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# REFINEMENTS FOR MEAN-INEQUALITIES VIA THE STABILIZABILITY CONCEPT

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Abstract. Exploring the stabilizability concept, recently introduced by the author in [9], we give an approach for obtaining refinements of mean-inequalities in a general point of view. Our theoretical study will be illustrated by a lot of examples showing the generality of our approach and the interest of the stabilizability concept.

## 1. Introduction

Stability and stabilizability concepts for binary means have been recently introduced by the author in [9]. The aim of this paper is to show that the above concepts are useful tool from the theoretical point of view as well as for practical purposes. Let us first recall some basic notions about binary means that will be needed throughout the paper. We understand by mean a binary map $m$ between positive real numbers satisfying the following statements.
(i) $m(a,a)=a$, for all $a>0$;
(ii) $m(a,b)=m(b,a)$, for all $a,b>0$;
(iii) $m(ta,tb)=tm(a,b)$, for all $a,b,t>0$;
(iv) $m(a,b)$ is an increasing function in $a$ (and in $b$);
(v) $m(a,b)$ is a continuous function of $a$ and $b$.

The set of all means can be equipped with a partial ordering, called point-wise order, defined by, $m_1 \leq m_2$ if and only if $m_1(a,b) \leq m_2(a,b)$ for every $a,b>0$. We write $m_1 < m_2$ if and only if $m_1(a,b) < m_2(a,b)$ for all $a,b>0$ with $a \neq b$. Clearly, $m_1 < m_2$ implies $m_1 \leq m_2$.

The standard examples of means satisfying the above requirements are recalled in the following.

$$A := A(a,b) = \frac{a+b}{2};\ G := G(a,b) = \sqrt{ab};\ H := H(a,b) = \frac{2ab}{a+b}; \tag{1.1}$$
$$L := L(a,b) = \frac{b-a}{ln\, b - ln\, a},\ L(a,a)=a;\ I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},\ I(a,a)=a,$$

respectively called the arithmetic, geometric, harmonic, logarithmic and identric means. These means satisfy the following inequalities

$$min < H < G < L < I < A < max, \tag{1.2}$$

where $min$ and $max$ are the trivial means $(a,b) \longmapsto min(a,b)$ and $(a,b) \longmapsto max(a,b)$.

For a given mean $m$, we set

$$m^*(a,b) = \Big(m\big(a^{-1},b^{-1}\big)\Big)^{-1}, \tag{1.3}$$

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and it is easy to see that $m^*$ is also a mean, called the dual mean of $m$. The symmetry and homogeneity axioms (ii),(iii) yield

$$m^*(a,b) = \frac{ab}{m(a,b)} \tag{1.4}$$

for all $a, b > 0$, which we briefly write $m^* = G^2/m$. Every mean $m$ satisfies $m^{**} = m$ and, if $m_1$ and $m_2$ are two means such that $m_1 \leq m_2$ (resp. $m_1 < m_2$) then $m_1^* \geq m_2^*$ (resp. $m_1^* > m_2^*$). A mean $m$ is called self-dual if $m^* = m$. It is clear that the arithmetic and harmonic means are mutually dual and the geometric mean is the unique self-dual mean. We recall that, the mean-map $m \longmapsto m^*$ is point-wise convex in the sense that the following inequality, [9]

$$\Big((1-t)m_1 + tm_2\Big)^* \leq (1-t)m_1^* + tm_2^* \tag{1.5}$$

holds true for every real number $t \in [0,1]$ and all means $m_1$ and $m_2$. Further, the inequality (1.5) is strict (in the above sense) if and only if $t \in ]0,1[$ and $m_1 \neq m_2$.

The dual of the logarithmic mean is given by

$$L^* := L^*(a,b) = ab\frac{ln\ b - ln\ a}{b-a}, \quad L^*(a,a) = a, \tag{1.6}$$

while that of the identric mean is

$$I^* := I^*(a,b) = e\left(\frac{a^b}{b^a}\right)^{1/b-a}, \quad I^*(a,a) = a. \tag{1.7}$$

The following inequalities are immediate from the above.

$$min < H < I^* < L^* < G < L < I < A < max. \tag{1.8}$$

A mean $m$ is called strict if $m(a,b)$ is strictly monotonic increasing in $a$ (and in $b$). Every strict mean $m$ satisfies that, $m(a,b) = a \Longrightarrow a = b$. It is easy to see that if $m$ is a strict mean then so is $m^*$. The means $min$ and $max$ are not strict while $H, G, A, L, L^*, I, I^*$ are strict means.

In the literature, there are some families of means, called power means, which include the above familiar means. Precisely, let $p$ be a real number, we recall the following:

• The power binomial mean:

$$\begin{cases} B_p(a,b) := B_p = \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, \\ \\ B_{-1} = H,\ B_1 = A,\ B_0 := lim_{p\to 0} B_p = G. \end{cases} \tag{1.9}$$

• The power logarithmic mean:

$$\begin{cases} L_p(a,b) := L_p = \left(\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right)^{1/p},\ L_p(a,a) = a, \\ \\ L_{-2} = G,\ L_{-1} = L,\ L_0 = I,\ L_1 = A. \end{cases} \tag{1.10}$$

• The power difference mean:

$$\begin{cases} D_p(a,b) := D_p = \dfrac{p}{p+1}\dfrac{a^{p+1} - b^{p+1}}{a^p - b^p},\ D_p(a,a) = a, \\ \\ D_{-2} = H,\ D_{-1} = L^*,\ D_{-1/2} = G,\ D_0 = L,\ D_1 = A. \end{cases} \tag{1.11}$$

• The power exponential mean:

$$\left\{\begin{array}{l} I_p(a,b) := I_p = exp\left(-\frac{1}{p} + \frac{a^p ln\ a - b^p ln\ b}{a^p - b^p}\right),\ I_p(a,a) = a, \\ \\ I_{-1} = I^*,\ I_0 = G,\ I_1 = I. \end{array}\right. \tag{1.12}$$

• The second power logarithmic mean:

$$\left\{\begin{array}{l} l_p(a,b) := l_p = \left(\frac{1}{p}\frac{b^p - a^p}{ln\ b - ln\ a}\right)^{1/p},\ l_p(a,a) = a, \\ \\ l_{-1} = L^*,\ l_0 = G,\ l_1 = L. \end{array}\right. \tag{1.13}$$

If $m_p$ stands for one of the above power means, it is well known that $m_{-\infty} = min$ and $m_{+\infty} = max$. Further, all the above power means (also called means of order $p$) are strictly monotonic increasing in $p$, for fixed $a, b > 0$. Otherwise, it is easy to see that $B_p^* = B_{-p}$ for all real number $p$. We notice that these power means are included in a generalized family of means (not needed here), namely the Stolarsky mean of order 2, see [6] for instance.

In the past years, enormous efforts by some authors has been devoted to refine various inequalities between means (called mean-inequalities), see [1, 2, 4, 5, 6, 7, 13, 14, 15] for instance and the related references cited therein. Our fundamental goal in this paper is to explore the stabilizability concept for obtaining a game of mean-inequalities whose certain of them have been differently discussed in the literature. Our approach stems its importance in the following items:
First, by a united procedure we find some known mean-inequalities and further other ones in a short and nice manner.
Secondly, by the same procedure, starting from an arbitrary lower and/or upper bounds of a stabilizable mean we show how to obtain in a recursive manner an infinity of lower and/or upper bounds of this mean. We also give, throughout a lot of examples, sufficient conditions for ensuring that the new bounds are refinements of the initial ones.

