Received 09/08/11

REFINEMENTS OF HERMITE-HADAMARD TYPE INEQUALITY FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE QUASI CONVEX

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ABSTRACT. In this paper, some new inequalities of Hermite-Hadamard type for quasiconvex functions are given. Applications to some special means are considered.

Keywords: Hermite-Hadamard inequality, quasiconvex functions, special means

1. INTRODUCTION AND PRELIMINARY

Let $f : I \to \mathbb{R}$ be a convex function on interval $I \subseteq \mathbb{R}$ and let $a, b \in I$ with $a < b$. We consider the well-known Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

In recent years, many authors established several inequalities connected to Hermite-Hadamard inequality, see [2, 3, 4, 5, 6, 7, 12, 14] and references therein. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In [8] Dragomir and Agarwal established the following results connected with the right hand side of (1) as well as to apply them for some elementary inequalities for real numbers and numerical integration.

Theorem 1.1. *Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{2}$$

Theorem 1.2. *Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}.\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{3}$$

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In [13] Pearce and Pečarić proved the following theorem:

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, for $q \geq 1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}. \quad (4)$$

In recent years several extensions and generalizations have been considered for classical convexity. We recall that the notion of quasiconvex functions generalizes the notion of convex functions. More exactly, a function $f : [a, b] \to \mathbb{R}$ is said to be quasiconvex if for every $x, y \in I$ we have

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}, \text{ for all } t \in [0, 1].$$

Ion in [11] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some quasiconvex functions are involved. The main results of [11] are given by the following theorems.

Theorem 1.4. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is quasiconvex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)\max\{|f'(a)|, |f'(b)|\}}{4}. \quad (5)$$

Theorem 1.5. *Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is quasiconvex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}.\left[\max\left\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\right\}\right]^{\frac{p-1}{p}}. \quad (6)$$

Note that, Theorems 1.4 and 1.5 are generalizations of Theorems 1.1 and 1.2, respectively. S.S. Dragomir in [9, 10] discussed inequalities for twice differentiable functions concerning with Hermite-Hadamard inequality on the basis of the following Lemma.

Lemma 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If f'' is integrable on $[a, b]$, then the following equality holds,*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt. \quad (7)$$

Then, Alomari, Darus and Dragomir in [1] by using Lemma 1.1 introduced the following theorems for twice differentiable quasiconvex functions which are generalizations of Theorems 1.3, 1.4 and 1.5.

Theorem 1.6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and f'' is integrable on $[a, b]$. If $|f''|$ is quasiconvex on $[a, b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\max\left\{\left|f''(a)\right| + \left|f''(b)\right|\right\}. \quad (8)$$

Theorem 1.7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and f'' is integrable on $[a, b]$. If $|f''|^{p/p-1}$ is quasiconvex on $[a, b]$, for $p > 1$, then the following* *inequality holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left(\max\left\{\left|f''(a)\right|^q+\left|f''(b)\right|^q\right\}\right)^{1/q},
\end{aligned} \tag{9}$$

where $1/p+1/q=1$.

Theorem 1.8. *Let* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *be a twice differentiable function on* I°, $a,b\in I^\circ$ *with* $a<b$ *and* f'' *is integrable on* $[a,b]$. *If* $|f''|^q$ *is quasiconvex on* $[a,b]$, *for* $q\geq 1$, *then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12}\left(\max\left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\right)^{1/q}. \tag{10}$$

The main purpose of this paper is to establish refinements inequalities of the results in [1]. We obtained new inequalities related to the right-hand side of Hermite-Hadamard inequality for functions whose second derivatives absolute values are quasiconvex. Then, we give some applications for special means of real numbers.

2. Hermite-Hadamard type inequalities

We will establish some new results connected with the right-hand side of (1) used the new following Lemma.

