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A COMPANION OF OSTROWSKI'S INEQUALITY FOR THE RIEMANN-STIELTJES INTEGRAL $\int_a^b f(t)\,du(t)$, WHERE f IS OF BOUNDED VARIATION AND u IS OF r-H-HÖLDER TYPE AND APPLICATIONS

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ABSTRACT. Some companions of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of bounded variation on $[a,b]$ and u is of r-H-Hölder type on $[a,b]$, are proved. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

1. Introduction

In [11], Dragomir has proved an Ostrowski inequality for the Riemann-Stieltjes integral, as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^r,\quad \forall x,y\in[a,b],$$

where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\left|f(x)(u(b)-u(a))-\int_a^b f(t)\,du(t)\right|\le H\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\cdot\bigvee_a^b(u) \tag{1.1}$$

for all $x\in[a,b]$, where, $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{2}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

In [12], Dragomir has proved the dual case as follows:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then we have the inequality*

$$\left|\left(u\left(b\right)-u\left(a\right)\right)f\left(x\right)-\int_a^b f\left(t\right)du\left(t\right)\right| \tag{1.2}$$
$$\leq H\left[\left(x-a\right)^r\cdot\bigvee_a^x\left(f\right)+\left(b-x\right)^r\cdot\bigvee_x^b\left(f\right)\right]$$
$$\leq H\begin{cases}\left[\left(x-a\right)^r+\left(b-x\right)^r\right]\left[\frac{1}{2}\bigvee_a^b\left(f\right)+\frac{1}{2}\left|\bigvee_a^x\left(f\right)-\bigvee_x^b\left(f\right)\right|\right]\\ \\ \left[\left(x-a\right)^{qr}+\left(b-x\right)^{qr}\right]^{1/q}\left[\left(\bigvee_a^x\left(f\right)\right)^p+\left(\bigvee_x^b\left(f\right)\right)^p\right]^{1/p}\\ \\ \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r\cdot\bigvee_a^b\left(f\right)\end{cases}$$

For other results concerning inequalities for Stieltjes integrals, see [4, 6, 7, 8, 9, 15, 17, 18, 20, 22, 23].

Motivated by [19], Dragomir in [14], established the following companion of the Ostrowski inequality for mappings of bounded variation.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{f\left(x\right)+f\left(a+b-x\right)}{2}-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\leq\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b\left(f\right), \tag{1.3}$$

for any $x\in[a,\frac{a+b}{2}]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

For recent results concerning the above companion of Ostrowski's inequality and other related results see [1, 2, 5, 14, 16, 21].

Recently, Alomari [3] has proved the following companion of Ostrowski's inequality for Riemann-Stieltjes integral holds.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a r-H–Hölder type mapping, where, $H>0$ and $r\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality*

$$\left|f\left(x\right)\left[u\left(\frac{a+b}{2}\right)-u\left(a\right)\right]+f\left(a+b-x\right)\left[u\left(b\right)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f\left(t\right)du\left(t\right)\right| \tag{1.4}$$
$$\leq H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b\left(u\right)$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. where, $\bigvee_a^b\left(u\right)$ denotes the total variation of u on $[a,b]$. Furthermore, the constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one, for all $r\in(0,1]$.

In this paper, we establish a companion of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is assumed to be of bounded variation on $[a,b]$ and u is of r-H-Hölder type on $[a,b]$, are given. Applications to the approximation problem of the Riemann-Stieltjes integral in terms of Riemann-Stieltjes sums are also pointed out.