## 2. Background Material about Stabilizable Means

For the sake of simplicity for the reader, we will recall in this section some basic notions and results stated by the author in an earlier paper [9].

**Definition 2.1.** Let $m_1, m_2$ and $m_3$ be three given means. For all $a, b > 0$, define

$$\mathcal{R}(m_1, m_2, m_3)(a,b) = m_1\Big(m_2\big(a, m_3(a,b)\big), m_2\big(m_3(a,b), b\big)\Big), \tag{2.1}$$

called the resultant mean-map of $m_1, m_2$ and $m_3$.

A study investigating the elementary properties of the resultant mean-map has been stated in [9]. Here, we just recall the following result needed later.

**Proposition 2.1.** *The map $(a,b) \longmapsto \mathcal{R}(m_1, m_2, m_3)(a,b)$ defines a mean, with the following properties:*
*(i) For every means $m_1, m_2, m_3$ we have*

$$\Big(\mathcal{R}(m_1, m_2, m_3)\Big)^* = \mathcal{R}(m_1^*, m_2^*, m_3^*). \tag{2.2}$$

*(ii) The mean-map $\mathcal{R}$ is point-wisely increasing with respect to each its mean variables, that is,*

$$\Big(m_1 \leq m_1^{'},\ m_2 \leq m_2^{'},\ m_3 \leq m_3^{'},\ \Big) \Longrightarrow \mathcal{R}(m_1, m_2, m_3) \leq \mathcal{R}\big(m_1^{'}, m_2^{'}, m_3^{'}\big). \tag{2.3}$$

The following result, which the proof is straightforward, is also of interest in what follows.

**Proposition 2.2.** *For all mean $M$, the mean-map $m \longmapsto \mathcal{R}(A, m, M)$ is point-wise affine in the sense that the mean-equality*

$$\mathcal{R}\Big(A, (1-t)m + tm^{'}, M\Big) = (1-t)\mathcal{R}(A, m, M) + t\mathcal{R}(A, m^{'}, M) \tag{2.4}$$

*holds for all real number $t \in [0, 1]$ and all means $m, m^{'}$.*

*Example* 2.1. Simple computations lead to

$$\mathcal{R}(H, H, A) = \Big(\frac{1}{2}A + \frac{1}{2}H\Big)^* = \frac{2AH}{A+H}, \quad \mathcal{R}(H, A, A) = \frac{3}{4}A + \frac{1}{4}H. \tag{2.5}$$

$$\mathcal{R}(A, G, G) = \Big(\frac{1}{2}AG + \frac{1}{2}G^2\Big)^{1/2}, \quad \mathcal{R}(A, A, G) = \frac{1}{2}A + \frac{1}{2}G. \tag{2.6}$$

$$\mathcal{R}(G, G, A) = \sqrt{AG}, \quad \mathcal{R}(G, A, A) = \Big(\frac{3}{4}A^2 + \frac{1}{4}G^2\Big)^{1/2}. \tag{2.7}$$

The following lemma will be needed in the sequel.

**Lemma 2.3.** *Let $m_1$ and $m_2$ be two means, then the following equality*

$$\mathcal{R}(m_1, m_2, G)(a, b) = m_1\big(\sqrt{a}, \sqrt{b}\big)m_2\big(\sqrt{a}, \sqrt{b}\big). \tag{2.8}$$

*holds for all $a, b > 0$.*

*Proof.* By definition of $\mathcal{R}$, with the homogeneity axiom for $m_2$, we obtain

$$\begin{aligned} \mathcal{R}(m_1, m_2, G)(a, b) &= m_1\Big(m_2\big(a, \sqrt{ab}\big), m_2\big(\sqrt{ab}, b\big)\Big) \\ &= m_1\Big(\sqrt{a}m_2\big(\sqrt{a}, \sqrt{b}\big), \sqrt{b}m_2\big(\sqrt{a}, \sqrt{b}\big)\Big). \end{aligned} \tag{2.9}$$

This, with the homogeneity axiom for $m_1$, yields the desired result. □

As proved in [9], and will be again shown throughout this paper, the resultant mean-map stems its importance in the fact that it is a tool for introducing the stability and stabilizability notions as recalled in the following.

**Definition 2.2.** A mean $m$ is said to be:
(a) Stable if $\mathcal{R}(m, m, m) = m$.
(b) Stabilizable if there exist two nontrivial stable means $m_1$ and $m_2$ satisfying the relation $\mathcal{R}(m_1, m, m_2) = m$. We then say that $m$ is $(m_1, m_2)$-stabilizable.

In [9], the author stated a developed study about the stability and stabilizability of the standard and power means. In particular, he proved that if $m$ is stable then so is $m^*$, and if $m$ is $(m_1, m_2)$-stabilizable then $m^*$ is $(m_1^*, m_2^*)$-stabilizable. About the power standard means, the summarized results stated in [9] are recited in the following theorem and its next corollary.

**Theorem 2.4.** *For all real number p, the following statements are met:*
*(1) The power binomial mean $B_p$ is stable.*
*(2) The power logarithmic mean $L_p$ is $(B_p, A)$-stabilizable while the power difference mean $D_p$ is $(A, B_p)$-stabilizable.*
*(3) The power exponential mean $I_p$ is $(G, B_p)$-stabilizable while the second power logarithmic mean $l_p$ is $(B_p, G)$-stabilizable.*

**Corollary 2.5.** *With the above, the following assertions are met:*
*(1) The arithmetic, geometric and harmonic means $A, G$ and $H$ are stable.*
*(2) The logarithmic mean $L$ is $(H, A)$-stabilizable and $(A, G)$-stabilizable while the identric mean $I$ is $(G, A)$-stabilizable.*
*(3) The mean $L^*$ is $(A, H)$-stabilizable and $(H, G)$-stabilizable while $I^*$ is $(G, H)$-stabilizable.*

**N.B.** Throughout what follows, we investigate some results of mean-inequalities, under convenient assumptions, for the strict symbol $<$ (in the above sense). By similar manner, all stated results still true when we replace $<$ by $\leq$ in the hypotheses as in the related conclusions. Of course, this is not immediate since $m_1 < m_2$ is, as hypothesis and as conclusion, stronger than $m_1 \leq m_2$.

## 3. Refinements for Mean-Inequalities: General Approach

As already pointed before, this section displays some important applications of the above concepts for refining mean-inequalities in a general point of view. Particular examples illustrating the generality of our approach and the interest of this work will be discussed. We first state the following result which is an improvement of that of Proposition 2.1..

