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HADAMARD-TYPE INEQUALITIES THROUGH $h-$CONVEXITY

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Abstract. Some new inequalities of Hadamard-type for h-convex mappings with general point of line segment are established. Two Lemmas, which allow us to establish new inequalities connected with lower and upper part of the celebrated Hermite-Hadamrd inequality, are pointed out.

Introduction

Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

refer as the Hermite-Hadamard inequality; (see[3],[8],[9]).In recent years, many authors established several inequalities connected to this famous integral inequality (1). For some results which generalize, improve and extend the inequality (1);(See[2],[3],[4],[5],[6],[7]).

In [4] the authors obtained inequalities for differentiable convex function which are connected with the inequality (1) , one of them is pointed out as:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$,then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}[|f'(a)| + |f'(b)|] \tag{2}$$

In [6], U. S. Kirmaci gave the following results:

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Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a, b]$,then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{8} [|f'(a)| + |f'(b)|] \tag{3}$$

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I^o$ with $a < b$ and let $p > 1$. If $|f'|^q$ is convex on $[a, b]$,then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{16} \left(\frac{4}{p+1}\right)^{1/p} \times \left[\left(|f'(a)|^q + 3|f'(b)|^q\right)^{1/q} + \left(3|f'(a)|^q + |f'(b)|^q\right)^{1/q} \right] \tag{4}$$

In [7] U.S. Kirmaci et al. established the following new Hadamard-type inequality for concave functions:

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $|f'|^q, q > 1$ is concave on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left(\frac{b-a}{4}\right) \left(\frac{q-1}{2q-1}\right)^{1-1/q} \times \left[\left| f'\left(\frac{3a+b}{4}\right) \right| + \left| f'\left(\frac{a+3b}{4}\right) \right| \right] \tag{5}$$

In [2] M. Alomari et. al established the following Hadamard-type inequality for concave functions:

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $|f'|^q, q > 1$ is concave on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left(\frac{b-a}{4}\right) \left(\frac{q-1}{2q-1}\right)^{1-1/q} \times \left[\left| f'\left(\frac{3a+b}{4}\right) \right| + \left| f'\left(\frac{a+3b}{4}\right) \right| \right] \tag{6}$$

In [1] M. Alomari et. al obtained the Hadamard-type inequalities for quasi-convex functions using the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$,then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{4}\left[\int_0^1 (-t)\times f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\, dt + \int_0^1 t\; f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\, dt\right].$$

In [13] S. Varošanec defined the concept of $h-$convexity as the following:

Definition 1. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function(or f belongs to the class $SX(h,I)$) if f is non-negative ,and*

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y)$$

, for all $x, y \in I$ and $t \in (0,1)$.

If the inequality is reversed then f is said to be $h-$concave and in this case f belongs to the class $SV(h,I)$

Remark 1. *Obviously, if*

- *$h(t) = t$, then all the non-negative convex functions belong to the class $SX(h,I)$ and all non-negative concave functions belong to the class $SV(h,I)$.*
- *$h(t) = \frac{1}{t}$, then $SX(h,I) = Q(I)$.*
- *$h(t) = 1$, then $SX(h,I) \supseteq P(I)$.*
- *$h(t) = t^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq {K_s}^2$.*

M. Z. Sarikaya et al. discussed in [11, 12] some new inequalities of Hadamard-type for h-convex functions.We presented here new results of Hadamard-type on the basis of Lemmas 2 and 3 for $h-$convex functions.

1. The Main Result

1.1. Right Estimation of Hadamard-Inequality.

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I^o where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ and $\lambda, \mu \in [0, \infty)$ with $\lambda + \mu \neq 0$,then*

$$\frac{\mu f(a) + \lambda f(b)}{\lambda + \mu} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{(\lambda+\mu)^2}\left[\int_0^\mu (-t)\times f'\left(\frac{\lambda+t}{\lambda+\mu}a + \frac{\mu-t}{\lambda+\mu}b\right)dt + \int_0^\lambda tf'\left(\frac{\lambda-t}{\lambda+\mu}a + \frac{\mu+t}{\lambda+\mu}b\right)dt\right]. \quad (7)$$

Proof. Let

$$I_1 = \frac{b-a}{(\lambda+\mu)^2}\int_0^\mu (-t)f'\left(\frac{\lambda+t}{\lambda+\mu}a + \frac{\mu-t}{\lambda+\mu}b\right)dt$$

