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HERMITE-HADAMARD TYPE INEQUALITY FOR OPERATOR CONVEX AND OPERATOR PREINVEX FUNCTIONS

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Abstract. In this paper we establish some estimates of the right hand side of a Hermite- Hadamard type inequality in which some operator convex functions and operator preinvex functions of selfadjoint operators in Hilbert spaces are involved.

1. INTRODUCTION

The following inequality holds for any convex function f defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Dragomir and Agarwal in [3] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some convex functions are involved. The main results of [3] are given by the following theorems.

Theorem 1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.2}$$

Theorem 2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the*

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following inequality holds true

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}.\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [8].

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := f(1-t)x + ty; t \in [0, 1].$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0; 1] \to \mathbb{R}, g(x, y)(t) := f[(1-t)x + ty], t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function de.ned on a segment $[x, y] \in X$, we have the Hermite- Hadamard integral inequality (see [3, p.2], [4, p.2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)dt \leq \frac{f(x)+f(y)}{2} \tag{1.4}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0; 1] \to \mathbb{R}$.

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator convex functions and operator preinvex functions .

In order to do that we need the following preliminary definitions and results. Let A be a bounded self adjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [7, p.3]): For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(*i*) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(*ii*) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

(*iii*) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(*iv*) $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the continuous functional calculus for a bounded selfadjoint operator A. If A is a bounded selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$(P) \qquad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order in $B(H)$. A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$(OC) \qquad f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every bounded selfadjoint operators A and B in $B(H)$ whose spectra are contained in I.

First Dragomir in [6] has proved the Hermite-Hadamard type inequality for operator convex function by the following Theorem:

Theorem 3. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\begin{aligned}\left(f\left(\frac{A+B}{2}\right) \leq\right) \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] &\leq \int_0^1 f((1-t)A+tB)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right]\left(\leq \frac{f(A)+f(B)}{2}\right).\end{aligned}$$

2. OPERATOR CONVEX FUNCTIONS

We start with the following Theorem, which is a generalization of Theorem 2.1 in [2].

Theorem 4. *Let $f : I \to \mathbb{R}^+$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $a, b \in (0,1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned}(2.1) \qquad &\left|\frac{1}{2}\int_0^a f((1-s)A+sB)ds + \frac{1}{2}\int_0^b f((1-s)A+sB)ds \right. \\ &\left. -\frac{1}{b-a}\int_a^b \left(\int_0^t f((1-s)A+sB)ds\right)dt\right| \\ &\leq \frac{b-a}{8}[f((1-a)A+aB) + f((1-b)A+bB)].\end{aligned}$$

Proof. Suppose that H is the Hilbert space related to the function $f : I \to \mathbb{R}^+$ and A, B are any selfadjoint operators with spectra in I. For every $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$, we have

$$\langle [(1-t)A + tB]\, x, x\rangle = (1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle \in I,$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Bx, x\rangle \in Sp(B) \subseteq I$.

This and continuity f imply that the operator valued integral $\int_0^1 f((1-t)A + tB)dt$ exists. We define

$$(2.2) \qquad \varphi : [0,1] \to \mathbb{R}^+, \quad \varphi(t) := \left\langle \int_0^t f\left((1-s)A + sB\right) ds\ x, x\right\rangle.$$

Obviously for every $t \in (0,1)$ we have

$$\varphi'(t) = \langle f((1-t)A + tB)x, x\rangle \geq 0,$$

hence, $|\varphi'(t)| = \varphi'(t)$. Since f is operator convex, then for any $t_1, t_2 \in [0,1]$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ we have

$$\begin{aligned}
\varphi'(\lambda_1 t_1 + \lambda_2 t_2) &= \langle f([1-(\lambda_1 t_1 + \lambda_2 t_2)]A + [\lambda_1 t_1 + \lambda_2 t_2]B)x, x\rangle \\
&= \langle f(\lambda_1[(1-t_1)A + t_1 B] + \lambda_2[(1-t_2)A + t_2 B])x, x\rangle \\
&\leq \lambda_1 \langle f((1-t_1)A + t_1 B)x, x\rangle + \lambda_2 \langle f((1-t_2)A + t_2 B)x, x\rangle \qquad (2.3)\\
&= \lambda_1 \varphi'(t_1) + \lambda_2 \varphi'(t_2),
\end{aligned}$$

showing that φ' is a convex function on $[0,1]$. Applying Theorem 1 to the function φ implies that

$$\begin{aligned}
(2.4)\quad &\left| \frac{1}{2}\left\langle \int_0^a f((1-s)A + sB)ds\ x, x\right\rangle + \frac{1}{2}\left\langle \int_0^b f((1-s)A + sB)ds\ x, x\right\rangle \right. \\
&\left. -\frac{1}{b-a}\int_a^b \left\langle \int_0^t f((1-s)A + sB)ds\ x, x\right\rangle dt \right| \\
&\leq \frac{b-a}{8}\left[\langle f((1-a)A + aB)x, x\rangle + \langle f((1-b)A + bB)x, x\rangle\right].
\end{aligned}$$