**Theorem 3.1.** *Let $m_1, m_1^{'}, m_2, m_2^{'}, m_3$ and $m_3^{'}$ be means such that*

$$m_1 \leq m_1^{'}, m_2 \leq m_2^{'} \text{ and } m_3 \leq m_3^{'}. \tag{3.1}$$

*Assume that one of the following three statements holds:*
*(i) $m_1 < m_1^{'}$, $m_2^{'}$ and $m_3^{'}$ are strict means,*
*(ii) $m_2 < m_2^{'}$, $m_1$ and $m_3^{'}$ are strict means,*
*(iii) $m_3 < m_3^{'}$, $m_1$ and $m_2$ are strict means.*
*Then we have*

$$\mathcal{R}\big(m_1, m_2, m_3\big) < \mathcal{R}\big(m_1^{'}, m_2^{'}, m_3^{'}\big), \tag{3.2}$$

*in the sense that*

$$\mathcal{R}\big(m_1, m_2, m_3\big)(a, b) < \mathcal{R}\big(m_1^{'}, m_2^{'}, m_3^{'}\big)(a, b) \tag{3.3}$$

*holds for all $a, b > 0$ with $a \neq b$.*

*Proof.* Assume that (3.1) holds:
(i) Without loss the generality, let $a, b > 0$ with $a < b$. Then we have

$$\begin{aligned} \mathcal{R}(m_1, m_2, m_3)(a, b) &= m_1\Big(m_2\big(a, m_3(a, b)\big), m_2\big(m_3(a, b), b\big)\Big) \\ &\leq m_1\Big(m_2^{'}\big(a, m_3^{'}(a, b)\big), m_2^{'}\big(m_3^{'}(a, b), b\big)\Big). \end{aligned} \tag{3.4}$$

Since $m_3^{'}$ and $m_2^{'}$ are assumed strict means then we have, respectively,

$$a < m_3^{'}(a, b) < b \text{ and } m_2^{'}\big(a, m_3^{'}(a, b)\big) < m_2^{'}\big(m_3^{'}(a, b), b\big). \tag{3.5}$$

This, with $m_1 < m_1^{'}$, yields the desired result.
(ii), (iii) Similar to (i). We left the detail to the reader as simple exercise. □

The following corollary, which will be usually needed in the sequel, is immediate from the above theorem.

**Corollary 3.2.** *Let $m_1, m_1^{'}, m_2, m_2^{'}, m_3$ and $m_3^{'}$ be strict means such that*

$$m_1 \le m_1^{'}, m_2 \le m_2^{'} \text{ and } m_3 \le m_3^{'}. \tag{3.6}$$

*Assume that there exists $i = 1, 2, 3$ such that $m_i < m_i^{'}$, then one has*

$$\mathcal{R}\big(m_1, m_2, m_3\big) < \mathcal{R}\big(m_1^{'}, m_2^{'}, m_3^{'}\big). \tag{3.7}$$

Now, we are in position to state the following result which gives a refinement of a mean-inequality $m_1 < m < m_2$ when $m$ is $(m_1, m_2)$-stabilizable or $(m_2, m_1)$-stabilizable.

**Theorem 3.3.** *Let $m$ be a $(m_1, m_2)$-stabilizable mean. Assume that $m_1 < m < m_2$, then the following refinement holds*

$$m_1 < \mathcal{R}(m_1, m_1, m_2) < m < \mathcal{R}(m_1, m_2, m_2) < m_2. \tag{3.8}$$

*If $m_2 < m < m_1$ then the role of $m_1$ and $m_2$ in the above inequalities is reversed.*

*Proof.* According to Theorem 3.1, with $m_1 < m < m_2$ and the fact that $m_1$ and $m_2$ are strict means, we obtain

$$\begin{aligned}\mathcal{R}(m_1, m_1, m_1) < \mathcal{R}(m_1, m_1, m_2) < \mathcal{R}(m_1, m, m_2) \\ < \mathcal{R}(m_1, m_2, m_2) < \mathcal{R}(m_2, m_2, m_2).\end{aligned} \tag{3.9}$$

This, with the fact that $m_1$ and $m_2$ are stable and $m$ is $(m_1, m_2)$-stabilizable, yields the desired result. □

Now, let us observe the following particular examples illustrating the situation of the above theorem.

*Example* 3.1. Knowing that $H < L < A$ with $L$ is $(H, A)$-stabilizable, the above theorem gives

$$H < \mathcal{R}(H, H, A) < L < \mathcal{R}(H, A, A) < A. \tag{3.10}$$

This, with (2.5), gives the following refinement of the arithmetic-logarithmic-harmonic mean inequality

$$H < \frac{2G^2}{A + H} < L < \frac{3}{4}A + \frac{1}{4}H < A. \tag{3.11}$$

*Example* 3.2. Starting from $G < L < A$ with $L$ is $(A, G)$-stabilizable, Theorem 3.3 implies that

$$G < \mathcal{R}(A, G, G) < L < \mathcal{R}(A, A, G) < A. \tag{3.12}$$

According to (2.6), we obtain the following inequalities which refine the arithmetic-logarithmic-geometric mean inequality

$$G < \Big(\frac{1}{2}AG + \frac{1}{2}G^2\Big)^{1/2} < L < \frac{1}{2}A + \frac{1}{2}G < A. \tag{3.13}$$

*Example* 3.3. Now, consider the known inequalities $G < I < A$ with $I$ is $(G, A)$-stabilizable. Similarly to the above we obtain

$$G < \mathcal{R}(G, G, A) < I < \mathcal{R}(G, A, A) < A. \tag{3.14}$$

This, when combined with (2.7), implies a refinement of the arithmetic-identric-geometric mean inequality given by

$$G < \sqrt{AG} < I < \Big(\frac{3}{4}A^2 + \frac{1}{4}G^2\Big)^{1/2} < A. \tag{3.15}$$

Refinements of mean-inequalities, even stronger than that of the above examples, are largely studied in the literature, see [6] and the related reference cited therein. As already pointed before, our approach gives a united procedure having a general point of view when we have to refine a mean double inequality $m_1 \leq m \leq m_2$ where the intermediary mean $m$ is $(m_1, m_2)$-stabilizable or $(m_2, m_1)$-stabilizable. Further, the next theorem shows that our approach can be successively repeated in the aim to obtain more lower and/or upper bounds of a given stabilizable mean.

**Theorem 3.4.** *Let $m$ be a $(m_1, m_2)$- stabilizable mean. Let $m_3$ and $m_4$ be two means such that*

$$m_3 < m < m_4. \tag{3.16}$$

*Then we have the following mean-inequalities*

$$\mathcal{R}(m_1, m_3, m_2) < m < \mathcal{R}(m_1, m_4, m_2). \tag{3.17}$$

*Proof.* By Theorem 3.1, with $m_3 < m < m_4$, we have

$$\mathcal{R}(m_1, m_3, m_2) < \mathcal{R}(m_1, m, m_2) < \mathcal{R}(m_1, m_4, m_2). \tag{3.18}$$

This, with the fact that $m$ is $(m_1, m_2)$-stabilizable, gives the desired result. □

As pointed in the above, Theorem 3.4 starts from an arbitrary lower and upper bounds of a stabilizable mean $m$ for giving other lower and upper bounds of the mean $m$, and so we can reiter the same procedure for obtaining an infinity of lower and upper bounds of $m$. An important question arises from this latter situation: Under what general conditions, (3.18) is a refinement of (3.17), that is,

$$m_3 < \mathcal{R}(m_1, m_3, m_2) \text{ and } \mathcal{R}(m_1, m_4, m_2) < m_4? \tag{3.19}$$

This makes appear in (3.19) weak conditions of stabilizability, which we call sub-stabilizability and super-stabilizability of $m_3$ and $m_4$, see [10]. For the moment, we will not give any answer about general sufficient conditions for ensuring the above refinement, but we just discuss (in the sections below) the response for some particular cases.