Integrating by parts,

$$I_1 = \frac{t}{\lambda+\mu} f\left(\frac{\lambda+t}{\lambda+\mu}a + \frac{\mu-t}{\lambda+\mu}b\right)\Bigg|_0^\mu - \frac{1}{\lambda+\mu}\int_0^\mu f\left(\frac{\lambda+t}{\lambda+\mu}a + \frac{\mu-t}{\lambda+\mu}b\right)dt = \frac{\mu f(a)}{\lambda+\mu} - \frac{1}{b-a}\int_a^{\frac{\lambda a+\mu b}{\lambda+\mu}} f(x)dx \quad (8)$$

Similarly,

$$I_2 = \frac{\lambda f(b)}{\lambda+\mu} - \frac{1}{b-a}\int_{\frac{\lambda a+\mu b}{\lambda+\mu}}^b f(x)dx \quad (9)$$

Identity (7) is obtained by adding (8) and (9). □

Remark 2. *For $\lambda + \mu = b - a$ we remain in the interval $[a, b]$.*
At mid-point i.e., $\lambda = \mu$, Lemma 1 is obtained.

We are now ready to give the main results of this section.

Theorem 6. *Let $|f'|$ be h-convex function, then under the assumption of Lemma 2 we have*

$$\left|\frac{\mu f(a) + \lambda f(b)}{\lambda + \mu} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\int_0^1 \left\{\left|f'(a)\left(\frac{\lambda}{\lambda+\mu} - t\right)\right| + \left|f'(b)\left(\frac{\mu}{\lambda+\mu} - t\right)\right|\right\} h(t)dt \quad (10)$$

Proof. From Lemma 2,

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{(\lambda+\mu)^2}\left[\int_0^{\mu}|-t|\times\right.$$
$$\left.\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt+\int_0^{\lambda}|t|\left|f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)\right|dt\right]. \quad (11)$$

Consider

$$I_5=\int_0^{\mu}|-t|\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt$$

Since $|f'|$ is h-convex,

$$I_5 \le |f'(a)|\int_0^{\mu} th\left(\frac{\lambda+t}{\lambda+\mu}\right)dt+|f'(b)|\int_0^{\mu} th\left(\frac{\mu-t}{\lambda+\mu}\right)dt$$

$$I_5 \le (\lambda+\mu)^2\left[|f'(a)|\int_{\frac{\lambda}{\lambda+\mu}}^{1}\left(t-\frac{\lambda}{\lambda+\mu}\right)h(t)dt+|f'(b)|\int_0^{\frac{\mu}{\lambda+\mu}}\left(\frac{\mu}{\lambda+\mu}-t\right)h(t)dt\right]$$

Similarly

$$I_6=\int_0^{\lambda} t\left|f'\left(\frac{\lambda-t}{\lambda+\mu}a+\frac{\mu+t}{\lambda+\mu}b\right)\right|dt \ \le (\lambda+\mu)^2$$
$$\left[|f'(b)|\int_{\frac{\mu}{\lambda+\mu}}^{1}\left(t-\frac{\mu}{\lambda+\mu}\right)h(t)dt+|f'(a)|\int_0^{\frac{\lambda}{\lambda+\mu}}\left(\frac{\lambda}{\lambda+\mu}-t\right)h(t)dt\right]$$

substitute the inequalities against I_5 and I_6 in (11)

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le (b-a)\left[\ |f'(a)|\times\right.$$
$$\left\{\int_0^{\frac{\lambda}{\lambda+\mu}}\left(\frac{\lambda}{\lambda+\mu}-t\right)h(t)dt+\int_{\frac{\lambda}{\lambda+\mu}}^{1}\left(t-\frac{\lambda}{\lambda+\mu}\right)h(t)dt\right\}+$$
$$\left.|f'(b)|\left\{\int_0^{\frac{\mu}{\lambda+\mu}}\left(\frac{\mu}{\lambda+\mu}-t\right)h(t)dt+\int_{\frac{\mu}{\lambda+\mu}}^{1}\left(t-\frac{\mu}{\lambda+\mu}\right)h(t)dt\right\}\right]$$

Which completes the proof. □

Corollary 1. *For mid-point i.e.,* $\lambda=\mu$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\ dx\right| \le (b-a)\left[|f'(a)|+|f'(b)|\right]\int_0^1\left|\frac{1}{2}-t\right|h(t)\ dt$$

Remark 3. *For mid-point and convex function, we get inequality* (2).

