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HERMITE-HADAMARD INEQUALITY FOR FUNCTIONS WHOSE DERIVATIVES ABSOLUTE VALUES ARE PREINVEX

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Abstract. In this paper we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for preinvex functions. Then, a generalization to functions of several variables on invex subsets of $\mathbb{R}^n$ is introduced.

Keywords: Hermite-Hadamard inequality, invex sets, preinvex functions

1. Introduction and preliminary

Let $I = [c, d]$ be an interval on the real line $\mathbb{R}$, let $f : I \to \mathbb{R}$ be a convex function and let $a, b \in [c, d], a < b$. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [4, 5, 6, 7]).

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

Dragomir and Agarwal [8] used the formula,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{2}$$

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to prove the following results.

Theorem 1.1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds true*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{3}$$

Theorem 1.2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the following inequality holds true*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \cdot \left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right]^{\frac{p-1}{p}}. \tag{4}$$

Ion in [10] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some quasi-convex functions are involved.

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [11]. Weir and Mond [15] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. In [2, 3] Aslam Noor introduced the Hermite-Hadamard inequality for preinvex and log-preinvex functions.

In This paper we generalize the results in [10] for functions whose first derivatives absolute values are preinvex. Also some results for functions whose second derivatives absolute values are preinvex will be given. Now, we recall some notions in invexity analysis which will be used throughout the paper (see [14, 16] and references therein).

Definition 1.1. *A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$y + t\eta(x, y) \in S. \tag{5}$$

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [14]). Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$ the $\eta-$path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y, x) : t \in [0, 1]\}.$$

Definition 1.2. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Then, the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y). \tag{6}$$

Every convex function is a preinvex with respect to the map $\eta(x,y) = x-y$ but the converse does not holds. For properties and applications of preinvex functions see [16, 13] and references therein.

The organization of the paper is as follows:
In section 2 some generalizations of Hermite-Hadamard type inequality for first order differentiable functions are given. Section 3 is devoted to a generalization to several variable preinvex functions. Hermite-Hadamard type inequality for second order differentiable functions are studied in section4.

2. First order differentiable functions

In this section we introduce some generalizations of Hermite-Hadamard type inequality for functions whose first derivatives absolute values are preinvex. We begin with the following lemma which is a generalization of Lemma 2.1 in [8] to invex setting.

Lemma 2.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\theta(a,b) \neq 0$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If f' is integrable on the $\theta-$path P_{bc}, $c = b + \theta(a,b)$ then, the following inequality holds*

$$\begin{aligned} &-\frac{f(b)+f(b+\theta(a,b))}{2} + \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx \\ &= \frac{\theta(a,b)}{2}\int_0^1 (1-2t)f'(b+t\theta(a,b))dt. \end{aligned} \tag{7}$$

Proof. Suppose that $a, b \in A$. Since A is an invex set with respect to θ, for every $t \in [0,1]$ we have $b + t\theta(a,b) \in A$. Integrating by parts implies that

$$\begin{aligned} &\int_0^1 (1-2t)f'(b+t\theta(a,b))dt \\ &= \left[\frac{(1-2t)f(b+t\theta(a,b))}{\theta(a,b)}\right]_0^1 + \frac{2}{\theta(a,b)}\int_0^1 f(b+t\theta(a,b))dt \\ &= -\frac{f(b)+f(b+\theta(a,b))}{\theta(a,b)} + \frac{2}{(\theta(a,b))^2}\int_b^{b+\theta(a,b)} f(x)dx, \end{aligned} \tag{8}$$

which completes the proof. □

Theorem 2.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $\mid f' \mid$ is preinvex on A then, for every $a, b \in A$ with $\theta(a,b) \neq 0$ the following inequality holds*

$$\begin{aligned} &\left|\frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx\right| \\ &\leq \frac{|\theta(a,b)|}{8}\left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{9}$$

Proof. Suppose that $a, b \in A$. Since A is an invex set with respect to θ, for every $t \in [0,1]$ we have $b + t\theta(a,b) \in A$. By preinvexity of $|f'|$ and Lemma 2.1 we get

$$\begin{aligned}
&\left| \frac{f(b) + f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)dx \right| \\
&= \left| \frac{\theta(a,b)}{2} \int_0^1 (1-2t) f'(b + t\theta(a,b))dt \right| \\
&\leq \frac{|\theta(a,b)|}{2} \left\{ \int_0^1 |1-2t| \left(t|f'(a)| + (1-t)|f'(b)| \right) dt \right\} \\
&= \frac{|\theta(a,b)|}{8} \left\{ \left|f'(a)\right| + \left|f'(b)\right| \right\},
\end{aligned} \tag{10}$$

where,

$$\int_0^1 |1-2t| \, (1-t)dt = \int_0^1 |1-2t| \, tdt = \frac{1}{4}.$$

□

Now, we give an example of an invex set with respect to an θ which is satisfies the conditions of Theorem 2.1.

