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On a Ky Fan Type Inequality due to H. Alzer

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Abstract

In this article, using an identity concerning arithmetic and harmonic means of positive numbers together with the Ky Fan's inequality, we get a very simple proof for the Ky Fan type inequality $A'_n - H'_n \leq A_n - H_n$, due to H. Alzer.

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1 Introduction

Throughout this article, given n arbitrary real numbers $x_1, \cdots, x_n > 0$, we denote by A_n, G_n and H_n the arithmetic, geometric and harmonic means of $x_1, \cdots, x_n$ respectively, i.e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{1/n}, \qquad H_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}. \tag{1}$$

Also, when $x_i \in (0, 1/2]$, we denote by A'_n, G'_n, and H'_n the arithmetic, geometric and harmonic means of $1 - x_1, \cdots, 1 - x_n$ respectively, i.e.

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1 - x_i), \qquad G'_n = \prod_{i=1}^{n}(1 - x_i)^{1/n}, \qquad H'_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{1-x_i}}. \tag{2}$$

The Ky Fan's inequality

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}, \tag{3}$$

was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [3, p. 5], and from then it has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [2] and the references therein, see also [5-11].
One of interesting additive analogue of Ky Fan's inequality was discovered by H. Alzer [1] in 1993, as follows:

$$A'_n - H'_n \leq A_n - H_n, \tag{4}$$

with equality holding if and only if $x_1 = \cdots = x_n$.
Using differentiation and Tchebyschef inequality, the proof of H. Alzer is elementary, but technical and cumbersome. In this article, using a recursive identity together with the Ky Fan's inequality (3), we give a very simple proof of (4).

2 Simple Proof of $A'_n - H'_n \leq A_n - H_n$

In this section, first we establish a recursive identity concerning arithmetic and harmonic means of positive numbers, which in turn yields a Rado type inequality, see e.g. [4], together with a representation of $A_n - H_n$ as a finite series of nonnegative terms, and then using this identity with the Ky Fan's inequality (3), we give an inductive proof of (4).

Lemma 2.1. *For $n \geq 2$, we have*

$$A_n - H_n = \frac{n-1}{n}\left[A_{n-1} - H_{n-1} + \frac{(x_n - H_{n-1})^2}{(n-1)x_n + H_{n-1}}\right]. \tag{5}$$

As a consequence, the following Rado type inequality holds

$$A_n - H_n \geq \frac{n-1}{n}\left(A_{n-1} - H_{n-1}\right), \tag{6}$$

with equality holding if and only if $x_n = H_{n-1}$.
Also,

$$A_n - H_n = \frac{1}{n}\sum_{k=2}^{n}(k-1)\frac{(x_k - H_{k-1})^2}{(k-1)x_k + H_{k-1}}. \tag{7}$$

Proof. We have

$$\begin{aligned} A_n - H_n &= \frac{(n-1)A_{n-1} + x_n}{n} - \frac{(n-1)H_{n-1} + x_n}{n} + \frac{(n-1)H_{n-1} + x_n}{n} - \frac{n}{\frac{n-1}{H_{n-1}} + \frac{1}{x_n}} \\ &= \frac{n-1}{n}(A_{n-1} - H_{n-1}) + \frac{n-1}{n}\frac{(x_n - H_{n-1})^2}{(n-1)x_n + H_{n-1}}, \end{aligned}$$

and (5) is obtained. The inequality (6) is clear. For (7), write (5) with respect to k instead of n, multiply each side by k and sum from k=2 to $k = n$.

Proof of the inequality (4). If $x_1 = \cdots = x_n$, evidently equality holds in (4). Let not all of x_i's be equal. We can suppose that $x_n = \max_{1 \leq i \leq n} x_i$. Changing the roles of x_i's with $(1 - x_i)$'s in the Lemma 2.1, we can write

$$A'_n - H'_n = \frac{n-1}{n}\left[A'_{n-1} - H'_{n-1} + \frac{(1 - x_n - H'_{n-1})^2}{(n-1)(1 - x_n) + H'_{n-1}}\right]. \tag{8}$$

But by the Ky Fan's inequality (3), we have

$$\frac{A'_{n-1}}{G'_{n-1}} \leq \frac{A'_{n-1}+A_{n-1}}{G'_{n-1}+G_{n-1}} = \frac{1}{G'_{n-1}+G_{n-1}}.$$

So,

$$H'_{n-1}+H_{n-1} \leq G'_{n-1}+G_{n-1} \leq \frac{G'_{n-1}}{A'_{n-1}} \leq 1 = (1-x_n)+x_n,$$

and therefore, $x_n - H_{n-1} \geq H'_{n-1} - (1-x_n) > 0$, and $(x_n - H_{n-1})^2 \geq (H'_{n-1} - (1-x_n))^2 > 0$. Now comparing (5) with (8), by using the induction hypothesis

$$A'_{n-1} - H'_{n-1} \leq A_{n-1} - H_{n-1},$$

and taking into account that

$$(n-1)x_n + H_{n-1} < (n-1)x_n + \frac{1}{2} < (n-1)(1-x_n) + H'_{n-1},$$

we get (4) with strict inequality.

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