Theorem 7. *Let $|f'|^q$ be h-convex function and the assumptions of Lemma 2 hold, then*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(\lambda+\mu)^{-1}}{\{(p+1)(\lambda+\mu)\}^{1/p}}\Bigg[\mu^{1+1/p}\times$$
$$\left(\int_0^{\frac{\mu}{\lambda+\mu}}\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}+$$
$$\lambda^{1+1/p}\left(\int_{\frac{\mu}{\lambda+\mu}}^1\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}\Bigg] \tag{12}$$

Proof. Consider

$$I_7=I_5=\int_0^\mu|-t|\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt=\int_0^\mu t\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|dt$$

By Hölder's inequality

$$I_7\leq\left(\int_0^\mu t^p dt\right)^{1/p}\int_0^\mu\left(\left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q dt\right)^{1/q}$$

$$I_7\leq\frac{\mu^{1+1/p}(\lambda+\mu)^{1/q}}{(p+1)^{1/p}}\left(|f'(a)|^q\int_0^{\frac{\mu}{\lambda+\mu}}h(1-t)dt+|f'(b)|^q\int_0^{\frac{\mu}{\lambda+\mu}}h(t)dt\right)^{1/q}$$

Analogously

$$I_8=I_6\leq\frac{\lambda^{1+1/p}(\lambda+\mu)^{1/q}}{(p+1)^{1/p}}\left(|f'(a)|^q\int_0^{\frac{\lambda}{\lambda+\mu}}h(t)dt+|f'(b)|^q\int_0^{\frac{\lambda}{\lambda+\mu}}h(1-t)dt\right)^{1/q}$$

Putting the inequalities against I_7 and I_8 in (11), we get (12). ∎

Corollary 2. *For mid-point i.e.,* $\lambda=\mu$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\ dx\right|\leq\frac{(b-a)}{2^{1+1/p}(1+p)^{1/p}}\Bigg[$$
$$\left(\int_0^{1/2}\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}\,dt\right)^{1/q}+$$
$$\left(\int_{1/2}^1\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}\,dt\right)^{1/q}\Bigg]$$

Corollary 3. *For mid-point and convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\ dx\right| \leq \frac{(b-a)}{4(1+p)^{1/p}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|}{4}\right)^{1/q}\right.$$
$$\left.+\left(\frac{|f'(a)|^q+3|f'(b)|^q|}{4}\right)^{1/q}\right]$$

Theorem 8. *Let $|f'|^q$ be h-convex function and the assumptions of Lemma 2 hold, then*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{[(\lambda+\mu)^2]^{1/p}}\left[\left(\frac{\mu^2}{2}\right)^{1/p}\times\right.$$
$$\left(\int_0^{\frac{\mu}{\lambda+\mu}}\left(\frac{\mu}{\lambda+\mu}-t\right)\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}$$
$$+\left(\frac{\lambda^2}{2}\right)^{1/p}\left(\int_{\frac{\mu}{\lambda+\mu}}^1\left(t-\frac{\mu}{\lambda+\mu}\right)\times\right.$$
$$\left.\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}\Bigg] \quad (13)$$

Corollary 4. *Under the assumptions of above Theorem 8*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)\ dx\right| \leq \frac{(b-a)}{2^{1/p}}\left[\left(\int_0^{\frac{\mu}{\lambda+\mu}}\left(\frac{\mu}{\lambda+\mu}-t\right)\right.\right.$$
$$\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt)^{1/q}+$$
$$\left.\left(\int_{\frac{\mu}{\lambda+\mu}}^1\left(t-\frac{\mu}{\lambda+\mu}\right)\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}\right]$$

Proof. This consequence of Theorem 8 because $\frac{\lambda^2}{(\lambda+\mu)^2}<1$ and $\frac{\mu^2}{(\lambda+\mu)^2}<1$. ∎

Corollary 5. *For mid-point i.e.,* $\lambda=\mu$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\ dx\right| \leq \frac{(b-a)}{2^{1/p}}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)\right.\right.$$
$$\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt)^{1/q}+$$
$$\left.\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}\right]$$

Corollary 6. *For mid-point and convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\ dx\right| \leq \frac{(b-a)}{2}\left[\left(\frac{5|f'(a)|^q+\ |f'(b)|^q}{24}\right)^{1/q}+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{24}\right)^{1/q}\right]$$

