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ON SOME NEW FEJÉR-TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper some new mappings associated with the Fejér inequality for double integrals are defined and as a consequence some new Fejér type inequalities for co-ordinated convex functions are established as well.

1. Introduction

It is well known in literature that a function $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$ if the inequality:

$$f(\lambda x+(1-\lambda)\,y)\leq\lambda f(x)+(1-\lambda)\,f(y),$$

holds for all $x,\,y\in[a,b]$ and $\lambda\in[0,1]$.

Many inequalities have been established for convex functions in past few years but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications [7, 8]:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following inequality gives the weighted version of (1.1), called Fejér's inequality [5]:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx\leq\frac{1}{b-a}\int_a^b f(x)p(x)dx\leq\frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{1.2}$$

for convex function $f:[a,b]\to\mathbb{R}$ and $p:[a,b]\to[0,\infty)$ an integrable and symmetric about $x=\frac{a+b}{2}$.

The inequalities (1.1) and (1.2) have been generalized, extended and refined in a number of ways (for instance see [2, 3, 4, 6, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein).

Consider now a bi-dimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\leq\alpha f(x,y)+(1-\alpha)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$.

In [2, 4], S. S. Dragomir introduced a new concept of convexity, called the co-ordinated convexity as:

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A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial functions $f_y : [a,b] \to \mathbb{R}$ and $f_x : [c,d] \to \mathbb{R}$ defined by $f_y(u) = f(u,y)$ and $f_x(v) = f(x,v)$ are convex for $(x,y) \in [a,b] \times [c,d]$.

A formal definition for co-ordinated convex functions may be stated in

Definition 1. *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if*

$$\begin{aligned}
&f(tx + (1-t)y, su + (1-s)w) \\
&\le tsf(x,u) + t(1-s)f(x,w) + s(1-t)f(y,u) + (1-t)(1-s)f(y,w)
\end{aligned}$$

holds for all $t, s \in [0,1]$ and $(x,u), (y,w) \in \Delta$.

Clearly, every convex function $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists a co-ordinated convex function which is not convex [2, 4].

In [4], an inequality of Hermite-Hadamard type for co-ordinated convex functions on a rectangle from the plane was established and some properties of functions associated to it were also proved. In [16], D. Y. Hwang et al. considered a monotonic nondecreasing function connected with Hadamard type inequalities in two variables and established some Hadamard type inequalities for Lipschizian function as well. Recently M. Alomari et al. [1], proved a Fejér inequality for double integrals and considered some functions associated to it to establish some inequalities for Lipschitzian functions.

The aim of this paper is to establish some new Fejér-type inequalities for co-ordinated convex functions on rectangle from the plane.

2. Main Results

Throughout the section let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinated convex function, $p : [a,b] \times [c,d] \to [0,\infty)$ be integrable and symmetric about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. We now define the following functions on $[0,1]^2$ associated with Fejér inequality for double integrals proved in [1]:

$$\begin{aligned}
G(t,s) = \frac{1}{4}\Bigg[& f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) + \\
& f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) \\
& + f\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) \\
& + f\left(tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right)\Bigg],
\end{aligned}$$

$$H(t,s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dydx,$$

$$H_p(t,s) = \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) p(x,y) dydx,$$

$$L(t,s)=\frac{1}{4(b-a)(d-c)}\int_a^b\int_c^d\left[f\left(ta+(1-t)x,sc+(1-s)y\right)\right.$$
$$+f\left(ta+(1-t)x,sd+(1-s)y\right)+f\left(tb+(1-t)x,sc+(1-s)y\right)$$
$$\left.+f\left(tb+(1-t)x,sd+(1-s)y\right)\right],$$

and

$$L_p(t,s)=\frac{1}{4}\int_a^b\int_c^d\left[f\left(ta+(1-t)x,sc+(1-s)y\right)\right.$$
$$+f\left(ta+(1-t)x,sd+(1-s)y\right)+f\left(tb+(1-t)x,sc+(1-s)y\right)$$
$$\left.+f\left(tb+(1-t)x,sd+(1-s)y\right)\right]p(x,y)dydx.$$