**N.B.** Let $m_p \in \{l_p, L_p, I_p, D_p\}$ be a power mean. Henceforth, when we say
"Let $m_1$ and $m_2$ be two means such that $m_1 < m_p < m_2$ for some $p$",
it should be understood in the following sense,
"Let $p$ be a real number and assume that there exist two means $m_1 := m_1(p)$ and $m_2 := m_2(p)$ satisfying that $m_1 < m_p < m_2$".

## 4. Expansions of $l_p$ and $L$ in terms of Infinite Products

This section displays a first application of the above concepts for obtaining interesting formulas which are not obvious to establish directly. Our first main result here is recited as follows.

**Theorem 4.1.** *Let $m_1$ and $m_2$ be two means satisfying that*

$$m_1 < l_p < m_2 \tag{4.1}$$

*for some $p$. Then, the following double inequality*

$$m_1\Big(a^{1/2^n}, b^{1/2^n}\Big)\prod_{i=1}^{n} B_p\Big(a^{1/2^i}, b^{1/2^i}\Big) < l_p(a,b) < m_2\Big(a^{1/2^n}, b^{1/2^n}\Big)\prod_{i=1}^{n} B_p\Big(a^{1/2^i}, b^{1/2^i}\Big) \tag{4.2}$$

*holds for all $a, b > 0$, with $a \neq b$, and every integer $n \geq 1$.*

*Proof.* Since $l_p$ is $(B_p, G)$-stabilizable then Theorem 3.3 yields

$$\mathcal{R}(B_p, m_1, G) < l_p < \mathcal{R}(B_p, m_2, G). \tag{4.3}$$

This, with Lemma 2.3, implies that for all $a, b > 0,\ a \neq b$, one has

$$m_1\Big(\sqrt{a}, \sqrt{b}\Big)B_p\Big(\sqrt{a}, \sqrt{b}\Big) < l_p(a,b) < m_2\Big(\sqrt{a}, \sqrt{b}\Big)B_p\Big(\sqrt{a}, \sqrt{b}\Big). \tag{4.4}$$

The desired double inequality follows by a simple mathematical induction, so completes the proof. □

Letting $n \to +\infty$ in (4.2) we immediately obtain the following result giving an interesting explicit formulae of $l_p(a,b)$ in terms of infinite products.

**Corollary 4.2.** *For every real number $p$ and for all $a, b > 0$, there holds*

$$l_p(a,b) = \prod_{i=1}^{\infty} B_p\Big(a^{1/2^i}, b^{1/2^i}\Big). \tag{4.5}$$

Taking $p = 1$ in the above theorem and its corollary, with the fact that $l_1 = L$ and $B_1 = A$, we obtain the next particular result.

**Corollary 4.3.** *Let $m_1$ and $m_2$ be two means such that $m_1 < L < m_2$. Then, for all $a, b > 0$, with $a \neq b$, and every $n \geq 0$, one has*

$$m_1\Big(a^{1/2^n}, b^{1/2^n}\Big)\prod_{i=1}^{n} A\Big(a^{1/2^i}, b^{1/2^i}\Big) < L(a,b) < m_2\Big(a^{1/2^n}, b^{1/2^n}\Big)\prod_{i=1}^{n} A\Big(a^{1/2^i}, b^{1/2^i}\Big). \tag{4.6}$$

*Furthermore, the next expansion*

$$L(a,b) = \prod_{i=1}^{\infty} A\Big(a^{1/2^i}, b^{1/2^i}\Big) \tag{4.7}$$

*is valid for all $a, b > 0$.*

The above expansions of $l_p(a,b)$ and $L(a,b)$ are very useful in the theoretical context as in the practical purposes. For instance, these expansions give an interesting idea in the aim to extend the means $l_p$ and $L$ from two variables to three or more arguments. See [11, 12], for further details about this latter point.

Otherwise, if the mean-bounds $m_1$ and $m_2$ in the double inequality $m_1 < l_p < m_2$ are explicitly given, more precisions about the new bounds of $l_p$ appearing in (4.2) can be obtained. That will be our aim in the section below.

## 5. Refinements for bounding the Means $l_p$ and $L$

Since $l_p$ is $(B_p, G)$-stabilizable, we then will be interested by bounds of $l_p$ in terms of $B_p$ and $G$. We begin by regarding bounds of $l_p$ in a convex-geometric form $B_p^{\alpha}G^{1-\alpha}$ as well:

**Theorem 5.1.** *Let $\alpha, \beta \in [0,1]$ be such that*

$$B_p^{\alpha}G^{1-\alpha} < l_p < B_p^{\beta}G^{1-\beta} \tag{5.1}$$

*for some $p$. Then there holds*

$$G^{\frac{1-\alpha}{2}} B_{p/2}^{\frac{1+\alpha}{2}} = G^{\frac{1-\alpha}{2}} \Big(\frac{B_p^p + G^p}{2}\Big)^{\frac{1+\alpha}{2p}} < l_p < G^{\frac{1-\beta}{2}} \Big(\frac{B_p^p + G^p}{2}\Big)^{\frac{1+\beta}{2p}} = G^{\frac{1-\beta}{2}} B_{p/2}^{\frac{1+\beta}{2}}. \tag{5.2}$$

*Proof.* Since $l_p$ is $(B_p, G)$-stabilizable then Theorem 3.3 gives

$$\mathcal{R}\big(B_p, B_p^{\alpha}G^{1-\alpha}, G\big) < l_p < \mathcal{R}\big(B_p, B_p^{\beta}G^{1-\beta}, G\big). \tag{5.3}$$

According to Lemma 2.3 we have, for all $a, b > 0$,

$$\begin{aligned}\mathcal{R}\big(B_p, B_p^{\alpha}G^{1-\alpha}, G\big)(a,b) &= B_p\big(\sqrt{a},\sqrt{b}\big)\big(B_p^{\alpha}G^{1-\alpha}\big)\big(\sqrt{a},\sqrt{b}\big)\\ &= B_p^{1+\alpha}\big(\sqrt{a},\sqrt{b}\big)G^{1-\alpha}\big(\sqrt{a},\sqrt{b}\big).\end{aligned} \tag{5.4}$$

For all $a, b > 0$, we can write $G(\sqrt{a},\sqrt{b}) = G^{1/2}(a,b)$ and it is easy to verify that,

$$B_p\big(\sqrt{a},\sqrt{b}\big) = \Big(\frac{B_p^p + G^p}{2}\Big)^{1/2p}(a,b) = B_{p/2}^{1/2}(a,b), \tag{5.5}$$

from which the desired double inequality (5.2) follows. □

**Corollary 5.2.** *Let $\alpha, \beta \in [0,1]$ be two real numbers such that*

$$A^{\alpha}G^{1-\alpha} < L < A^{\beta}G^{1-\beta}. \tag{5.6}$$

*Then there holds*

$$G^{\frac{1-\alpha}{2}}\Big(\frac{A+G}{2}\Big)^{\frac{1+\alpha}{2}} < L < G^{\frac{1-\beta}{2}}\Big(\frac{A+G}{2}\Big)^{\frac{1+\beta}{2}}. \tag{5.7}$$

*Proof.* Taking $p = 1$ in the above theorem, with the fact that $l_1 = L$ and $B_1 = A$, we immediately obtain the announced result. □

Let us now examine the next examples in the aim to illustrate the above theoretical results.