2. The Results

The following companion of Ostrowski's inequality for Riemann-Stieltjes integral holds.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ be of r-H–Hölder type on $[a,b]$, $r\in(0,1]$. Then we have the inequality*

(2.1)
$$\left|\left(u\left(\frac{a+b}{2}\right)-u(a)\right)f(x)+\left(u(b)-u\left(\frac{a+b}{2}\right)\right)f(a+b-x)-\int_a^b f(t)\,du(t)\right|$$
$$\leq H\left[(x-a)^r\cdot\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)^r\cdot\bigvee_x^{a+b-x}(f)+(x-a)^r\cdot\bigvee_{a+b-x}^b(f)\right]$$
$$\leq H\begin{cases}\left[2(x-a)^r+\left(\frac{a+b}{2}-x\right)^r\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}\\[2ex] \left[2^q(x-a)^{qr}+\left(\frac{a+b}{2}-x\right)^{qr}\right]^{1/q}\\ \times\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^{a+b-x}(f)\right)^p+\left(\bigvee_{a+b-x}^b(f)\right)^p\right]^{1/p},\ \text{if } \frac{1}{p}+\frac{1}{q}=1,\ p>1.\\[2ex] \left[\frac{1}{4}(b-a)+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b(f)\end{cases}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$. where, $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

Proof. As u is continuous and f is of bounded variation on $[a,b]$, the following Riemann-Stieltjes integrals exist and, by the integration by parts formula, we can state that

$$\int_a^x(u(t)-u(a))\,df(t)=(u(x)-u(a))f(x)-\int_a^x f(t)\,du(t),$$

$$\int_x^{a+b-x}\left(u(t)-u\left(\frac{a+b}{2}\right)\right)df(t)$$
$$=\left(u(a+b-x)-u\left(\frac{a+b}{2}\right)\right)f(a+b-x)-\left(u(x)-u\left(\frac{a+b}{2}\right)\right)f(x)-\int_x^{a+b-x}f(t)\,du(t)$$

and

$$\int_{a+b-x}^b(u(t)-u(b))\,df(t)$$
$$=(u(b)-u(a+b-x))f(a+b-x)-\int_{a+b-x}^b f(t)\,du(t).$$

If we add the above three identities, we obtain

$$\left(u\left(\frac{a+b}{2}\right)-u\left(a\right)\right)f\left(x\right)+\left(u\left(b\right)-u\left(\frac{a+b}{2}\right)\right)f\left(a+b-x\right)-\int_a^b f\left(t\right)du\left(t\right)$$
$$=\int_a^x \left(u\left(t\right)-u\left(a\right)\right)df\left(t\right)+\int_x^{a+b-x}\left(u\left(t\right)-u\left(\frac{a+b}{2}\right)\right)df\left(t\right)+\int_{a+b-x}^b \left(u\left(t\right)-u\left(b\right)\right)df\left(t\right),$$

for all $x\in[a,\frac{a+b}{2}]$.

Now, using the properties of absolute value, we have:

(2.2)
$$\left|\left(u\left(\frac{a+b}{2}\right)-u\left(a\right)\right)f\left(x\right)+\left(u\left(b\right)-u\left(\frac{a+b}{2}\right)\right)f\left(a+b-x\right)-\int_a^b f\left(t\right)du\left(t\right)\right|$$
$$\le\left|\int_a^x \left(u\left(t\right)-u\left(a\right)\right)df\left(t\right)\right|+\left|\int_x^{a+b-x}\left(u\left(t\right)-u\left(\frac{a+b}{2}\right)\right)df\left(t\right)\right|$$
$$+\left|\int_{a+b-x}^b \left(u\left(t\right)-u\left(b\right)\right)df\left(t\right)\right|$$
$$\le\sup_{t\in[a,x]}\left|u\left(t\right)-u\left(a\right)\right|\cdot\bigvee_a^x\left(f\right)+\sup_{t\in[x,a+b-x]}\left|u\left(t\right)-u\left(\frac{a+b}{2}\right)\right|\cdot\bigvee_x^{a+b-x}\left(f\right)$$
$$+\sup_{t\in[a+b-x,b]}\left|u\left(t\right)-u\left(b\right)\right|\cdot\bigvee_{a+b-x}^b\left(f\right),$$

and for the last inequality we have used the well-known property if $p:[c,d]\to\mathbb{R}$ is continuous and $\nu:[c,d]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)d\nu(t)$ exists and the following inequality holds:

$$\left|\int_c^d p\left(t\right)d\nu\left(t\right)\right|\le\sup_{t\in[c,d]}\left|p\left(t\right)\right|\bigvee_c^d\left(\nu\right).$$