To prove our results we need:

Lemma 1. [1] *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and let*

$$a\le y_1\le x_1\le x_2\le y_2\le b \text{ with } x_1+x_2=y_1+y_2,$$

$$a\le w_1\le v_1\le v_2\le w_2\le b \text{ with } v_1+v_2=w_1+w_2.$$

Then, for the convex partial functions $f_y:[a,b]\to\mathbb{R}$ and $f_x:[c,d]\to\mathbb{R}$ defined by $f_y(u)=f(u,y)$ and $f_x(v)=f(x,v)$ for $(x,y)\in[a,b]\times[c,d]$, the followings hold:

$$f(x_1,v)+f(x_2,v)\le f(y_1,v)+f(y_2,v), \text{ for all } v\in[c,d] \tag{2.1}$$

and

$$f(u,v_1)+f(u,v_2)\le f(u,w_1)+f(u,w_2), \text{ for all } u\in[a,b]. \tag{2.2}$$

We begin with following results:

Theorem 1. *Let f, p, H_p be defined as above, then*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx$$
$$\le 4\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y)p\left(2x-\frac{a+b}{2},2y-\frac{c+d}{2}\right)dydx\le\int_0^1\int_0^1 H_p(t,s)dsdt$$
$$\le\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx+\right.$$
$$\left.\int_a^b\int_c^d\left\{f(x,y)+f\left(\frac{a+b}{2},y\right)+f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dydx\right]. \tag{2.3}$$

Proof. By using the simple techniques of integration, under the assumptions on p, the following identities hold:

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx$$
$$=16\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)p(x,y)dsdtdydx, \tag{2.4}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y)p\left(2x-\frac{a+b}{2},2y-\frac{c+d}{2}\right)dydx$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right)\right.$$
$$+f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3(c+d)}{4}-\frac{y}{2}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)$$
$$\left.+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right]p(x,y)dsdtdydx, \quad (2.5)$$

$$\int_0^1\int_0^1 H_p(s,t)dsdt$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(t\frac{a+b}{2}+(1-t)x,s\frac{c+d}{2}+(1-s)y\right)\right.$$
$$+f\left(t\frac{a+b}{2}+(1-t)x,sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tx+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right)$$
$$\left.+f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dsdtdydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)\right.$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[\left(t\frac{a+b}{2}+(1-t)x,s\frac{c+d}{2}+(1-s)(c+d-y)\right)\right.$$
$$+f\left(t\frac{a+b}{2}+(1-t)x,s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tx+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)$$
$$\left.+f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dsdtdydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right) \quad (2.6)$$

$$+ f\left(t\frac{a+b}{2}+(1-t)(a+b-x), sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t\frac{a+b}{2}+(1-t)(a+b-x), s\frac{c+d}{2}+(1-s)y\right)\Bigg] p(x,y)dsdtdydx \quad (2.7)$$

and

$$\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx+\int_a^b\int_c^d\left\{f(x,y)+f\left(\frac{a+b}{2},y\right)\right.\right.$$
$$\left.\left.+f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dydx\right]$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{f(x,y)+f\left(\frac{a+b}{2},y\right)+f\left(x,\frac{c+d}{2}\right)\right.$$
$$\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$+f(a+b-x,c+d-y)+f\left(\frac{a+b}{2},c+d-y\right)$$
$$\left.+f\left(a+b-x,\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{f(x,c+d-y)+f\left(\frac{a+b}{2},c+d-y\right)+f\left(x,\frac{c+d}{2}\right)\right.$$
$$\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\{f(a+b-x,y)$$
$$\left.f\left(\frac{a+b}{2},y\right)+f\left(a+b-x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}p(x,y)dsdtdydx. \quad (2.8)$$