Theorem 9. *Let $|f'|^q$ be concave function and the assumptions of Lemma 2 hold, then*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{(\lambda+\mu)^2(1+p)^{1/p}}\left[\mu^2 \left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{2(\lambda+\mu)}a+\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|+ \lambda^2\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}b+\frac{\lambda^2}{2(\lambda+\mu)}a-\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|\right] \quad (14)$$

Proof. Considering

$$I_9 = I_5 \leq \frac{\mu^{1+1/p}}{(p+1)^{1/p}}\left(\int_0^\mu \left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q\right)^{1/q}$$

Since $|f'|^q$ is concave on $[a,b]$, By integral Jensen inequality

$$\int_0^\mu \left|f'\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)\right|^q\ dt \leq \left(\int_0^\mu t^0\ dt\right) \left|f'\left(\frac{1}{\int_0^\mu t^0\ dt}\int_0^\mu\left(\frac{\lambda+t}{\lambda+\mu}a+\frac{\mu-t}{\lambda+\mu}b\right)dt\right)\right|^q = \mu\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{2(\lambda+\mu)}a+\frac{\mu^2}{2(\lambda+\mu)b}\right)\right)\right|^q$$

Therefore

$$I_9 \leq \frac{\mu^2}{(p+1)^{1/p}}\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{\lambda+\mu}a+\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|$$

Analogously

$$I_{10} = I_6 \leq \frac{\lambda^2}{(p+1)^{1/p}}\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}b+\frac{\lambda^2}{2(\lambda+\mu)}a-\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|$$

On putting inequalities against I_9 and I_{10} in (11) we get (14). ∎

Corollary 7. *Let $|f'|^q$ be concave function and the assumptions of Lemma 2 hold, then*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{(1+p)^{1/p}}$$
$$\left[\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}a-\frac{\lambda^2}{2(\lambda+\mu)}a+\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|\right.$$
$$\left.+\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}b+\frac{\lambda^2}{2(\lambda+\mu)}a-\frac{\mu^2}{2(\lambda+\mu)}b\right)\right)\right|\right] \quad (15)$$

Proof. This consequence of Theorem 9 because $\frac{\lambda^2}{(\lambda+\mu)^2}<1$ and $\frac{\mu^2}{(\lambda+\mu)^2}<1$. □

Remark 4. *For $\lambda=\mu$ in (14) , we get (5).*

1.2. Left Estimation of Hadamard- ineqality.

Lemma 3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be differentiable function on I^o where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$ and $a<\frac{\mu a+\lambda b}{\lambda+\mu}<b$, for $\lambda,\mu\in[0,\infty)$ with $\lambda+\mu\neq 0$,then*

$$\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)=\frac{b-a}{(\lambda+\mu)^2}\left[\int_0^\lambda (t-\lambda)f'\left(\frac{\mu+t}{\lambda+\mu}a+\frac{\lambda-t}{\lambda+\mu}b\right)dt+\right.$$
$$\left.\int_0^\mu (\mu-t)f'\left(\frac{\mu-t}{\lambda+\mu}a+\frac{\lambda+t}{\lambda+\mu}b\right)dt\right]. \quad (16)$$

Proof. Let

$$I_3=\frac{b-a}{(\lambda+\mu)^2}\int_0^\lambda (t-\lambda)f'\left(\frac{\mu+t}{\lambda+\mu}a+\frac{\lambda-t}{\lambda+\mu}b\right)dt$$

Integrating by parts,

$$I_3=\frac{t-\lambda}{\lambda+\mu}f\left(\frac{\mu+t}{\lambda+\mu}a+\frac{\lambda-t}{\lambda+\mu}b\right)\Big|_0^\lambda-\frac{1}{\lambda+\mu}\int_0^\lambda f\left(\frac{\mu+t}{\lambda+\mu}a+\frac{\lambda-t}{\lambda+\mu}b\right)dt$$
$$=-\frac{\lambda}{\lambda+\mu}f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)+\frac{1}{b-a}\int_a^{\frac{\mu a+\lambda b}{\lambda+\mu}} f(x)dx \quad (17)$$

Similarly,

$$I_4=-\frac{\mu}{\lambda+\mu}f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)+\frac{1}{b-a}\int_{\frac{\mu a+\lambda b}{\lambda+\mu}}^b f(x)dx \quad (18)$$