As u is of r-H-Hölder type on $[a,b]$, we can state that

$$\sup_{t\in[a,x]}\left|u\left(t\right)-u\left(a\right)\right|\le\sup_{t\in[a,x]}\left[H\left(t-a\right)^r\right]=H\left(x-a\right)^r,$$

$$\sup_{t\in[x,a+b-x]}\left|u\left(t\right)-u\left(\frac{a+b}{2}\right)\right|\le\sup_{t\in[x,a+b-x]}\left[H\left|t-\frac{a+b}{2}\right|^r\right]=H\left(\frac{a+b}{2}-x\right)^r,$$

and

$$\sup_{t\in[a+b-x,b]}\left|u\left(t\right)-u\left(b\right)\right|\le\sup_{t\in[a+b-x,b]}\left[H\left(b-t\right)^r\right]=H\left(x-a\right)^r.$$

Now, using (3.2), we have

$$\left|\left(u\left(\frac{a+b}{2}\right)-u\left(a\right)\right)f\left(x\right)+\left(u\left(b\right)-u\left(\frac{a+b}{2}\right)\right)f\left(a+b-x\right)-\int_a^b f\left(t\right)du\left(t\right)\right|$$
$$\leq H\left[\left(x-a\right)^r\cdot\bigvee_a^x\left(f\right)+\left(\frac{a+b}{2}-x\right)^r\cdot\bigvee_x^{a+b-x}\left(f\right)+\left(x-a\right)^r\cdot\bigvee_{a+b-x}^b\left(f\right)\right]:=M(x),$$

for all $x\in[a,\frac{a+b}{2}]$, and the first inequality in (2.1) is proved.

Now, we observe that

$$M(x)\leq H\max\left\{\bigvee_a^x\left(f\right),\bigvee_x^{a+b-x}\left(f\right),\bigvee_{a+b-x}^b\left(f\right)\right\}\left[\left(x-a\right)^r+\left(\frac{a+b}{2}-x\right)^r+\left(x-a\right)^r\right]$$
$$=H\max\left\{\bigvee_a^x\left(f\right),\bigvee_x^{a+b-x}\left(f\right),\bigvee_{a+b-x}^b\left(f\right)\right\}\left[2\left(x-a\right)^r+\left(\frac{a+b}{2}-x\right)^r\right].$$

and the first part of the second inequality is proved. Also, we may observe that

$$M(x)\leq H\max\left\{\left(x-a\right)^r,\left(\frac{a+b}{2}-x\right)^r\right\}\left[\bigvee_a^x\left(f\right)+\bigvee_x^{a+b-x}\left(f\right)+\bigvee_{a+b-x}^b\left(f\right)\right]$$
$$=H\left[\max\left\{\left(x-a\right),\left(\frac{a+b}{2}-x\right)\right\}\right]^r\cdot\bigvee_a^b\left(f\right)$$
$$=H\left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b\left(f\right).$$

and the third part of the second inequality is proved.

Using the elementary inequality of Hölder type,

$$0\leq\alpha\beta+\gamma\delta\leq\left(\alpha^p+\gamma^p\right)^{1/p}\left(\beta^q+\delta^q\right)^{1/q},\quad\alpha,\beta,\gamma,\delta\geq0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

we obtain

$$M\left(x\right)\leq H\left[2^q\left(x-a\right)^{qr}+\left(\frac{a+b}{2}-x\right)^{qr}\right]^{1/q}\left[\left(\bigvee_a^x\left(f\right)\right)^p+\left(\bigvee_x^{a+b-x}\left(f\right)\right)^p+\left(\bigvee_{a+b-x}^b\left(f\right)\right)^p\right]^{1/p},$$

and the second part of the second inequality is proved. □

The following inequalities are hold:

Corollary 1. *Let f as in Theorem 5. In (2.1) choose*

(1) *$x=a$, then we get the following trapezoid inequality*

$$\left|f\left(a\right)\left[u\left(\frac{a+b}{2}\right)-u\left(a\right)\right]+f\left(b\right)\left[u\left(b\right)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f\left(t\right)du\left(t\right)\right|\tag{2.3}$$
$$\leq H\left(\frac{b-a}{2}\right)^r\begin{cases}\bigvee_a^b\left(f\right)\\ 2^{1/p}\bigvee_a^b\left(f\right)\end{cases}.$$

(2) $x=\frac{a+b}{2}$, *then we get the following mid-point inequality*

$$\left|\left(u\left(b\right)-u\left(a\right)\right)f\left(\frac{a+b}{2}\right)-\int_a^b f\left(t\right)dt\right| \tag{2.4}$$
$$\leq H\left(\frac{b-a}{2}\right)^r\begin{cases}\left[\bigvee_a^b\left(f\right)+\left|\bigvee_a^{\frac{a+b}{2}}\left(f\right)-\bigvee_{\frac{a+b}{2}}^b\left(f\right)\right|\right]\\ 2\left[\left(\bigvee_a^{\frac{a+b}{2}}\left(f\right)\right)^p+\left(\bigvee_{\frac{a+b}{2}}^b\left(f\right)\right)^p\right]^{1/p}\\ \bigvee_a^b\left(f\right)\end{cases}$$

where, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

We may state the following Ostrowski type inequality:

Corollary 2. *Let* f *and* u *be as in Theorem 5. Additionally, if* f *is symmetric about the x-axis, i.e.,* $f\left(a+b-x\right)=f\left(x\right)$, *then we have*

$$\left|\left(u\left(b\right)-u\left(a\right)\right)f\left(x\right)-\int_a^b f\left(t\right)dt\right|$$
$$\leq H\left[\left(x-a\right)^r\cdot\bigvee_a^x\left(f\right)+\left(\frac{a+b}{2}-x\right)^r\cdot\bigvee_x^{a+b-x}\left(f\right)+\left(x-a\right)^r\cdot\bigvee_{a+b-x}^b\left(f\right)\right]$$
$$\leq H\begin{cases}\left[2\left(x-a\right)^r+\left(\frac{a+b}{2}-x\right)^r\right]\max\left\{\bigvee_a^x\left(f\right),\bigvee_x^{a+b-x}\left(f\right),\bigvee_{a+b-x}^b\left(f\right)\right\}\\ \left[2^q\left(x-a\right)^{qr}+\left(\frac{a+b}{2}-x\right)^{qr}\right]^{1/q}\\ \times\left[\left(\bigvee_a^x\left(f\right)\right)^p+\left(\bigvee_x^{a+b-x}\left(f\right)\right)^p+\left(\bigvee_{a+b-x}^b\left(f\right)\right)^p\right]^{1/p},\ if\ \frac{1}{p}+\frac{1}{q}=1,\ p>1.\\ \left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot\bigvee_a^b\left(f\right)\end{cases}$$

for all $x\in[a,\frac{a+b}{2}]$.

Corollary 3. *Let* f *as in Theorem 5, and* $u:[a,b]\to\mathbb{R}$ *be an L-Lipschitzian mapping on* $[a,b]$, *that is,*

$$\left|u\left(x\right)-u\left(y\right)\right|\leq L\left|x-y\right|,\quad\forall x,y\in[a,b],$$

where, $L > 0$ is fixed. Then, for all $x \in [a, \frac{a+b}{2}]$, we have the inequality

$$\left| f(x)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]+f(a+b-x)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,du(t)\right|$$
$$\leq L\left[(x-a)\cdot\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\cdot\bigvee_x^{a+b-x}(f)+(x-a)\cdot\bigvee_{a+b-x}^b(f)\right]$$
$$\leq L\begin{cases}(b-a)\cdot\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}\\[2ex] \left[2^q(x-a)^q+\left(\frac{a+b}{2}-x\right)^q\right]^{1/q}\\ \times\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^{a+b-x}(f)\right)^p+\left(\bigvee_{a+b-x}^b(f)\right)^p\right]^{1/p}, \ if\ \frac{1}{p}+\frac{1}{q}=1,\ p>1.\\[2ex] \left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]\cdot\bigvee_a^b(f)\end{cases}$$