By Lemma 1, the following inequalities hold for all $(t,s)\in\left[0,\frac{1}{2}\right]^2$, $(x,y)\in[a,\frac{a+b}{2}]\times[c,\frac{c+d}{2}]$:
By setting $y_1=\frac{x}{2}+\frac{a+b}{4}$, $x_1=x_2=\frac{a+b}{2}$, $y_2=\frac{3(a+b)}{4}-\frac{x}{2}$ in (2.1), for all $v\in[c,\frac{c+d}{2}]$, we observe that

$$2f\left(\frac{a+b}{2},v\right)\le f\left(\frac{x}{2}+\frac{a+b}{4},v\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},v\right), \quad (2.9)$$

holds.
Multiplying both sides of the inequality (2.9) by 2, replacing $v=\frac{c+d}{2}$ and then applying (2.2) for $w_1=\frac{y}{2}+\frac{c+d}{4}$, $v_1=v_2=\frac{c+d}{2}$, $w_2=\frac{3(c+d)}{4}-\frac{x}{2}$ to both of the expressions on right-side of (2.9), we have that the following inequality:

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\le\left[f\left(\frac{x}{2}+\frac{a+b}{4},\frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{x}{2}+\frac{a+b}{4},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right.$$
$$\left.+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\right], \quad (2.10)$$

holds.
Now by choosing $y_1 = t\frac{a+b}{2} + (1-t)\,x$, $x_1 = x_2 = \frac{x}{2} + \frac{a+b}{4}$, $y_2 = tx + (1-t)\,\frac{a+b}{2}$ in (2.1), the following inequality holds:

$$2f\left(\frac{x}{2} + \frac{a+b}{4}, v\right) \leq f\left(t\frac{a+b}{2} + (1-t)\,x, v\right) + f\left(tx + (1-t)\,\frac{a+b}{2}, v\right), \quad (2.11)$$

for all $v \in [c, \frac{c+d}{2}]$.
By replacing $y_1 = t\,(a+b-x) + (1-t)\,\frac{a+b}{2}$, $x_1 = x_2 = \frac{3(a+b)}{4} - \frac{x}{2}$, $y_2 = t\frac{a+b}{2} + (1-t)\,(a+b-x)$ in (2.1), we notice that

$$2f\left(\frac{3\,(a+b)}{4} - \frac{x}{2}, v\right) \leq f\left(t\,(a+b-x) + (1-t)\,\frac{a+b}{2}, v\right) + f\left(t\frac{a+b}{2} + (1-t)\,(a+b-x)\,, v\right), \quad (2.12)$$

holds for all $v \in [c, \frac{c+d}{2}]$.
Multiplying both sides of (2.11) and (2.12) by 2, setting $v = \frac{y}{2} + \frac{c+d}{2}$ and $v = \frac{3(c+d)}{4} - \frac{y}{2}$, then using (2.2) for the particular choices of w_1, w_2, v_1 and v_2, we have that the following inequalities hold:

$$\begin{aligned} 4f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) &\leq f\left(t\frac{a+b}{2} + (1-t)\,x, s\frac{c+d}{2} + (1-s)\,y\right) \\ &+ f\left(t\frac{a+b}{2} + (1-t)\,x, sy + (1-s)\,\frac{c+d}{2}\right) \\ &+ f\left(tx + (1-t)\,\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)\,y\right) \\ &+ f\left(tx + (1-t)\,\frac{a+b}{2}, sy + (1-s)\,\frac{c+d}{2}\right), \quad (2.13) \end{aligned}$$

$$\begin{aligned} 4f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{3\,(c+d)}{4} - \frac{y}{2}\right) &\leq f\left(t\frac{a+b}{2} + (1-t)\,x, s\frac{c+d}{2} + (1-s)\,(c+d-y)\,y\right) \\ &+ f\left(t\frac{a+b}{2} + (1-t)\,x, s\,(c+d-y) + (1-s)\,\frac{c+d}{2}\right) \\ &+ f\left(tx + (1-t)\,\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)\,(c+d-y)\right) \\ &+ f\left(tx + (1-t)\,\frac{a+b}{2}, s\,(c+d-y) + (1-s)\,\frac{c+d}{2}\right), \quad (2.14) \end{aligned}$$

$$\begin{aligned}
4f&\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{3(c+d)}{4}-\frac{y}{2}\right)\\
&\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)(c+d-y)\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)(c+d-y)\right) \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
4f&\left(\frac{3(a+b)}{4}-\frac{x}{2},\frac{y}{2}+\frac{c+d}{4}\right)\\
&\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\frac{c+d}{2}+(1-s)y\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),sy+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),s\frac{c+d}{2}+(1-s)y\right). \qquad (2.16)
\end{aligned}$$