*Example* 5.1. It is not hard to verify that $G < l_p < B_p$ for every $p > 0$, with reversed double inequality for $p < 0$. Theorem 5.1 immediately gives (with $\alpha = 0, \beta = 1$ for $p > 0$; $\alpha = 1, \beta = 0$ for $p < 0$)

$$G^{1/2}\Big(\frac{B_p^p + G^p}{2}\Big)^{1/2p} < l_p < \Big(\frac{B_p^p + G^p}{2}\Big)^{1/p} \tag{5.8}$$

for each real number $p \neq 0$. It is easy to verify that the double inequality (5.8) refines the initial one. In particular, we have

$$G < \Big(\frac{AG+G^2}{2}\Big)^{1/2} < L < \frac{A+G}{2} < A, \tag{5.9}$$

which refines the arithmetic-logarithmic-geometric mean inequality $G < L < A$.

**Theorem 5.3.** *Let $\alpha \in [0,1]$ be such that*

$$B_p^{\alpha} G^{1-\alpha} < (>) l_p \tag{5.10}$$

*for some $p > (<) 0$, respectively. Then one has*

$$B_p^{\frac{1+\alpha}{4}} G^{\frac{3-\alpha}{4}} < (>) l_p. \tag{5.11}$$

*If moreover $\alpha < (>) 1/3$ then (5.11) refines (5.10).*

*Proof.* Assume that

$$B_p^{\alpha} G^{1-\alpha} < l_p \tag{5.12}$$

for some $p > 0$. According to Theorem 5.1, the first inequality of (5.2) holds and the arithmetic-geometric mean inequality gives

$$\Big(\frac{B_p^p + G^p}{2}\Big)^{\frac{1+\alpha}{2p}} > B_p^{\frac{1+\alpha}{4}} G^{\frac{1+\alpha}{4}}. \tag{5.13}$$

The desired inequality follows after a simple reduction. Further, the inequality

$$B_p^{\alpha} G^{1-\alpha} < B_p^{\frac{1+\alpha}{4}} G^{\frac{3-\alpha}{4}} \tag{5.14}$$

for $p > 0$ is reduced to

$$G^{\frac{1-3\alpha}{4}} < B_p^{\frac{1-3\alpha}{4}} \tag{5.15}$$

which holds when $\alpha < 1/3$. For the reversed inequalities, the same arguments as previous work, so completes the proof. □

If we get $p = 1$ in the above theorem we immediately obtain the following result.

**Corollary 5.4.** *Let $\alpha \in [0,1]$ be a real number satisfying that*

$$A^{\alpha} G^{1-\alpha} < L. \tag{5.16}$$

*Then one has*

$$A^{\frac{1+\alpha}{4}} G^{\frac{3-\alpha}{4}} < L. \tag{5.17}$$

*If moreover $\alpha < 1/3$ then (5.17) refines (5.16).*

Theorem 5.3 tells us that every given bound of $l_p$ in a convex-geometric form yields another bound of $l_p$ in an analogous, but different, form. Illustrating this latter point, we will deduce a better bound of $l_p$ than the above ones. Precisely, we may state the next result.

**Theorem 5.5.** *Let $p$ be a real number. If $p > 0$ then one has*

$$B_p^{1/3} G^{2/3} < l_p. \tag{5.18}$$

*If $p < 0$ then the above inequality is reversed. In particular the following inequality holds true*

$$A^{1/3} G^{2/3} < L. \tag{5.19}$$

*Proof.* Assume that $p > 0$. Starting from $G < l_p$ (see Example 5.1), we are in the situation of Theorem 5.3 with $\alpha = 0$, and so we have $B_p^{1/4} G^{3/4} < l_p$. Let us reiter successively this procedure: if in the step $n$ we have

$$B_p^{\alpha_n} G^{1-\alpha_n} < l_p \tag{5.20}$$

then in the step $n+1$ we obtain

$$B_p^{\frac{1+\alpha_n}{4}} G^{\frac{3-\alpha_n}{4}} < l_p, \tag{5.21}$$

that is,

$$B_p^{\alpha_{n+1}} G^{1-\alpha_{n+1}} < l_p \ \text{ with } \ \alpha_{n+1} = \frac{1+\alpha_n}{4},\ \alpha_0 = \alpha. \tag{5.22}$$

It is easy to see that the real sequences $(\alpha_n)_n$ converges to $1/3$ for every given initial data $\alpha_0 \in [0,1]$. The desired inequality follows by letting $n \to +\infty$ in the recursive inequality

$$B_p^{\alpha_n} G^{1-\alpha_n} < l_p. \tag{5.23}$$

The proof is similar for $p < 0$. Taking $p = 1$ in (5.18) we obtain (5.19), so completes the proof. □

To understand the interest of the above theorem, let us observe the following example.

*Example* 5.2. Let us apply Theorem 5.3 to the previous inequality $B_p^{1/3} G^{2/3} < (>) l_p$. Then, the next inequality

$$l_p^{3p} > G^p \left( \frac{B_p^p + G^p}{2} \right)^2 \tag{5.24}$$

holds true for each real number $p$ ($p \neq 0$). In particular, taking $p = 1$ we obtain

$$G\Big(\frac{A+G}{2}\Big)^2 < L^3, \tag{5.25}$$

which refines $A^{1/3} G^{2/3} < L$.

*Remark* 5.1. The inequality (5.19) was proved by Leach and Sholander in [5] while (5.25) has been shown by Sāndor in [15]. These two inequalities were proved by different methods therein while together obtained here via the same approach. In the same sense, other examples will be seen later (see Remarks 5.3, 5.4, 6.1).

*Remark* 5.2. As far as we know, inequality (5.18) appears to be new. As well known, inequality (5.19) is the best possible in the sense that the constant $\alpha = 1/3$ cannot be improved in $A^\alpha G^{1-\alpha} < L$. This latter point rejoins the fact that if we apply Corollary 5.4 to (5.19) we obtain the same inequality. Also, if we apply Theorem 5.3 to (5.18) we obtain the same. This allows us to conjecture that inequality (5.18) is also the best possible.

Now, we will be interested by bounds of $l_p$ in a convex-arithmetic expression as well:

**Theorem 5.6.** *Let $\alpha, \beta \in [0,1]$ be two real numbers such that*

$$\alpha B_p + (1-\alpha) G < l_p < \beta B_p + (1-\beta) G, \tag{5.26}$$

*for some real number $p$. Then there holds*

$$\begin{aligned} \alpha\Big(\frac{B_p^p + G^p}{2}\Big)^{1/p} + (1-\alpha) G^{1/2} \Big(\frac{B_p^p + G^p}{2}\Big)^{1/2p} &< l_p \\ < \beta\Big(\frac{B_p^p + G^p}{2}\Big)^{1/p} + (1-\beta) G^{1/2} \Big(\frac{B_p^p + G^p}{2}\Big)^{1/2p}&. \end{aligned} \tag{5.27}$$

*Proof.* By the same arguments as previous, we have

$$\mathcal{R}\Big(B_p, \alpha B_p + (1-\alpha)G, G\Big) < l_p < \mathcal{R}\Big(B_p, \beta B_p + (1-\beta)G, G\Big). \tag{5.28}$$

Again, thanks to Lemma 2.3, we obtain

$$\alpha\Big(B_p(\sqrt{a},\sqrt{b})\Big)^2 + (1-\alpha)G^{1/2}B_p\big(\sqrt{a},\sqrt{b}\big) < l_p < \beta\Big(B_p(\sqrt{a},\sqrt{b})\Big)^2 + (1-\beta)G^{1/2}B_p\big(\sqrt{a},\sqrt{b}\big). \tag{5.29}$$

By virtue of the identity (5.5) we obtain the desired result after simple manipulations. □

As in the above, taking $p = 1$ in the latter theorem we immediately obtain the following result.