Example 2.1. Suppose that $K := (-3,-1) \cup (1,4)$ and the function $\theta : K \times K \to \mathbb{R}$ is defined by

$$\theta(x,y) = \begin{cases} x - y & x > 0, \ y > 0, \\ x - y & x < 0, \ y < 0, \\ 3 - y & x < 0, \ y > 0, \\ -2 - y & x > 0, \ y < 0. \end{cases}$$

Clearly K is an open invex set with respect to θ. Suppose that $a \in (-3,-1)$ and $b \in (1,4), \ b \neq 3$ hence, $\theta(a,b) = 3 - b \neq 0$. Now,

$$P_{bc} = [b, 3], \ b < 3,$$

and

$$P_{bc} = [3, b], \ b > 3,$$

where $c = b + \theta(a,b)$.

Another similar result is embodied in the following theorem.

Theorem 2.2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. Assume that $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is preinvex on A then, for every $a, b \in A$*

with $\theta(a,b)\neq 0$ the following inequality holds

$$\begin{aligned}&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right|\\&\le\frac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[|f'(a)|^{p/p-1}+|f'(b)|^{p/p-1}\}\right]^{\frac{p-1}{p}}.\end{aligned}\tag{11}$$

Proof. Suppose that $a,b\in A$. By assumption, Hölder's inequality and the proof of Theorem 4.1 we have

$$\begin{aligned}&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right|\\&\le\frac{|\theta(a,b)|}{2}\int_0^1|1-2t|\left|f'(b+t\theta(a,b))\right|dt\\&\le\frac{|\theta(a,b)|}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{1/p}\left(\int_0^1\left|f'(b+t\theta(a,b))\right|^q dt\right)^{1/q}\\&=\frac{|\theta(a,b)|}{2(p+1)^{1/p}}\left(\int_0^1\left|f'(b+t\theta(a,b))\right|^q dt\right)^{1/q}\\&\le\frac{|\theta(a,b)|}{2(p+1)^{1/p}}\left(\int_0^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{1/q}\\&=\frac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[\left|f'(a)\right|^q+\left|f'(a)\right|^q\right]^{1/q},\end{aligned}\tag{12}$$

where $q:=p/(p-1)$. □

Note that if $A=[a,b]$ and $\theta(x,y)=x-y$ for every $x,y\in A$ then, we can deduce Theorems 1.1 and 1.2, from Theorems 2.1 and 2.2, respectively.

3. An extension to several variables functions

The aim of this section is to extend the Proposition 1 in [10] and Theorem 2.2 to functions of several variables defined on invex subsets of $\mathbb{R}^n$.

The mapping $\eta:S\times S\to\mathbb{R}^n$ is said to be satisfies the condition C if for every $x,y\in S$ and $t\in[0,1]$,

$$\eta(y,y+t\eta(x,y))=-t\eta(x,y),$$
$$\eta(x,y+t\eta(x,y))=(1-t)\eta(x,y).$$

Note that, in Example 2.1, θ satisfies the condition C.

For every $x,y\in S$ and every $t_1,t_2\in[0,1]$ from condition C we have

$$\eta(y+t_2\eta(x,y),y+t_1\eta(x,y))=(t_2-t_1)\eta(x,y),\tag{13}$$

see [16] for details.

Proposition 3.1. *Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$ and $f : S \to \mathbb{R}$ is a function. Suppose that η satisfies condition C on S. Then, for every $x, y \in S$ the function f is preinvex with respect to η on $\eta-$path P_{xv} if and only if the function $\varphi : [0,1] \to \mathbb{R}$ defined by*

$$\varphi(t) := f(x + t\eta(y,x)),$$

is convex on $[0,1]$.