Lemma 2.1. *Suppose that* $f: I\to\mathbb{R}$ *is a twice differentiable function on* I°, *the interior of* I. *Assume that* $a,b\in I^\circ$, *with* $a<b$ *and* f'' *is integrable on* $[a,b]$. *Then, the following equality holds,*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \\
&=\frac{(b-a)^2}{16}\int_0^1(1-t^2)\left(f''(\frac{1+t}{2}a+\frac{1-t}{2}b)+f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right)dt.
\end{aligned} \tag{11}$$

Proof. It suffices to note that

$$\begin{aligned}
I_1 &= \int_0^1(1-t^2)f''(\frac{1+t}{2}a+\frac{1-t}{2}b)dt \\
&= \left[\frac{2}{a-b}(1-t^2)f'(\frac{1+t}{2}a+\frac{1-t}{2}b)\right]_0^1+\frac{4}{a-b}\int_0^1 tf'(\frac{1+t}{2}a+\frac{1-t}{2}b)dt \\
&= -\frac{2}{a-b}f'(\frac{a+b}{2})+\frac{4}{a-b}\int_0^1 tf'(\frac{1+t}{2}a+\frac{1-t}{2}b)dt \\
&= -\frac{2}{a-b}f'(\frac{a+b}{2})+\frac{4}{a-b}\left(\left[\frac{2}{a-b}tf(\frac{1+t}{2}a+\frac{1-t}{2}b)\right]_0^1\right. \\
&\left.-\frac{2}{a-b}\int_0^1 f(\frac{1+t}{2}a+\frac{1-t}{2}b)dt\right) \\
&= -\frac{2}{a-b}f'(\frac{a+b}{2})+\frac{8}{(a-b)^2}f(a)-\frac{8}{(a-b)^2}\int_0^1 f(\frac{1+t}{2}a+\frac{1-t}{2}b)dt.
\end{aligned}$$

Let $x=\frac{1+t}{2}a+\frac{1-t}{2}b$, hence $dx=\frac{a-b}{2}dt$ and we get

$$I_1=\frac{2}{b-a}f'(\frac{a+b}{2})+\frac{8}{(b-a)^2}f(a)-\frac{16}{(b-a)^3}\int_a^{\frac{a+b}{2}}f(x)dx. \tag{12}$$

Similarly, we can show that

$$\begin{aligned}I_2&=\int_0^1(1-t^2)f''(\frac{1-t}{2}a+\frac{1+t}{2}b)dt\\&=-\frac{2}{b-a}f'(\frac{a+b}{2})+\frac{8}{(b-a)^2}f(b)-\frac{16}{(b-a)^3}\int_{\frac{a+b}{2}}^b f(x)dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{(b-a)^2}{16}\left[I_1+I_2\right]\\&=\frac{(b-a)^2}{16}\left[\frac{8}{(b-a)^2}\left(f(b)+f(b)\right)-\frac{16}{(b-a)^3}\int_a^b f(x)dx\right]\\&=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx,\end{aligned}$$

which is required. □

In the following theorem, we shall propose some new upper bound for the right-hand side of Hermite-Hadamard inequality for functions whose second derivatives absolute values are quasi-convex, which is better than the inequality had done in [1].

Theorem 2.1. *Let $f:I\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L_1[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f''|$ is quasiconvex on $[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\le\frac{(b-a)^2}{24}\left[\max\left\{\left|f''(a)\right|,\left|f''(\frac{a+b}{2})\right|\right\}+\max\left\{\left|f''(b)\right|,\left|f''(\frac{a+b}{2})\right|\right\}\right].\end{aligned} \tag{13}$$

Proof. By Lemma 2.1 and quasiconvexity of $|f''|$ we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\le\left|\frac{(b-a)^2}{16}\int_0^1(1-t^2)\left(f''(\frac{1+t}{2}a+\frac{1-t}{2}b)+f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right)dt\right|\\&\le\frac{(b-a)^2}{16}\int_0^1\left|1-t^2\right|\left(\left|f''(\frac{1+t}{2}a+\frac{1-t}{2}b)\right|+\left|f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right|\right)dt\\&\le\frac{(b-a)^2}{16}\left[\max\left\{\left|f''(a)\right|,\left|f''(\frac{a+b}{2})\right|\right\}\right.\\&\left.+\max\left\{\left|f''(b)\right|,\left|f''(\frac{a+b}{2})\right|\right\}\right]\int_0^1\left(1-t^2\right)dt\\&=\frac{(b-a)^2}{24}\left[\max\left\{\left|f''(a)\right|,\left|f''(\frac{a+b}{2})\right|\right\}+\max\left\{\left|f''(b)\right|,\left|f''(\frac{a+b}{2})\right|\right\}\right].\end{aligned}$$