**Corollary 5.7.** *Let $\alpha, \beta \in [0,1]$ be two real numbers such that*

$$\alpha A + (1-\alpha)G < L < \beta A + (1-\beta)G. \tag{5.30}$$

*Then there holds*

$$\alpha\Big(\frac{A+G}{2}\Big) + (1-\alpha)\Big(\frac{AG+G^2}{2}\Big)^{1/2} < L < \beta\Big(\frac{A+G}{2}\Big) + (1-\beta)\Big(\frac{AG+G^2}{2}\Big)^{1/2}. \tag{5.31}$$

Theorem 5.6 has many interesting consequences. For instance, we give the two next corollaries.

**Corollary 5.8.** *Let $\alpha \in [0,1]$ be such that*

$$L < \alpha A + (1-\alpha)G. \tag{5.32}$$

*Then we have*

$$L < \frac{1+\alpha}{4}A + \frac{3-\alpha}{4}G. \tag{5.33}$$

*If $\alpha > 1/3$ then (5.33) refines (5.32).*

*Proof.* According to Theorem 5.6 we have

$$L < \alpha\Big(\frac{A+G}{2}\Big) + (1-\alpha)\Big(\frac{AG+G^2}{2}\Big)^{1/2}. \tag{5.34}$$

If we write

$$\Big(\frac{AG+G^2}{2}\Big)^{1/2} = G^{1/2}\Big(\frac{A+G}{2}\Big)^{1/2} \tag{5.35}$$

and we apply the arithmetic-geometric mean inequality, i.e.

$$G^{1/2}\Big(\frac{A+G}{2}\Big)^{1/2} < \frac{1}{2}G + \frac{1}{2}\frac{A+G}{2}, \tag{5.36}$$

we obtain the announced result after substituting this latter inequality in (5.34). If $\alpha > 1/3$, it is easy to see by similar manner as previous that (5.33) refines (5.32) and the proof is completed. □

**Corollary 5.9.** *The following inequality holds true*

$$L < \frac{1}{3}A + \frac{2}{3}G. \tag{5.37}$$

*Proof.* Similarly to the above, it is sufficient to see that the sequence $(\alpha_n)$ defined by

$$\alpha_{n+1} = \frac{1+\alpha_n}{4}, \quad \text{with} \quad \alpha_0 \in [0,1], \tag{5.38}$$

converges to 1/3 and the desired result follows as previous. We omit the routine detail here. ☐

*Remark* 5.3. The inequality (5.37) was differently proved by B.C.Carlson in [3] and here obtained by the same approach as (5.19) and (5.25).

Let us illustrate the above theoretical examples with the following examples.

*Example* 5.3. Consider the above mean-inequality $L < (1/3)A+(2/3)G$ which corresponds to $\beta = 1/3$ in Corollary 5.8. With this, the obtained refinement is given by

$$L < \frac{1}{3}\Big(\frac{A+G}{2}\Big) + \frac{2}{3}\Big(\frac{AG+G^2}{2}\Big)^{1/2} < \frac{1}{3}A + \frac{2}{3}G. \tag{5.39}$$

Of course, we can combine some the above results to improve the lower and upper bounds of $L$. The next example explains this situation.

*Example* 5.4. Let us consider the following double inequality

$$A^{1/3}G^{2/3} < L < \frac{1}{3}A + \frac{2}{3}G. \tag{5.40}$$

Combining Theorem 5.1 and Theorem 5.6 we immediately obtain

$$G^{1/3}\Big(\frac{1}{2}A + \frac{1}{2}G\Big)^{2/3} < L < \frac{1}{3}\Big(\frac{1}{2}A + \frac{1}{2}G\Big) + \frac{2}{3}\Big(\frac{1}{2}AG + \frac{1}{2}G^2\Big)^{1/2}. \tag{5.41}$$

The reader can easily verify that this latter double inequality refines the initial one, so proving our desired aim.

**Theorem 5.10.** *Let $\alpha \in [0,1]$ be such that*

$$l_p < \alpha B_p + (1-\alpha)G \tag{5.42}$$

*for some $p \le 1$. Then there holds*

$$l_p < \frac{1+\alpha}{4}B_p + \frac{3-\alpha}{4}G. \tag{5.43}$$

*Proof.* If (5.42) holds then Theorem 5.6 gives

$$l_p < \alpha\Big(\frac{B_p^p + G^p}{2}\Big)^{1/p} + (1-\alpha)G^{1/2}\Big(\frac{B_p^p + G^p}{2}\Big)^{1/2p}. \tag{5.44}$$

This, with $p \le 1$ and the fact that the map $x \longmapsto x^{1/p}$ is convex on $]0,+\infty[$, yields

$$l_p < \alpha\Big(\frac{B_p + G}{2}\Big) + (1-\alpha)G^{1/2}\Big(\frac{B_p + G}{2}\Big)^{1/2}. \tag{5.45}$$

The arithmetic-geometric mean inequality gives

$$G^{1/2}\Big(\frac{B_p + G}{2}\Big)^{1/2} < \frac{1}{2}G + \frac{1}{2}\frac{B_p + G}{2}, \tag{5.46}$$

and the desired inequality follows by combining (5.45) and (5.46) with a simple reduction. ☐

Taking $p = -1$ in the above theorem, with the fact that $l_{-1} = L^* = G^2/L$ and $B_{-1} = H = G^2/A$, we immediately obtain the next result.

**Corollary 5.11.** *Let $\alpha \in [0,1]$ be such that*

$$\frac{1}{L} < \frac{\alpha}{A} + \frac{1-\alpha}{G}. \tag{5.47}$$

*Then one has*

$$\frac{1}{L} < \frac{1+\alpha}{4}\frac{1}{A} + \frac{3-\alpha}{4}\frac{1}{G}. \tag{5.48}$$

*If moreover $\alpha > 1/3$ then (5.48) refines (5.47).*

**Theorem 5.12.** *For all real number $p \leq 1$ with $p \neq 0$, we have*

$$l_p < \frac{1}{3}B_p + \frac{2}{3}G. \tag{5.49}$$

*In particular, the following inequality holds*

$$\frac{1}{L} < \frac{1}{3}\frac{1}{A} + \frac{2}{3}\frac{1}{G}. \tag{5.50}$$

*Proof.* We left it to the reader as an interesting exercise. □

We end this section by stating another result showing how to obtain a lower bound of the logarithmic mean $L$ when we start from an upper bound of its dual $L^*$. In fact, since $L^*$ is $(A,H)$-stabilizable then we search bounds of $L^*$ in terms of $A$ and $H$. Precisely, we have the following.