By setting $y_1=x$, $x_1=t\frac{a+b}{2}+(1-t)x$, $x_2=tx+(1-t)\frac{a+b}{2}$, $y_2=\frac{a+b}{2}$ in (2.1), the following holds:

$$\begin{aligned}
f&\left(t\frac{a+b}{2}+(1-t)x,v\right)+f\left(tx+(1-t)\frac{a+b}{2},v\right)\\
&\leq f(x,v)+f\left(\frac{a+b}{2},v\right), \qquad (2.17)
\end{aligned}$$

for all $v\in[c,\frac{c+d}{2}]$.

By the choice of $y_1=\frac{a+b}{2}$, $x_1=t(a+b-x)+(1-t)\frac{a+b}{2}$, $x_2=t\frac{a+b}{2}+(1-t)(a+b-x)$, $y_2=a+b-x$ in (2.1), we have that the following inequality:

$$\begin{aligned}
f&\left(t(a+b-x)+(1-t)\frac{a+b}{2},v\right)+f\left(t\frac{a+b}{2}+(1-t)(a+b-x),v\right)\\
&\leq f(a+b-x,v)+f\left(\frac{a+b}{2},v\right), \qquad (2.18)
\end{aligned}$$

holds for all $v\in[c,\frac{c+d}{2}]$.

By respective settings $v=s\frac{c+d}{2}+(1-s)y$, $v=sy+(1-s)\frac{c+d}{2}$, $v=s\frac{c+d}{2}+(1-s)(c+d-y)$ and $v=s(c+d-y)+(1-s)\frac{c+d}{2}$ in (2.17) and (2.18), using (2.2) for the particular choices of w_1, w_2, v_1 and v_2 and then summing up the resulting inequalities we obtain

$$
\begin{aligned}
f&\left(t\frac{a+b}{2}+(1-t)\,x, s\frac{c+d}{2}+(1-s)\,y\right)\\
&\quad+f\left(t\frac{a+b}{2}+(1-t)\,x, sy+(1-s)\,\frac{c+d}{2}\right)\\
&\quad+f\left(tx+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,y\right)\\
&\quad+f\left(tx+(1-t)\,\frac{a+b}{2}, sy+(1-s)\,\frac{c+d}{2}\right)\\
&\leq f\,(x,y)+f\left(x,\frac{c+d}{2}\right)\\
&\quad+f\left(\frac{a+b}{2},y\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right), \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
f&\left(t\frac{a+b}{2}+(1-t)\,x, s\frac{c+d}{2}+(1-s)\,(c+d-y)\right)\\
&\quad+f\left(t\frac{a+b}{2}+(1-t)\,x, s\,(c+d-y)+(1-s)\,\frac{c+d}{2}\right)\\
&\quad+f\left(tx+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,(c+d-y)\right)\\
&\quad+f\left(tx+(1-t)\,\frac{a+b}{2}, s\,(c+d-y)+(1-s)\,\frac{c+d}{2}\right)\\
&\leq f\,(x,c+d-y)+f\left(x,\frac{c+d}{2}\right)\\
&\quad+f\left(\frac{a+b}{2},c+d-y\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right), \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
f&\left(t\frac{a+b}{2}+(1-t)\,(a+b-x)\,, s\frac{c+d}{2}+(1-s)\,y\right)\\
&\quad+f\left(t\frac{a+b}{2}+(1-t)\,(a+b-x)\,, sy+(1-s)\,\frac{c+d}{2}\right)\\
&\quad+f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,y\right)\\
&\quad+f\left(t\,(a+b-x)+(1-t)\,\frac{a+b}{2}, sy+(1-s)\,\frac{c+d}{2}\right)\\
&\leq f\,(a+b-x,y)+f\left(a+b-x,\frac{c+d}{2}\right)\\
&\quad+f\left(\frac{a+b}{2},y\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (2.21)
\end{aligned}
$$