□

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let f as in Theorem 2.1, if in addition*
(i) $|f''|$ is increasing, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{24}\left[\left|f''(b)\right|+\left|f''(\frac{a+b}{2})\right|\right]. \tag{14}$$

(ii) $|f''|$ is decreasing, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{24}\left[\left|f''(a)\right|+\left|f''(\frac{a+b}{2})\right|\right]. \tag{15}$$

Remark 2.1. We note that the inequalities (14) and (15) are two new refinements of the trapezoid inequality for quasiconvex functions, and thus for convex functions.

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following theorem.

Theorem 2.2. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^{p/p-1}$ is quasiconvex on $[a,b]$, for $p > 1$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{(b-a)^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left[\left(\max\left\{\left|f''(a)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right. \\ &\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right],\end{aligned} \tag{16}$$

where $1/p + 1/q = 1$.

Proof. By lemma 2.1 and using well known Hölder integral inequality we have,

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \left|\frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left(f''(\frac{1+t}{2}a+\frac{1-t}{2}b)+f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right)dt\right| \\ &\le \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p\,dt\right)^{1/p}\left[\left(\int_0^1\left|f''(\frac{1+t}{2}a+\frac{1-t}{2}b)\right|^q dt\right)^{1/q}\right. \\ &\left.+\left(\int_0^1\left|f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right|^q dt\right)^{1/q}\right]dt \\ &\le \frac{(b-a)^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left[\left(\max\left\{\left|f''(a)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right. \\ &\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right],\end{aligned}$$

where $1/p + 1/q = 1$. We note that, the Beta and Gamma functions are defined respectively, as follows

$$\Gamma(x) = \int_0^1 e^{-x} t^{x-1} dt, \qquad x > 0,$$

and

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \qquad x > 0,\ y > 0,$$

are used to evaluate the integral $\int_0^1 \left(1 - t^2\right)^p dt$. Indeed, by setting $t^2 = u$, we get

$$dt = \frac{1}{2} u^{-1/2} du,$$

and using property

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},$$

of Beta function, we obtain

$$\begin{aligned}\int_0^1 \left(1 - t^2\right)^p dt &= \frac{1}{2} \int_0^1 u^{-1/2} (1 - u)^p du = \frac{1}{2} \beta(\frac{1}{2}, p + 1) \\ &= 2^{-1} \frac{\Gamma(\frac{1}{2})\Gamma(1 + p)}{\Gamma(\frac{3}{2} + p)} = 2^{-1} \frac{\sqrt{\pi}\ \Gamma(1 + p)}{\Gamma(\frac{3}{2} + p)} \\ &= \left(\frac{\sqrt{\pi}}{2}\right) \frac{\Gamma(1 + p)}{\Gamma(\frac{3}{2} + p)},\end{aligned}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and proof is completed. □

Corollary 2.2. *Let f as in Theorem 2.2, if in addition*
(i) $|f''|^{p/p-1}$ is increasing, then we have

$$\begin{aligned}&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \\ &\leq \frac{(b - a)^2}{16} \left(\frac{\sqrt{\pi}}{2}\right)^{1/p} \left(\frac{\Gamma(1 + p)}{\Gamma(\frac{3}{2} + p)}\right)^{1/p} \left[|f''(b)| + \left| f''(\frac{a + b}{2}) \right| \right].\end{aligned} \tag{17}$$

(ii) $|f''|^{p/p-1}$ is decreasing, then we have

$$\begin{aligned}&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x)dx \right| \\ &\leq \frac{(b - a)^2}{16} \left(\frac{\sqrt{\pi}}{2}\right)^{1/p} \left(\frac{\Gamma(1 + p)}{\Gamma(\frac{3}{2} + p)}\right)^{1/p} \left[|f''(a)| + \left| f''(\frac{a + b}{2}) \right| \right].\end{aligned} \tag{18}$$

A more general inequality is given using Lemma 2.1, as follows:

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is quasiconvex on $[a, b]$, for $q \geq 1$, then the following inequality* *holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{24}\left[\left(\max\left\{\left|f''(a)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right. \tag{19}$$
$$\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right],$$

where $1/p+1/q=1$.