Corollary 4. *In Theorem 5, if f is monotonic on $[a,b]$, and u is of r-H-Hölder type. Then we have the inequality*

(2.5)
$$\left|\left(u\left(\frac{a+b}{2}\right)-u(a)\right)f(x)+\left(u(b)-u\left(\frac{a+b}{2}\right)\right)f(a+b-x)-\int_a^b f(t)\,du(t)\right|$$
$$\leq H\left[(x-a)^r\cdot\left[|f(x)-f(a)|+|f(b)-f(a+b-x)|\right]+\left(\frac{a+b}{2}-x\right)^r\cdot|f(a+b-x)-f(x)|\right]$$
$$\leq H\begin{cases}\max\left\{|f(x)-f(a)|,|f(a+b-x)-f(x)|,|f(b)-f(a+b-x)|\right\}\\ \times\left[2(x-a)^r+\left(\frac{a+b}{2}-x\right)^r\right]\\[2ex] \left[2^q(x-a)^{qr}+\left(\frac{a+b}{2}-x\right)^{qr}\right]^{1/q}, \ if\ \frac{1}{p}+\frac{1}{q}=1,\ p>1.\\ \times\left[|f(x)-f(a)|^p+|f(a+b-x)-f(x)|^p+|f(b)-f(a+b-x)|^p\right]^{1/p},\\[2ex] \left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]^r\cdot|f(b)-f(a)|\end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Now, we point out some results for the Riemann integral of a product.

Corollary 5. *Let f be a mappings of bounded variation and $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Put $\|g\|_\infty := \sup_{t\in[a,b]} |g(t)|$ Then we have the inequality*

$$\left| f(x)\int_a^{\frac{a+b}{2}} g(s)ds + f(a+b-x)\int_{\frac{a+b}{2}}^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \|g\|_\infty \left[(x-a)\cdot \bigvee_a^x (f) + \left(\frac{a+b}{2}-x\right)\cdot \bigvee_x^{a+b-x}(f) + (x-a)\cdot \bigvee_{a+b-x}^b (f) \right]$$
$$\leq \|g\|_\infty \begin{cases} (b-a)\cdot \max\left\{\bigvee_a^x(f), \bigvee_x^{a+b-x}(f), \bigvee_{a+b-x}^b(f)\right\} \\ \left[2^q(x-a)^q + \left(\frac{a+b}{2}-x\right)^q\right]^{1/q} \\ \times\left[\left(\bigvee_a^x(f)\right)^p + \left(\bigvee_x^{a+b-x}(f)\right)^p + \left(\bigvee_{a+b-x}^b(f)\right)^p\right]^{1/p}, \; if \; \frac{1}{p}+\frac{1}{q}=1,\; p>1. \\ \left[\frac{b-a}{4} + \left|x-\frac{3a+b}{4}\right|\right]\cdot \bigvee_a^b(f) \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Proof. Define the mapping $u : [a,b] \to \mathbb{R}$, $u(t) = \int_a^t g(s)ds$. Then u is differentiable on (a,b) and $u'(t) = g(t)$. Therefore, u is L-Lipschitzian with the constant $L = \|g\|_\infty$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)\,g(t)\,dt,$$

then, by Corollary 3, we deduce the desired inequality. □

Remark 1. *In Corollary 5, if f is symmetric about the x-axis, i.e., $f(a+b-x) = f(x)$, then we have*