We deduce the inequality (2.1). □

Theorem 5. *Let $f : I \to \mathbb{R}^+$ be a continuous function on the interval I. Assume $q \in \mathbb{R}$ with $q > 1$. If f^q is operator convex on I then for any selfadjoint operators A and B with spectra in I and $a, b \in (0,1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned}
(2.5)\quad &\left| \frac{1}{2}\int_0^a f^q((1-s)A + sB)ds + \frac{1}{2}\int_0^b f^q((1-s)A + sB)ds \right. \\
&\left. -\frac{1}{b-a}\int_a^b \left(\int_0^t f^q((1-s)A + sB)ds\right) dt \right| \\
&\leq \frac{b-a}{2(\frac{2q-1}{q-1})^{\frac{q-1}{q}}}\left[\frac{f^q((1-a)A + aB) + f^q((1-b)A + bB)}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Since the function $f^q : I \to \mathbb{R}^+$ is continuous, the operator valued integral $\int_0^1 f^q((1-t)A + tB)dt$ exists for any selfadjoint operators A and B with spectra in I. For $x \in H$ with $\|x\| = 1$ we define

$$(2.6)\qquad \varphi : [0,1] \to \mathbb{R}^+, \quad \varphi(t) := \left\langle \int_0^t f^q\left((1-s)A + sB\right) ds\ x, x\right\rangle.$$

Obviously for every $t \in (0,1)$ we have

$$\varphi'(t) = \langle f^q((1-t)A + tB)x, x\rangle \geq 0,$$

hence, $|\varphi'(t)| = \varphi'(t)$. Applying Theorem 2 to the function φ implies that

$$\begin{aligned}(2.7)\quad &\Bigg|\frac{1}{2}\left\langle\int_0^a f^q((1-s)A+sB)ds\ x,x\right\rangle+\frac{1}{2}\left\langle\int_0^b f^q((1-s)A+sB)ds\ x,x\right\rangle\\ &\qquad -\frac{1}{b-a}\int_a^b\left\langle\int_0^t f^q((1-s)A+sB)ds\ x,x\right\rangle dt\Bigg|\\ &\leq \frac{b-a}{2(\frac{2q-1}{q-1})^{\frac{q-1}{q}}}\left[\frac{\langle f^q((1-a)A+aB)x,x\rangle+\langle f^q((1-b)A+bB)x,x\rangle}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

We deduce the inequality (2.5). □

3. OPERATOR PREINVEX FUNCTIONS

Definition 1. *Let X be a real vector space, a set $S \subseteq X$ is said to be invex with respect to the map $\eta : S \times S \to X$, if for every $x, y \in S$ and $t \in [0,1]$,*

$$(3.1)\qquad y + t\eta(x,y) \in S.$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x-y$, but there exist invex sets which are not convex (see [1]).

Let $S \subseteq X$ be an invex set with respect to $\eta : S \times S \to X$. For every $x, y \in S$ the $\eta-$path P_{xv} joining the points x and $v := x + \eta(y,x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y,x) : t \in [0,1]\}.$$

The mapping η is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0,1]$,

$$\begin{aligned}(C)\qquad &\eta(y, y+t\eta(x,y)) = -t\eta(x,y),\\ &\eta(x, y+t\eta(x,y)) = (1-t)\eta(x,y).\end{aligned}$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition C we have

$$(3.2)\qquad \eta(y+t_2\eta(x,y), y+t_1\eta(x,y)) = (t_2-t_1)\eta(x,y),$$

see [10, 9] for details.

Let $\mathcal{A}$ be a C^*-algebra, denote by $\mathcal{A}_{sa}$ the set of all self adjoint elements in $\mathcal{A}$.

Definition 2. *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S\times S \to B(H)_{sa}$. Then, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to η on S, if for every $A, B \in S$ and $t \in [0,1]$,*

$$(3.3)\qquad f(A + t\eta(B,A)) \leq (1-t)tf(A) + tf(B).$$

in the operator order in $B(H)$.

Every operator convex function is an operator preinvex with respect to the map $\eta(A,B) = A - B$ but the converse does not holds.

Proposition 1. *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S\times S \to B(H)_{sa}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose that η satisfies condition C on S. Then for every $A, B \in S$ and $V = A + \eta(B,A)$ the function f is operator preinvex with respect to η on $\eta-$path P_{AV} if and only if the function $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ defined by*

$$\varphi_{x,A,B}(t) := \langle f(A + t\eta(B,A))x, x\rangle$$

is convex on $[0,1]$ *for every* $x \in H$ *with* $\|x\| = 1$.