Proof. Suppose that φ is convex on $[0,1]$ and $z_1 := x + t_1\eta(y,x) \in P_{xv}$, $z_2 := x + t_2\eta(y,x) \in P_{xv}$. Fix $\lambda \in [0,1]$. Since η satisfies condition C, by (13) we have

$$\begin{aligned} f(z_1 + \lambda\eta(z_2, z_1)) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\ &= \varphi((1-\lambda)t_1 + \lambda t_2) \\ &\leq (1-\lambda)\varphi(t_1) + \lambda\varphi(t_2) \\ &= (1-\lambda)f(z_1) + \lambda f(z_2). \end{aligned} \tag{14}$$

Hence, f is preinvex with respect to η on $\eta-$path P_{xv}.

Conversely, let $x, y \in S$ and the function f be preinvex with respect to η on $\eta-$path P_{xv}. Suppose that $t_1, t_2 \in [0,1]$. Then, for every $\lambda \in [0,1]$ we have

$$\begin{aligned} \varphi((1-\lambda)t_1 + \lambda t_2) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y,x)) \\ &= f\left(x + t_1\eta(y,x) + \lambda\eta(x + t_2\eta(y,x), x + t_1\eta(y,x)\right) \\ &\leq \lambda f(x + t_2\eta(y,x)) + (1-\lambda)f(x + t_1\eta(y,x)) \\ &= \lambda\varphi(t_2) + (1-\lambda)\varphi(t_1). \end{aligned} \tag{15}$$

Therefore, φ is quasiconvex on $[0,1]$. □

The following Theorem is a generalization of Proposition 1 in [10].

Theorem 3.1. *Let $S \subseteq \mathbb{R}^n$ be an open invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. Assume that η satisfies condition C. Suppose that for every $x, y \in S$ the function $f : S \to \mathbb{R}^+$ is preinvex with respect to η on $\eta-$path P_{xv}. Then, for every $a, b \in (0,1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned} &\left|\frac{1}{2}\int_0^a f(x + s\eta(y,x))ds + \frac{1}{2}\int_0^b f(x + s\eta(y,x))ds \right. \\ &\left. - \frac{1}{b-a}\int_a^b \left(\int_0^s f(x + t\eta(y,x))dt\right) ds\right| \\ &\leq \frac{b-a}{8}\{f(x + a\eta(y,x)) + f(x + b\eta(y,x))\}. \end{aligned} \tag{16}$$

Proof. Let $x, y \in S$ and $a, b \in (0,1)$ with $a < b$. Since f is preinvex with respect to η on $\eta-$path P_{xv} by Proposition 3.1 the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f(x + t\eta(y,x)),$$

is convex on $[0,1]$. Now, we define the function $\phi : [0,1] \to \mathbb{R}^+$ as follows

$$\phi(t) := \int_0^t \varphi(s)ds = \int_0^t f(x + s\eta(y,x))ds.$$

Obviously for every $t \in (0,1)$ we have

$$\phi'(t) = \varphi(t) = f(x + t\eta(y,x)) \geq 0,$$

hence, $|\phi'(t)| = \phi'(t)$. Applying Theorem 1.1 to the function ϕ implies that

$$\left|\frac{\phi(a)+\phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(s)ds\right| \leq \frac{(b-a)\left(\phi'(a)+\phi'(b)\right)}{8},$$

and we deduce that (16) holds. □

4. Second order differentiable functions

In this section we introduce some generalizations of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are preinvex. We begin with the following lemma (see Lemma 1 in [1] and Lemma 4 in [9]).

Lemma 4.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\theta(a,b) \neq 0$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If f'' is integrable on the $\theta-$path P_{bc}, $c = b + \theta(a,b)$ then, the following inequality holds*

$$\begin{aligned}\frac{f(b)+f(b+\theta(a,b))}{2} &- \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx \\ &= \frac{\theta(a,b)^2}{2}\int_0^1 t(1-t)f''(b+t\theta(a,b))dt.\end{aligned} \tag{17}$$