$$\left| f(x)\int_a^b g(s)ds - \int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \|g\|_\infty \left[(x-a)\cdot \bigvee_a^x (f) + \left(\frac{a+b}{2}-x\right)\cdot \bigvee_x^{a+b-x}(f) + (x-a)\cdot \bigvee_{a+b-x}^b (f) \right]$$
$$\leq \|g\|_\infty \begin{cases} (b-a)\cdot \max\left\{\bigvee_a^x(f), \bigvee_x^{a+b-x}(f), \bigvee_{a+b-x}^b(f)\right\} \\ \left[2^q(x-a)^q + \left(\frac{a+b}{2}-x\right)^q\right]^{1/q} \\ \times\left[\left(\bigvee_a^x(f)\right)^p + \left(\bigvee_x^{a+b-x}(f)\right)^p + \left(\bigvee_{a+b-x}^b(f)\right)^p\right]^{1/p}, \; if \; \frac{1}{p}+\frac{1}{q}=1,\; p>1. \\ \left[\frac{b-a}{4} + \left|x-\frac{3a+b}{4}\right|\right]\cdot \bigvee_a^b(f) \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Remark 2. *Similar results for f monotonic or f Lipschitzian with a constant $K > 0$ apply, but we omit the details.*

Corollary 6. *Let f and g be as in Corollary 5. Put*

$$\|g\|_p := \left(\int_a^b |g(s)|^p \, ds\right)^{1/p}, \ p > 1.$$

Then we have the inequality

$$\left| f(x) \int_a^{\frac{a+b}{2}} g(s)ds + f(a+b-x) \int_{\frac{a+b}{2}}^b g(s)\,ds - \int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \|g\|_p \left[(x-a)^{\frac{p-1}{p}} \cdot \bigvee_a^x (f) + \left(\frac{a+b}{2} - x\right)^{\frac{p-1}{p}} \cdot \bigvee_x^{a+b-x} (f) + (x-a)^{\frac{p-1}{p}} \cdot \bigvee_{a+b-x}^b (f) \right]$$
$$\leq \|g\|_p \begin{cases} \left[2(x-a)^{\frac{(p-1)}{p}} + \left(\frac{a+b}{2} - x\right)^{\frac{(p-1)}{p}} \right] \cdot \max\left\{ \bigvee_a^x(f), \bigvee_x^{a+b-x}(f), \bigvee_{a+b-x}^b(f) \right\} \\ \left[2^\alpha (x-a)^{\alpha\frac{(p-1)}{p}} + \left(\frac{a+b}{2} - x\right)^{\alpha\frac{(p-1)}{p}} \right]^{1/\alpha} \\ \times \left[\left(\bigvee_a^x(f)\right)^\beta + \left(\bigvee_x^{a+b-x}(f)\right)^\beta + \left(\bigvee_{a+b-x}^b(f)\right)^\beta \right]^{1/\beta}, \ if \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \ \alpha > 1. \\ \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{\frac{p-1}{p}} \cdot \bigvee_a^b(f) \end{cases}$$

for all $x \in [a, \frac{a+b}{2}]$.

Proof. Consider the mapping u as in the proof of Corollary 5. Then, by Hölder's integral inequality, we can state that

$$|u(t) - u(s)| \leq \left| \int_s^t g(z)\,dz \right| \leq |t-s|^{\frac{1}{q}} \left(\int_s^t |g(z)|^p \, dz \right)^{\frac{1}{p}} \leq |t-s|^{\frac{p-1}{p}} \|g\|_p,$$

for all $t, s \in [a,b]$, which shows that the mapping u is of r-H-Hölder type with $r = \frac{p-1}{p}$ and $H = \|g\|_p < \infty$.

The mapping u is differentiable on (a,b) and $u'(t) = g(t)$, therefore, by the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t)\,du(t) = \int_a^b f(t)\,g(t)\,dt,$$

then, by Corollary 5, we deduce the desired inequality. □

Remark 3. *Similar results for f monotonic or f Lipschitzian with a constant $K > 0$ apply, but we omit the details.*

3. An Approximation for the Riemann-Stieltjes Integral

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a division of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $(i = 0,1,2,\cdots,n-1)$ and $\nu(h) := \max\{h_i | i = 0,1,2,\ldots,n-1\}$. Define the general Riemann-Stieltjes sum

(3.1)

$$S(f,u,I_n,\xi) = \sum_{i=0}^{n-1} f(\xi_i)\left[u\left(\frac{x_i+x_{i+1}}{2}\right) - u(x_i)\right] + f(x_i+x_{i+1}-\xi_i)\left[u(x_{i+1}) - u\left(\frac{x_i+x_{i+1}}{2}\right)\right]$$

In the following, we establish some upper bounds for the error approximation of the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by its Riemann-Stieltjes sum $S(f,u,I_n,\xi)$ using the third part of the second inequality (2.1).

