be a probability sequence in $\mathbb{R}$ i.e., $p_i \geq 0 \quad (i \in \mathbb{N})$ and $\sum_{i=1}^{\infty} p_i = 1$. If X is a semi-inner product $\mathcal{A}$-module and $x = (x_i), y = (y_i) \in \ell^2_p(X)$ we put

$$G_p(x, y) = \sum_{i=1}^{\infty} p_i\langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{\infty} p_i x_i, \sum_{i=1}^{\infty} p_i y_i \right\rangle$$

we use $G_p(x)$ instead of $G_p(x, x)$.

Theorem 3. *Let X be a semi-inner product C^*-module, $x = (x_i), y = (y_i) \in \ell^2_p(X)$ and $p = (p_i)$ a probability vector. If $z, t \in X, r \geq 0, s \geq 0$ such that*

$$\|x - \overline{z}\|_p \leq r, \quad \|y - \overline{t}\|_p \leq s, \tag{2.1}$$

then we have the inequality

$$\|G_p(x, y)\| \leq rs - \sqrt{r^2 - \|x - \overline{z}\|_p}\sqrt{s^2 - \|y - \overline{t}\|_p} \leq rs. \tag{2.2}$$

The constant 1 coefficient of rs in the inequalities (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. A simple calculation shows that

$$\sum_{i=1}^{\infty} p_i\langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{\infty} p_i x_i, \sum_{i=1}^{\infty} p_i y_i \right\rangle = \frac{1}{2}\sum_{i,j=1}^{\infty} p_i p_j \left\langle x_i - x_j, y_i - y_j\right\rangle, \tag{2.3}$$

therefore

$$G_p(x) = \frac{1}{2}\sum_{i,j=1}^{\infty} p_i p_j \left\langle x_i - x_j, x_i - x_j\right\rangle \geq 0. \tag{2.4}$$

It is easy to show that $G_p(.,.)$ is an $\mathcal{A}$-value semi-inner product on $\ell^2_p(X)$, so Schwarz inequality holds i.e.,

$$\|G_p(x, y)\|^2 \leq \|G_p(x)\|\|G_p(y)\|. \tag{2.5}$$

Also a simple calculation shows that for every $z, t \in X$

$$\begin{aligned} \sum_{i=1}^{\infty} p_i \left\langle x_i - z, y_i - t\right\rangle - \left\langle \sum_{i=1}^{\infty} p_i(x_i - z), \sum_{i=1}^{\infty} p_i(y_i - t) \right\rangle \\ = \sum_{i=1}^{\infty} p_i\langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{\infty} p_i x_i, \sum_{i=1}^{\infty} p_i y_i \right\rangle = G_p(x, y). \end{aligned} \tag{2.6}$$

particular

$$G_p(x) = \sum_{i=1}^{\infty} p_i \langle x_i - z, x_i - z\rangle - \left\langle \sum_{i=1}^{\infty} p_i(x_i - z), \sum_{i=1}^{\infty} p_i(x_i - z) \right\rangle \leq \sum_{i=1}^{\infty} p_i \langle x_i - z, x_i - z\rangle . \tag{2.7}$$

This implies that

$$\|G_p(x)\| \leq \|x - \overline{z}\|_p^2 \leq r^2, \tag{2.8}$$

similarly

$$\|G_p(x)\| \leq \|y - \overline{t}\|_p^2 \leq s^2. \tag{2.9}$$

Now using the elementary inequality for real numbers

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2 \tag{2.10}$$

on

$$m = r, \qquad n^2 = r^2 - \|x - \overline{z}\|_p^2$$
$$p = s, \qquad q^2 = s^2 - \|y - \overline{t}\|_p^2,$$

we get the inequality (2.2). To prove the sharpness of the constant 1 in the inequalities in (2.2), let us assume that, under the assumptions of the theorem, the inequalities hold with a constant $c > 0$, i.e.,

$$\|G_p(x, y)\| \leq crs - \sqrt{r^2 - \|x - \overline{z}\|_p^2}\sqrt{s^2 - \|y - \overline{t}\|_p^2} \leq crs. \tag{2.11}$$