Identity (16) is obtained by adding (17) and (18). □

Remark 5. *To remain in $[a,b]$, we choose $\lambda+\mu=b-a$.*

Theorem 10. *Let $|f'|$ be h-convex function, then under the assumption of Lemma 2 we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \leq (b-a)\left[|f'(a)|\left\{\int_0^{\frac{\mu}{\lambda+\mu}} t\ h(t)dt + \int_{\frac{\mu}{\lambda+\mu}}^1 (1-t)h(t)dt\right\} + |f'(b)|\left\{\int_0^{\frac{\lambda}{\lambda+\mu}} t\ h(t)dt + \int_{\frac{\lambda}{\lambda+\mu}}^1 (1-t)h(t)dt\right\}\right] \tag{19}$$

Corollary 8. *For mid-point i.e., $\lambda = \mu$*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left[\int_0^{\frac{1}{2}} t\ h(t)\ dt + \int_{\frac{1}{2}}^1 (1-t)\ h(t)\ dt\right] (|f'(a)|+|f'(b)|)$$

Remark 6. *For mid-point and convex function, we get inequality* (3).

Theorem 11. *Let $|f'|^q$ be h-convex function and the assumptions of Lemma 2 hold, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \leq \frac{(b-a)(\lambda+\mu)^{-1}}{\{(p+1)(\lambda+\mu)\}^{1/p}}\left[\mu^{1+1/p}\left(\int_0^{\frac{\lambda}{\lambda+\mu}} \{|f'(a)|^q h(1-t) + |f'(b)|^q h(t)\}\, dt\right)^{1/q} + \lambda^{1+1/p}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1 \{|f'(a)|^q h(1-t) + |f'(b)|^q h(t)\}\, dt\right)^{1/q}\right] \tag{20}$$

Corollary 9. *For mid-point i.e., $\lambda = \mu$*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{2^{1+1/p}(p+1)^{1/p}}\left[\left(\int_0^{\frac{1}{2}} \{|f'(a)|^q h(1-t) + |f'(b)|^q h(t)\}\ dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 \{|f'(a)|^q h(1-t) + |f'(b)|^q h(t)\}\ dt\right)^{1/q}\right]$$

Remark 7. *For mid-point and convex function, we get inequality* (4).

Theorem 12. *Let $|f'|^q$ be h-convex function and the assumptions of Lemma 2 hold, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right)\right| \leq \frac{(b-a)}{\{(\lambda+\mu)^2\}^{1/p}}\left[\left(\frac{\lambda^2}{2}\right)^{1/p}\left(\int_0^{\frac{\lambda}{\lambda+\mu}} t\times\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q} + \left(\frac{\mu^2}{2}\right)^{1/p}\left(\int_{\frac{\lambda}{\lambda+\mu}}^1 (1-t)\{|f'(a)|^q h(1-t)+\ |f'(b)|^q h(t)\}\,dt\right)^{1/q}\right] \quad (21)$$

Corollary 10. *For mid-point i.e., $\lambda=\mu$*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8^{1/p}}\left[\left(\int_0^{\frac{1}{2}} t\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}\ dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)\{|f'(a)|^q h(1-t)+|f'(b)|^q h(t)\}\ dt\right)^{1/q}\right]$$

Corollary 11. *For mid-point and convex function*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}\left[\left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{1/q} + \left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{1/q}\right]$$

Theorem 13. *Let $|f'|^q$ be concave function and the assumptions of Lemma 2 hold, then*

$$\left|f\left(\frac{\mu a+\lambda b}{\lambda+\mu}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{(\lambda+\mu)^2(1+p)^{1/p}}\left[\lambda^2\left|f'\left(\frac{1}{\lambda}\left(\frac{\lambda+\mu}{2}a - \frac{\mu^2}{2(\lambda+\mu)}a + \frac{\lambda^2}{2(\lambda+\mu)}b\right)\right)\right| + \mu^2\left|f'\left(\frac{1}{\mu}\left(\frac{\lambda+\mu}{2}b + \frac{\mu^2}{2(\lambda+\mu)}a - \frac{\lambda^2}{2(\lambda+\mu)}b\right)\right)\right|\right] \quad (22)$$

Remark 8. *For $\lambda=\mu$ in (22) , we get (6).*

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