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ be of r-H–Hölder type on $[a,b]$. Then*

$$\int_a^b f(t)\,du(t) = S(f,u,I_n,\xi) + R(f,u,I_n,\xi)$$

where, $S(f,u,I_n,\xi)$ is given in (3.1) and the remainder $R(f,u,I_n,\xi)$ satisfies the bound

$$|R(f,u,I_n,\xi)| \le H\left[\frac{1}{4}\nu(h) + \max_{i=0,1,\ldots,n-1}\left|\xi_i - \frac{3x_i+x_{i+1}}{4}\right|\right]^r \cdot \bigvee_a^b(f) \le H\left[\frac{1}{2}\nu(h)\right]^r \cdot \bigvee_a^b(f) \tag{3.2}$$

Proof. Applying Theorem 5 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\left|f(\xi_i)\left[u\left(\frac{x_i+x_{i+1}}{2}\right) - u(x_i)\right] + f(x_i+x_{i+1}-\xi_i)\left[u(x_{i+1}) - u\left(\frac{x_i+x_{i+1}}{2}\right)\right] - \int_{x_i}^{x_{i+1}} f(t)\,du(t)\right| \le H\left[\frac{1}{4}h_i + \left|\xi_i - \frac{3x_i+x_{i+1}}{4}\right|\right]^r \cdot \bigvee_{x_i}^{x_{i+1}}(f),$$

for all $i \in \{0,1,2,\cdots,n-1\}$.

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\begin{aligned}
&\left|R\left(f,u,I_n,\xi\right)\right| \\
&= \sum_{i=0}^{n-1}\left|f\left(\xi_i\right)\left[u\left(\frac{x_i+x_{i+1}}{2}\right)-u\left(x_i\right)\right]+f\left(x_i+x_{i+1}-\xi_i\right)\left[u\left(x_{i+1}\right)-u\left(\frac{x_i+x_{i+1}}{2}\right)\right]\right. \\
&\qquad \left.-\int_{x_i}^{x_{i+1}} f\left(t\right)du\left(t\right)\right| \\
&\le H\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\cdot\bigvee_{x_i}^{x_{i+1}}\left(f\right) \\
&\le H\sup_{i=0,1,\dots,n-1}\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\cdot\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}\left(f\right).
\end{aligned}$$

However,

$$\sup_{i=0,1,\dots,n-1}\left[\frac{1}{4}h_i+\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r\le\left[\frac{1}{4}\nu\left(h\right)+\sup\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\right]^r,$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}\left(f\right)=\bigvee_a^b\left(f\right).$$

which completely proves the first inequality in (3.2).

For the second inequality, we observe that

$$\left|\xi_i-\frac{3x_i+x_{i+1}}{4}\right|\le\frac{1}{4}h_i$$

for all $i\in\{0,1,2,\cdots,n-1\}$. which completes the proof. □

Corollary 7. *In Theorem 6, additionally, if f is symmetric about the x-axis, then we have $S\left(f,u,I_n,\xi\right)$ reduced to be*

$$S\left(f,u,I_n,\xi\right)=\sum_{i=0}^{n-1}f\left(\xi_i\right)\left[u\left(x_{i+1}\right)-u\left(x_i\right)\right]. \tag{3.3}$$

Then

$$\int_a^b f\left(t\right)du\left(t\right)=S\left(f,u,I_n,\xi\right)+R\left(f,u,I_n,\xi\right)$$

where, $S\left(f,u,I_n,\xi\right)$ is given in (3.3) and the remainder $R\left(f,u,I_n,\xi\right)$ satisfies the bound in (3.2).

Remark 4. *One may use the remaining inequalities in (2.1), to obtain other bounds for $R\left(f,u,I_n,\xi\right)$. We omit shall left the details.*

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