Assume that $p = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, ...)$ and e is an element of X such that $\|\langle e, e\rangle\| = 1$. We put

$$x = (z + re, z - re, z, z, z, ...)$$
$$y = (t + se, t - se, t, t, t, ...)$$

then, obviously,

$$\|x - \overline{z}\|_p \leq r, \quad \|y - \overline{t}\|_p \leq s,$$

which shows that the condition (2.1) holds. If we replace p, x, y in (2.11), we obtain

$$\|G_p(x, y)\| = rs \leq crs.$$

from where we deduce that $c \geq 1$, which proves the sharpness of the constant 1. □

If we consider X^n as a subspace of $\ell_p^2(X)$, then we can deduce [6, Theorem 1] from Theorem 3 as follows.

Corollary 1. *Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n$ and $p = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $z, t \in X, r \geq 0, s \geq 0$ such that*

$$\|x_i - z\| \leq r, \quad \|y_i - t\| \leq s, \text{ for all } i \in \{1, ..., n\}, \tag{2.12}$$

then we have the inequality

$$\|G_p(x,y)\| \;\le\; rs \;-\; \sqrt{r^2 - \sum_{i=1}^{n} p_i\|x_i - a\|^2}\sqrt{s^2 - \sum_{i=1}^{n} p_i\|y_i - b\|^2} \;\le\; rs. \tag{2.13}$$

The constant 1 coefficient of rs in the inequalities (2.13) is best possible in the sense that it cannot be replaced by a smaller constant.

Now we give applications of Corollary 1 for some discrete transforms.

3. Applications

Let X be a semi-inner product C^*-module on C^*-algebra $\mathcal{A}$, $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n$

For a given $\omega \in \mathbb{R}$, define the discrete Fourier Transform

$$\mathcal{F}_\omega(x)(m) = \sum_{k=1}^{n} \exp(2\omega imk) \times x_k, \quad m = 1, ..., n. \tag{3.1}$$

The element $\sum_{k=1}^{n} \exp(2\omega imk) \times \langle x_k, y_k\rangle$ of $\mathcal{A}$ is called Fourier transform of the vector $(\langle x_1, y_1\rangle, ..., \langle x_k, y_k\rangle) \in \mathcal{A}^n$ and will be denoted by

$$\mathcal{F}_\omega(x.y)(m) = \sum_{k=1}^{n} \exp(2\omega imk) \times \langle x_k, y_k\rangle \quad m = 1, ..., n. \tag{3.2}$$

The following Theorems 4, 5, 6 are versions of [1, Theorems 66, 67, 68] for semi-inner product C^*-modules respectively, (also see [3]).

Theorem 4. *Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n$. If $z, t \in X, r \ge 0, s \ge 0$ such that*

$$\|x_k - z\| \le r, \quad \|\exp(2\omega imk)y_k - t\| \le s, \text{ for all } k,m \in \{1, ..., n\}, \tag{3.3}$$

then we have the inequality

$$\left\| \mathcal{F}_\omega(x.y)(m) - \left\langle \frac{1}{n}\sum_{k=1}^{n} x_k, \mathcal{F}_\omega(y)(m) \right\rangle \right\| \le nrs, \tag{3.4}$$

for all $m \in \{1, ..., n\}$.

The proof follows by Corollary 1 applied for $p_k = \frac{1}{n}$ and for the vectors $\exp(2\omega imk)x_k$ and $y_k (k = 1, ..., n)$. We omit the details.

We can also consider the Mellin transform

$$\mathcal{M}(x)(m) = \sum_{k=1}^{n} k^{m-1} x_k, \quad m = 1, ..., n. \tag{3.5}$$

of the vector $x = (x_1, ..., x_n) \in X^n$. The Mellin transform of the vector $(\langle x_1, y_1\rangle, ..., \langle x_k, y_k\rangle) \in \mathcal{A}^n$ is defined by $\sum_{k=1}^{n} k^{m-1}\langle x_k, y_k\rangle$ and will be denoted by

$$\mathcal{M}(x.y)(m) = \sum_{k=1}^{n} k^{m-1}\langle x_k, y_k\rangle. \tag{3.6}$$