Proof. Suppose that $a, b \in A$. Since A is an invex set with respect to θ, for every $t \in [0,1]$ we have $b + t\theta(a,b) \in A$. Integrating by parts implies that

$$\begin{aligned}&\int_0^1 t(1-t)f''(b+t\theta(a,b))dt \\ &= \left[\frac{t(1-t)f'(b+t\theta(a,b))}{\theta(a,b)}\right]_0^1 - \frac{1}{\theta(a,b)}\int_0^1 (1-2t)f'(b+t\theta(a,b))dt \\ &= \frac{f(b)+f(b+\theta(a,b))}{\theta(a,b)^2} - \frac{2}{(\theta(a,b))^3}\int_b^{b+\theta(a,b)} f(x)dx,\end{aligned} \tag{18}$$

which completes the proof. □

Theorem 4.1. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and $\theta(a,b) \neq 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is a twice* *differentiable function on A. If $\mid f'' \mid$ is preinvex on A and f'' is integrable on the $\theta-$path P_{bc}, $c = b + \theta(a,b)$ then, the following inequality holds*

$$\begin{aligned}&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right| \\ &\leq \frac{\theta(a,b)^2}{24}\left(|f''(a)|+|f''(b)|\right).\end{aligned} \tag{19}$$

Proof. Suppose that $a, b \in A$. Since A is an invex set with respect to θ, for every $t \in [0,1]$ we have $b + t\theta(a,b) \in A$. By preinvexity of $|f''|$ and Lemma 4.1 we get

$$\begin{aligned}&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right| \\ &=\left|\frac{\theta(a,b)^2}{2}\int_0^1 t(1-t)f''(b+t\theta(a,b))dt\right| \\ &\leq \frac{\theta(a,b)^2}{2}\left[\int_0^1 t(1-t)\left(t|f''(a)|+(1-t)|f''(b)|\right)dt\right] \\ &=\frac{\theta(a,b)^2}{24}\left[|f''(a)|+|f''(b)|\right],\end{aligned} \tag{20}$$

which completes the proof. □

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following theorem.

Theorem 4.2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A \times A \to \mathbb{R}$ and $\theta(a,b) \neq 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is a twice differentiable function on A and $\mid f'' \mid^{\frac{p}{p-1}}$ is preinvex on A, for $p > 1$. If f'' is integrable on the $\theta-$path $P_{bc}, c = b + \theta(a,b)$ then, the following inequality holds*

$$\begin{aligned}&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right| \\ &\leq \frac{\theta(a,b)^2}{16}\left(\sqrt{\pi}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{21}$$

Proof. By preinvexity of $|f''|$, Lemma 4.1 and using the well known Hölder integral inequality, we get

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right| \\
&\leq \frac{\theta(a,b)^2}{2}\int_0^1 t(1-t)|f''(b+\theta(a,b))|dt \\
&\leq \frac{\theta(a,b)^2}{2}\left(\int_0^1 (t-t^2)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(b+\theta(a,b))|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{\theta(a,b)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\int_0^1 (t|f''(a)|^q+(1-t)|f''(b)|^q)dt\right)^{\frac{1}{q}} \\
&= \frac{\theta(a,b)^2}{16}\left(\sqrt{\pi}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

A more general inequality is given using Lemma 4.1, as follows:

Theorem 4.3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\theta : A\times A \to \mathbb{R}$ and $\theta(a,b) \neq 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is a twice differentiable function on A and $\mid f'' \mid^q$ is preinvex on A, for $q > 1$. If f'' is integrable on the $\theta-$path P_{bc}, $c = b+\theta(a,b)$ then, the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right| \\
&\leq \frac{\theta(a,b)^2}{12}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}}
\end{aligned} \tag{22}$$

Proof. By preinvexity of $|f''|$, Lemma 4.1 and using the well known weighted power mean inequality, we get

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)}f(x)dx\right|\\
&\leq \frac{\theta(a,b)^2}{2}\int_0^1 t(1-t)|f''(b+\theta(a,b))|dt\\
&\leq \frac{\theta(a,b)^2}{2}\left(\int_0^1 (t-t^2)dt\right)^{1-1/q}\left(\int_0^1 (t-t^2)|f''(b+\theta(a,b))|^q dt\right)^{1/q}\\
&\leq \frac{\theta(a,b)^2}{2}\left(\frac{1}{6}\right)^{1-1/q}\left(\int_0^1 (t-t^2)\left[t|f''(a)|^q+(1-t)|f''(b)|^q\right]dt\right)^{1/q}\\
&\leq \frac{\theta(a,b)^2}{2}\left(\frac{1}{6}\right)^{1-1/q}\left(\frac{1}{12}\left(|f''(a)|^q+|f''(b)|^q\right)\right)^{1/q}\\
&= \frac{\theta(a,b)^2}{12}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

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