Proof. By lemma 2.1 and using well known power mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \left|\frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left(f''(\frac{1+t}{2}a+\frac{1-t}{2}b)+f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right)dt\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)\,dt\right)^{1-1/q}\left[\left(\int_0^1 (1-t^2)\left|f''(\frac{1+t}{2}a+\frac{1-t}{2}b)\right|^q dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 (1-t^2)\left|f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right|^q dt\right)^{1/q}\right] dt$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-1/q}\left[\left(\frac{2}{3}\max\left\{\left|f''(a)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right.$$
$$\left.+\left(\frac{2}{3}\max\left\{\left|f''(b)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right]$$
$$=\frac{(b-a)^2}{24}\left[\left(\max\left\{\left|f''(a)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right.$$
$$\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right],$$

which completes the proof. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.3. *Let* f *as in Theorem 2.3, if in addition*
(i) $|f''|$ *is increasing, then (14) holds.*
(ii) $|f''|^{p/p-1}$ *is decreasing, then (15) holds.*

3. Applications to special means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers $\alpha,\beta(\alpha\neq\beta)$. We take
(1) Arithmetic mean:

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\qquad \alpha,\beta\in\mathbb{R}.$$

(2) Logarithmic mean:

$$L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|},\quad |\alpha|\neq|\beta|,\ \alpha,\beta\neq 0,\ \alpha,\beta\in\mathbb{R}.$$

(3) Generalized log-mean:

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\quad n\in\mathbb{N},\ \alpha,\beta\in\mathbb{R},\alpha\neq\beta.$$

Now, using the results of Section 2, we give some applications for special means of real numbers.

Proposition 3.1. *Let $a,b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then, we have*

$$\begin{aligned}&|L_n^n(a,b)-A(a^n,b^n)|\\ &\leq\frac{n(n-1)}{24}(b-a)^2\left[\max\left\{|a|^{n-2},\left|\frac{a+b}{2}\right|^{n-2}\right\}\right.\\ &\left.+\max\left\{|b|^{n-2},\left|\frac{a+b}{2}\right|^{n-2}\right\}\right].\end{aligned}$$

Proof. The assertion follows from Theorem 2.1 applied to the quasiconvex function $f(x)=x^n$, $x\in\mathbb{R}$. □

Proposition 3.2. *Let $a,b\in\mathbb{R}$, $a<b$ and $0\notin[0,1]$. Then, for all $p>1$ we have*

$$\begin{aligned}&\left|L^{-1}(a,b)-A(a^{-1},b^{-1})\right|\\ &\leq\frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\\ &\left[\left(\max\left\{|a|^{-3q},\left|\frac{a+b}{2}\right|^{-3q}\right\}\right)^{1/q}+\left(\max\left\{|b|^{-3q},\left|\frac{a+b}{2}\right|^{-3q}\right\}\right)^{1/q}\right],\end{aligned}$$

where $1/p+1/q=1$.

Proof. The assertion follows from Theorem 2.2 applied to the quasiconvex function $f(x)=\frac{1}{x}$, $x\in[a,b]$. □

Proposition 3.3. *Let $a,b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\geq 2$. Then, for all $q>1$ we have*

$$\begin{aligned}&|L_n^n(a,b)-A(a^n,b^n)|\\ &\leq\frac{n(n-1)}{24}(b-a)^2\left[\max\left\{|a|^{(n-2)q},\left|\frac{a+b}{2}\right|^{(n-2)q}\right\}\right.\\ &\left.+\max\left\{|b|^{(n-2)q},\left|\frac{a+b}{2}\right|^{(n-2)q}\right\}\right].\end{aligned}$$

Proof. The assertion follows from Theorem 2.3 applied to the quasiconvex function $f(x)=x^n$, $x\in\mathbb{R}$. □

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