Theorem 5. *Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n$. If $z, t \in X, r \geq 0, s \geq 0$ such that*

$$\|x_k - z\| \leq r, \quad \left\|k^{m-1} y_k - t\right\| \leq s, \text{ for all } k,m \in \{1, ..., n\}, \tag{3.7}$$

then we have the inequality

$$\left\| \mathcal{M}(x.y)(m) - \left\langle \frac{1}{n} \sum_{k=1}^{n} x_k, \mathcal{M}(y)(m) \right\rangle \right\| \leq nrs, \tag{3.8}$$

for all $m \in \{1, ..., n\}$.

The proof follows by Corollary 1 applied for $p_k = \frac{1}{n}$ and for the vectors x_k and $k^{m-1} y_k (k = 1, ..., n)$. We omit the details.

Another result which connects the Fourier transforms for different parameters ω also holds.

Theorem 6. *Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X^n$. If $z, t \in X, r \geq 0, s \geq 0$ such that*

$$\| \exp(2\omega_1 imk) x_k - z\| \leq r, \ \| \exp(2\omega_2 imk) y_k - t\| \leq s, \tag{3.9}$$

for all $k, m \in \{1, ..., n\}$, then we have the inequality

$$\left\| \frac{1}{n} \mathcal{F}_{\omega_2 - \omega_1}(x.y)(m) - \left\langle \frac{1}{n} \mathcal{F}_{\omega_1}(x)(m), \frac{1}{n} \mathcal{F}_{\omega_2}(y)(m) \right\rangle \right\| \leq rs, \tag{3.10}$$

for all $m \in \{1, ..., n\}$.

The proof follows by Corollary 1 applied for $p_k = \frac{1}{n}$ and for the vectors $\exp(2\omega_1 imk) x_k$ and $\exp(2\omega_2 imk) y_k (k = 1, ..., n)$. We omit the details.

In [6, Remark 1(ii)] the authors show that if X is a semi-inner product C^*-module, $x = (x_1, ..., x_n), \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$ and $p = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector, and if $z \in X, r \geq 0$, such that

$$\|x_i - z\| \leq r, \text{ for all i } \in \{1, ..., n\},$$

holds, then

$$\begin{aligned} \left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| &\leq r \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{i=1}^{n} p_j \alpha_j \right| \\ &\leq r \left[\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{3.11}$$

The following approximation result for the Fourier transform (3.1) which is a version of [1, Theorems 59] for semi-inner product C^*-modules holds, (see [2]).

Proposition 1. *Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n) \in X^n$. If $z \in X, r \geq 0$, such that*

$$\|x_i - z\| \leq r, \ \text{for all } i \in \{1, ..., n\},$$

then we have the inequality

$$\left\|\mathcal{F}_\omega(x)(m) - \frac{\sin(\omega mn)}{\sin(\omega m)} \exp[\omega(n+1)im] \times \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \leq r\left[n^2 - \frac{\sin^2(\omega mn)}{\sin^2(\omega m)}\right]^{\frac{1}{2}}, \tag{3.12}$$

for all $m \in \{1, ..., n\}$ *and* $\omega \in \mathbb{R}, \omega \neq \frac{l}{m}\pi, l \in \mathbb{Z}$.

Proof. From the inequality (3.10) we can state that,

$$\left\|\frac{1}{n}\sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n}\sum_{i=1}^{n} \alpha_i . \frac{1}{n}\sum_{i=1}^{n} x_i\right\| \leq r\left[\frac{1}{n}\sum_{i=1}^{n} |\alpha_i|^2 - \left|\frac{1}{n}\sum_{i=1}^{n} \alpha_i\right|^2\right]^{\frac{1}{2}}, \tag{3.13}$$

for all $\alpha_i \in \mathbb{C},\ x_i \in X\ (i = 1, ..., n)$. Consequently, we conclude that

$$\left\|\sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} \alpha_i . \frac{1}{n}\sum_{i=1}^{n} x_i\right\| \leq r\left[n\sum_{i=1}^{n} |\alpha_i|^2 - \left|\sum_{i=1}^{n} \alpha_i\right|^2\right]^{\frac{1}{2}}. \tag{3.14}$$

A simple calculation shows that (see the proof of Theorem 59 in [1]),

$$\sum_{k=1}^{n} \exp(2\omega imk) = \frac{\sin(\omega mn)}{\sin(\omega m)} \times \exp[\omega(n+1)im].$$

Putting $\alpha_k = \exp(2\omega imk)$, we get the desired result (3.12). □

The following approximation result for the Mellin transform (3.5) holds, (see [2]).