**Theorem 5.13.** *Let $\alpha$ be a real number satisfying that*

$$L^* < \alpha A + (1-\alpha)H. \tag{5.51}$$

*Then we have*

$$L^* < \Big(\frac{1+\alpha}{4}\Big)A + \Big(\frac{3-\alpha}{4}\Big)H. \tag{5.52}$$

*If moreover $\alpha > 1/3$ then (5.52) refines (5.51).*

*Proof.* Since $L^*$ is $(A,H)$-stabilizable then we obtain, with Proposition 2.2,

$$L^* < \mathcal{R}\big(A, \alpha A + (1-\alpha)H, H\big) = \alpha\mathcal{R}(A,A,H) + (1-\alpha)\mathcal{R}(A,H,H). \tag{5.53}$$

Thanks to relationships (2.5) for obtaining

$$L^* < \alpha\Big(\frac{A+H}{2}\Big) + (1-\alpha)\Big(\frac{3}{4}A + \frac{1}{4}H\Big)^*. \tag{5.54}$$

Due to the p-convexity of the mean-map $m \longmapsto m^*$, with $A^* = H$ and $H^* = A$, we obtain

$$L^* < \alpha\Big(\frac{A+H}{2}\Big) + (1-\alpha)\Big(\frac{3}{4}H + \frac{1}{4}A\Big), \tag{5.55}$$

which after reduction yields the desired result. □

**Corollary 5.14.** *The following inequality holds true*

$$\frac{1}{L} < \frac{2}{3}\frac{1}{A} + \frac{1}{3}\frac{1}{H}. \tag{5.56}$$

*Proof.* Similarly to the same idea as in the above we have

$$L^* < \Big(\frac{1+\alpha_n}{4}\Big)A + \Big(\frac{3-\alpha_n}{4}\Big)H, \tag{5.57}$$

where $(\alpha_n)_n$ is the sequence defined by the same recursive relation as in the proof of Corollary 5.9. Letting $n \to +\infty$ we obtain

$$L^* < \frac{1}{3}A + \frac{2}{3}H. \tag{5.58}$$

The general relation $m^* = G^2/m$ valid for all mean $m$, gives in particular, $L^* = G^2/L,\ H = G^2/A$ and $A = G^2/H$. Substituting this in the latter inequality, we obtain the desired result. □

*Remark* 5.4. The inequality (5.56) was differently proved by Chen in [4] and shown here by the same approach as (5.19), (5.25) and (5.37), so proving the interest of this work. Further, we notice that it is easy to verify that (5.56) is stronger than (5.50).

## 6. Refinements for bounding the Means $I_p$ and $I$

In this section, we will state some refinements for the power exponential mean $I_p$ in a parallel manner to those for $l_p$ already presented in the above section. We immediately deduce some refinements for the identric mean $I$. The proofs of the results announced here are often similar to that of the above and we omit the routine details to not lengthen this paper.

We begin by stating the next lemma which will be needed later.

**Lemma 6.1.** *Let $m_1$ and $m_2$ be two means such that*

$$m_1 < I_p < m_2 \tag{6.1}$$

*for some $p$. Then, for all $a, b > 0$, one has*

$$m_1\big(a, B_p\big)m_1\big(B_p, b\big) < I_p^2(a, b) < m_2\big(a, B_p\big)m_2\big(B_p, b\big), \tag{6.2}$$

*where we set $B_p := B_p(a, b)$ for the sake of simplicity.*

*Proof.* Since $I_p$ is $(G, B_p)$-stabilizable then Theorem 3.3 yields

$$\mathcal{R}\big(G, m_1, B_p\big) < I_p < \mathcal{R}\big(G, m_2, B_p\big). \tag{6.3}$$

By computations as previous we easily deduce the desired result. □

Starting from a double inequality $m_1 < I_p < m_2$, we may choose convenient means $m_1$ and $m_2$ giving easy computations with the fact that $I_p$ is $(G, B_p)$-stabilizable. More precisely, the next result is an analogue of Theorem 5.1 from $l_p$ to $I_p$.

**Theorem 6.2.** *Let $\alpha, \beta \in [0, 1]$ be two real numbers such that*

$$B_p^{\alpha}G^{1-\alpha} < I_p < B_p^{\beta}G^{1-\beta}, \tag{6.4}$$

*for some $p$. Then the following double inequality holds*

$$B_p^{1-\alpha}G^{1-\alpha}\Big(\frac{3}{4}B_p^{2p} + \frac{1}{4}G^{2p}\Big)^{\frac{\alpha}{p}} < I_p^2 < B_p^{1-\beta}G^{1-\beta}\Big(\frac{3}{4}B_p^{2p} + \frac{1}{4}G^{2p}\Big)^{\frac{\beta}{p}}. \tag{6.5}$$

*Proof.* Since $I_p$ is $(G, B_p)$-stabilizable then similarly to the above we have

$$\mathcal{R}\big(G, B_p^{\alpha}G^{1-\alpha}, B_p\big) < I_p < \mathcal{R}\big(G, B_p^{\beta}G^{1-\beta}, B_p\big). \tag{6.6}$$

Computing as previous and using Lemma 6.1 we obtain

$$\begin{aligned} B_p^{1-\alpha}G^{1-\alpha}\Big(\frac{a^p + B_p^p}{2}\Big)^{\alpha/p}\Big(\frac{b^p + B_p^p}{2}\Big)^{\alpha/p} &< I_p^2 \\ &< B_p^{1-\beta}G^{1-\beta}\Big(\frac{a^p + B_p^p}{2}\Big)^{\beta/p}\Big(\frac{b^p + B_p^p}{2}\Big)^{\beta/p}. \end{aligned} \tag{6.7}$$

The desired result follows after a simple reduction, with the fact that

$$\Big(\frac{a^p+B_p^p}{2}\Big)\Big(\frac{b^p+B_p^p}{2}\Big)=\frac{3}{4}B_p^{2p}+\frac{1}{4}G^{2p}, \tag{6.8}$$

so completes the proof. □

Taking $p=1$ in the above theorem, with the fact that $B_1=A$ and $I_1=I$, we deduce the following result for bounding the identric mean $I$.

**Corollary 6.3.** *Let $\alpha,\beta$ be two real numbers such that*

$$A^{\alpha}G^{1-\alpha}<I<A^{\beta}G^{1-\beta}. \tag{6.9}$$

*Then there holds*

$$A^{1-\alpha}G^{1-\alpha}\Big(\frac{3}{4}A^2+\frac{1}{4}G^2\Big)^{\alpha}<I^2<A^{1-\beta}G^{1-\beta}\Big(\frac{3}{4}A^2+\frac{1}{4}G^2\Big)^{\beta}. \tag{6.10}$$

*Example* 6.1. Let $p>0$ be a real number, then we have $I_p<B_p$ and so the above theorem with $\beta=1$ immediately implies that

$$I_p^{2p}<\frac{3}{4}B_p^{2p}+\frac{1}{4}G^{2p}. \tag{6.11}$$

In particular, for $p=1$ the double inequality (6.10) is reduced to

$$AG<I^2<\frac{3}{4}A^2+\frac{1}{4}G^2, \tag{6.12}$$

which refines the arithmetic-identric-geometric mean inequality $G<I<A$.