Proof. Suppose that $x \in H$ with $\|x\| = 1$ and $\varphi_{x,A,B}$ is convex on $[0,1]$ and $C_1 := A + t_1\eta(B,A) \in P_{AV}, C_2 := A + t_2\eta(B,A) \in P_{AV}$. Fix $\lambda \in [0,1]$. By (**??**) we have

$$\begin{aligned}\langle f(C_1 + \lambda\eta(C_2, C_1))x, x\rangle &= \langle f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B,A))x, x\rangle \\ &= \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) \\ &\le (1-\lambda)\varphi_{x,A,B}(t_1) + \lambda\varphi_{x,A,B}(t_2) \\ &= (1-\lambda)\langle f(C_1)x, x\rangle + \lambda\langle f(C_2)x, x\rangle.\end{aligned} \tag{3.4}$$

Hence, f is operator preinvex with respect to η on $\eta-$path P_{AV}.

Conversely, let $A, B \in S$ and the function f be operator preinvex with respect to η on $\eta-$path P_{AV}. Suppose that $t_1, t_2 \in [0,1]$. Then, for every $\lambda \in [0,1]$ and $x \in H$ with $\|x\| = 1$ we have

$$\begin{aligned}\varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) &= \langle f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B,A))x, x\rangle \\ &= \langle f\left(A + t_1\eta(B,A) + \lambda\eta(A + t_2\eta(B,A), A + t_1\eta(B,A)\right)x, x\rangle \\ &\le \lambda\langle f(A + t_2\eta(B,A))x, x\rangle + (1-\lambda)\langle f(A + t_1\eta(B,A))x, x\rangle \\ &= \lambda\varphi_{x,A,B}(t_2) + (1-\lambda)\varphi_{x,A,B}(t_1).\end{aligned} \tag{3.5}$$

Therefore, $\varphi_{x,A,B}$ is convex on $[0,1]$. □

The following Theorem is a generalization of Theorem 3.1 in [2].

Theorem 6. *Let the function* $f : I \to \mathbb{R}^+$ *is continuous,* $S \subseteq B(H)_{sa}$ *be an open invex set with respect to* $\eta : S \times S \to B(H)_{sa}$ *and* η *satisfies condition* C*. If for every* $A, B \in S$ *and* $V = A + \eta(B,A)$ *the function* f *is operator preinvex with respect to* η *on* $\eta-$*path* P_{AV} *with spectra of* A *and spectra of* V *in* I*. Then, for every* $a, b \in (0,1)$ *with* $a < b$ *the following inequality holds,*

$$\begin{aligned}&\left|\frac{1}{2}\int_0^a f(A + s\eta(B,A))ds + \frac{1}{2}\int_0^b f(A + s\eta(B,A))ds\right. \\ &\qquad\left. - \frac{1}{b-a}\int_a^b \left(\int_0^s f(A + t\eta(B,A))dt\right) ds\right| \\ &\qquad\qquad \le \frac{b-a}{8}\{f(A + a\eta(B,A)) + f(A + b\eta(B,A))\}.\end{aligned} \tag{3.6}$$

Proof. Let $A, B \in S$ and $a, b \in (0,1)$ with $a < b$. We define the function $\varphi : [0,1] \to \mathbb{R}^+$ by

$$\varphi(t) := \left\langle \int_0^t f(A + s\eta(B,A))ds \; x, x \right\rangle.$$

Obviously for every $t \in (0,1)$ we have

$$\varphi'(t) = \langle f(A + t\eta(B,A))x, x\rangle \ge 0,$$

hence, $|\varphi'(t)| = \varphi'(t)$. Since f is operator preinvex with respect to η on $\eta-$path P_{AV}, by Proposition 1 the function φ' is convex. Applying Theorem 1 to the function φ implies that

$$\left|\frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(s)ds\right| \le \frac{(b-a)\left(\varphi'(a) + \varphi'(b)\right)}{8},$$

and we deduce that (3.6) holds. □

The similar to Theorem 6, we have the following Theorem for operator preinvex function f^q.

Theorem 7. *Let the function $f : I \to \mathbb{R}^+$ is continuous, $S \subseteq B(H)_{sa}$ be an open invex set with respect to $\eta : S \times S \to B(H)_{sa}$. Assume that η satisfies condition C and $q \in \mathbb{R}$ with $q > 1$. If for every $A, B \in S$ and $V = A + \eta(B, A)$ with spectra of A and spectra of V in I, the function f^q is operator preinvex with respect to η on $\eta-$path P_{AV} . Then, for every $a, b \in (0, 1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned}(3.7)\quad & \left| \frac{1}{2}\int_0^a f^q(A + s\eta(B, A))ds + \frac{1}{2}\int_0^b f^q(A + s\eta(B, A))ds \right.\\ & \left. - \frac{1}{b-a}\int_a^b \left(\int_0^s f^q(A + t\eta(B, A))dt\right) ds \right| \\ & \le \frac{b-a}{2(\frac{2q-1}{q-1})^{\frac{q-1}{q}}} \left[\frac{\langle f^q((1-a)A + aB)x, x\rangle + \langle f^q((1-b)A + bB)x, x\rangle}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

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