Proposition 2. *Let* X *be a semi-inner product* C^**-module,* $x = (x_1, ..., x_n) \in X^n$*. If* $z \in X, r \geq 0$*, such that*

$$\|x_i - z\| \leq r, \text{ for all } i \in \{1, ..., n\},$$

then we have the inequality

$$\left\|\mathcal{M}(x)(m) - S_{m-1}(n) . \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \leq r[nS_{2m-2}(n) - S_{m-1}^2(n)]^{\frac{1}{2}}, m \in \{1, ..., n\}, \tag{3.15}$$

where $S_P(n), p \in \mathbb{R}, n \in \mathbb{N}$ *is the p-powered sum of the first n natural numbers, i.e.,*

$$S_p(n) := \sum_{k=1}^{n} k^p.$$

Consider the following particular values of Mellin Transform

$$\mu_1(x) := \sum_{k=1}^{n} kx_k$$

and

$$\mu_2(x) := \sum_{k=1}^{n} k^2 x_k$$

The following corollary holds.

Corollary 2. *Let X be a semi-inner product C^*-module, $x = (x_1, ..., x_n) \in X^n$. If $z \in X, r \geq 0$, such that*

$$\|x_i - z\| \leq r, \text{ for all } i \in \{1, ..., n\},$$

then we have the inequality

$$\left\| \mu_1(x) - \frac{n+1}{2}.\sum_{k=1}^{n} x_k \right\| \leq \frac{rn}{2} \left[\frac{(n-1)(n+1)}{3} \right]^{\frac{1}{2}}, \tag{3.16}$$

and

$$\left\| \mu_2(x) - \frac{(n+1)(2n+1)}{6}.\sum_{k=1}^{n} x_k \right\| \leq \frac{rn}{6\sqrt{5}} \sqrt{(n-1)(n+1)(2n+1)(8n+11)}. \tag{3.17}$$

Other inequalities related to the Grüss type discrete inequalities for polynomials with coefficients in a Hilbert space such as Theorem 61, Theorem 62, Corollary 52 in [1], have versions that are valid for polynomials with coefficients in a C^*-module. However, the details are omitted.

References

[1] S. S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science puplishers Inc., New York, 2005.

[2] S.S. DRAGOMIR, *A Grüss type discrete inequality in inner product spaces and applications*, J. Math. Anal. Appl., 250 (2000), 494-511.

[3] S.S. DRAGOMIR, *A Grüss type inequality for sequences of vectors in inner product spaces and applications*, Journal of Inequalities in Pure and Applied Mathematics, 1(2), 2000, Article 12. [ON LINE: http://jipam.vu.edu.au/v1n2/002 00.pdf]

[4] A. G. GHAZANFARI, S. S. DRAGOMIR, *Schwarz and Grüss type inequalities for C^*-seminorms and positive linear functionals on Banach $*$-modules*, Linear Algebra and Appl. **434** (2011), 944-956.

[5] A. G. GHAZANFARI, S. S. DRAGOMIR, *Bessel and Grüss type inequalities in inner product modules over Banach $*$-algebras*, J.I.A. vol(2011), Article ID 562923.

[6] A. G. GHAZANFARI, B. GHAZANFARI, A. BARANI, *Some Grüss type inequalities for n-tuples of vectors*, (submitted).

[7] G. J. MURPHY, *C^*-Algebra and Operator Theory*, Academic Press, 1990.

[8] N. E. WEGGE-OLSEN, *K-theory and C^*-algebras- A Friendly Approach*, Oxford University Press, Oxford, 1993.