**Theorem 6.4.** *Let $\alpha\in[0,1]$ be such that*

$$B_p^{\alpha}G^{1-\alpha}<(>)I_p \tag{6.13}$$

*for some $p>(<)0$, respectively. Then one has*

$$B_p^{\frac{2+\alpha}{4}}G^{\frac{2-\alpha}{4}}<(>)I_p. \tag{6.14}$$

*If moreover $\alpha<(>)2/3$ then (6.14) refines (6.13).*

*Proof.* Similar to that of the above. We left the detail for the reader as an interesting exercise. □

As previously, taking $p=1$ in the above theorem we immediately obtain the following result.

**Corollary 6.5.** *Let $\alpha$ be a real number satisfying that*

$$A^{\alpha}G^{1-\alpha}<I. \tag{6.15}$$

*Then one has*

$$A^{\frac{2+\alpha}{4}}G^{\frac{2-\alpha}{4}}<I. \tag{6.16}$$

*If moreover $\alpha<2/3$ then (6.16) refines (6.15).*

**Corollary 6.6.** *Let $p$ be a real number. If $p>0$ then one has*

$$B_p^{2/3}G^{1/3}<I_p. \tag{6.17}$$

*If $p<0$ then the above inequality is reversed. In particular the following inequality holds true*

$$A^{2/3}G^{1/3}<I. \tag{6.18}$$

*Proof.* We proceed by similar manner as previous. The obtained sequence here is $(\alpha_n)_n$ such that

$$\alpha_{n+1} = \frac{2+\alpha_n}{4}, \quad \text{with} \quad \alpha_0 = \alpha, \tag{6.19}$$

which converges to 2/3. We conclude by analogous arguments as previous. □

*Remark* 6.1. The inequality (6.18) has been proved by different methods, see [4] for comparison. We left the reader to state analogous ways about inequality (6.17) as in Remark 5.2.

As the reader can verify it, analogue of Theorem 5.6 for $I_p$ has length expression and makes appear hard computations.

We left to the reader the routine task for considering other mean-inequalities, involving the standard means, in the aim to obtain more lower and/or upper bounds for a stabilizable mean, eventually with some related refinements. As example, we can state the following.

**Theorem 6.7.** *Let $\alpha \in [0,1]$ be a real number such that*

$$A^{\alpha}G^{1-\alpha} < (>) L_p \tag{6.20}$$

*for some $p \geq (\leq) 0$. Then there holds*

$$A^{1-\alpha}G^{1-\alpha}\Big(\frac{3}{4}A^2 + \frac{1}{4}G^2\Big)^{\alpha} < (>) L_p^2. \tag{6.21}$$

**Theorem 6.8.** *Let $\alpha, \beta \in [0,1]$ be two real numbers such that*

$$\alpha A + (1-\alpha)G < D_p < \beta A + (1-\beta)G \tag{6.22}$$

*for some $p$. Then we have*

$$\alpha\frac{A+B_p}{2} + (1-\alpha)\Big(\frac{1}{2}AB_p + \frac{1}{2}GB_p\Big)^{1/2} < D_p < \beta\frac{A+B_p}{2} + (1-\beta)\Big(\frac{1}{2}AB_p + \frac{1}{2}GB_p\Big)^{1/2}. \tag{6.23}$$

We omit the proofs of the above results here. Of course, for the proof of Theorem 6.7 we use the fact that $L_p$ is $(B_p, A)$-stabilizable while that of Theorem 6.8 uses $D_p$ is $(A, B_p)$-stabilizable. Some consequences can also be derived from the above theorems in a similar manner as previous. In particular for $p = 0$, Theorem 6.7 coincides with Corollary 6.3 while Theorem 6.8 is reduced to Corollary 5.7. We left all these details to the interested reader.

In summary, the stability and stabilizability concepts are good tool for obtaining a lot of mean-inequalities in a short and nice manner. In particular, some mean-inequalities, already differently proved by many authors in the literature, have been here obtained as consequences via a procedure having a general point of view. This shows the interest of this work derived from the stabilizability concept.

Finally, as the reader can remark it, some other means known in the literature have not been mentioned in the above. As example, the Steiffert's mean $P$ defined by

$$P(a,b) = \frac{b-a}{4Arctan\sqrt{\frac{b}{a}} - \pi}, \quad P(a,a) = a, \tag{6.24}$$

has not been considered here. In fact, the author [9] conjectured that the mean $P$ defined by (6.24) is not stabilizable and such problem still open. In this direction, we indicate a recent paper [11] for further comments about this latter point.

## References


[1] H. Alzer and S.-Qiu, Inequalities for Means in two Variables, Archiv der Mathematik, Vol.80, No.2 (2003), 201-215.
[2] H. Alzer and W. Janous, Solution of Problem 8*, Crux Mathematicorum, Vol.13 (1987), 173-178.
[3] B.C. Carlson, The Logarithmic Mean, The American Mathematical Monthly, Vol.79 (1972), 615-618.
[4] Ch.-P. Chen, On some Inequalities for Means and the second Gautschi-Kershaw's Inequality, RGMIA, Vol. 11 Supplement (2008), Art. 6.
[5] E.B. Leach and M.C. Sholander, Extended mean values II,J. Math. Anal. Appl. 92 (1983), No. 1, 207-223.
[6] W-M. Qian and N-G. Zheng, An Optimal Double Inequality for Means, Journal of Inequalities and Applications, Vol. 2010, Art. ID 578310, 11 pages.
[7] M. Raïssouli, United Explicit Form for a Game of Monotone and Chaotic Matrix Means, International Electronic Journal of Pure and Applied Mathematics, 1 (2010), No.4, 475-493.
[8] M. Raïssouli, Approaching the Power Logarithmic and Difference Means by Algorithms involving the Power Binomial Mean, International Journal of Mathematics and Mathematical Sciences, Volume 2011 (2011), Article ID 687825, 12 pages.
[9] M. Raïssouli, Stability and Stabilizability for Means, Applied Mathematics E-Notes, 11 (2011), 159-174.
[10] M. Raïssouli, Stabilizability of the Stolarsky Mean and its Approximation in terms of the Power Binomial Mean. Submitted to Rocky Mountain Journal of Mathematics, May 2011.
[11] M. Raïssouli, Sub-Stabilizable and Super-Stabilizable Means. In preparation.
[12] M. Raïssouli, Stability and Stabilizability for Means involving several Arguments. In preparation.
[13] J. Sāndor, A note on some Inequalities for Means, Archiv der Mathematik, Vol.56, No.5 (1991), 471-473.
[14] J. Sāndor, Two inequalities for means, Internat. J. Math. Math. Sci. 18, (1995), No.3, 621-623.
[15] J. Sāndor, On certain Inequalities for Means. II, J.Math.Anal.Appl. 199 (1996), No.2, 629-635.
[16] K.B. Stolarsky, Generalizations of the logarithmic mean, Math. Mag. 48 (1975), 87-92.
[17] K.B. Stolarsky, The power and generalized logarithmic means, Amer. Math. Monthly, 87 (1980), 545-548.



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