and

$$\begin{aligned}
&f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2},s\frac{c+d}{2}+\left(1-s\right)\left(c+d-y\right)\right)\\
&\quad+f\left(t\left(a+b-x\right)+\left(1-t\right)\frac{a+b}{2},s\left(c+d-y\right)+\left(1-s\right)\frac{c+d}{2}\right)\\
&\quad+f\left(t\frac{a+b}{2}+\left(1-t\right)\left(a+b-x\right),s\left(c+d-y\right)+\left(1-s\right)\frac{c+d}{2}\right)\\
&\quad+f\left(t\frac{a+b}{2}+\left(1-t\right)\left(a+b-x\right),s\frac{c+d}{2}+\left(1-s\right)\left(c+d-y\right)\right)\\
&\qquad\le f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(a+b-x,c+d-y\right)\\
&\qquad\quad+f\left(\frac{a+b}{2},c+d-y\right)+f\left(a+b-x,\frac{c+d}{2}\right). \qquad (2.22)
\end{aligned}$$

Multiplying the inequalities (2.10)-(2.22) by $p(x,y)$ and integrating respectively over t on $[0,\frac{1}{2}]$, over s on $[0,\frac{1}{2}]$, over x on $[a,\frac{a+b}{2}]$ and over y on $[c,\frac{c+d}{2}]$ and using the identities (2.4)-(2.8), we derive (2.3). □

Theorem 2. *Let f, p, H_p be defined as above. Let f be twice differentiable on $[a,b]\times[c,d]$ such that the second order partial derivatives are continuous. If the first order partial derivatives of f are co-ordinated convex and p is bounded on $[a,b]\times[c,d]$, then*

$$\begin{aligned}
0&\ge H_p(t,s)-\int_a^b\int_c^d f(x,y)p(x,y)dydx\\
&\le\left(1-t\right)\left(1-s\right)\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\left(b-a\right)\left(d-c\right)\right.\\
&\qquad\left.-\int_a^b\int_c^d f\left(x,y\right)dydx\right]\left\|p\right\|_\infty, \qquad (2.23)
\end{aligned}$$

for $(s,t)\in[0,1]^2$ and $\left\|p\right\|_\infty=\sup_{(x,y)\in[a,b]\times[c,d]}\left|p(x,y)\right|$. And

$$\begin{aligned}
0&\ge 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx-H_p(t,s)\\
&\quad-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx\\
&\le\frac{\left(b-a\right)\left(d-c\right)}{16}\left[\frac{\partial^2 f(a,d)}{\partial y\partial x}+\frac{\partial^2 f(b,c)}{\partial y\partial x}-\frac{\partial^2 f(a,c)}{\partial y\partial x}-\frac{\partial^2 f(b,d)}{\partial y\partial x}\right]\\
&\qquad\times\int_a^b\int_c^d p(x,y)dydx. \qquad (2.24)
\end{aligned}$$

Proof. Using the substitution rules for integration, under the assumptions on p, the following identities hold:

$$\int_a^b\int_c^d f(x,y)p(x,y)dydx$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(a+b-x,y)p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,c+d-y)p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(a+b-x,c+d-y)p(x,y)dydx \quad (2.25)$$

and

$$H_p(t,s)$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]p(x,y)dydx. \quad (2.26)$$

By integrating by parts, we also have the following identity:

$$\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\left[\frac{\partial^2 f(a+b-x,c+d-y)}{\partial x\partial y}\right.$$
$$\left.-\frac{\partial^2 f(a+b-x,y)}{\partial x\partial y}-\frac{\partial^2 f(x,c+d-y)}{\partial x\partial y}+\frac{\partial^2 f(x,y)}{\partial x\partial y}\right]dydx$$
$$=\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx$$
$$=\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c)\right.$$
$$\left.+\int_a^b\int_c^d f(x,y)\,dydx\right]$$
$$-\frac{d-c}{2}\int_a^b[f(x,c)+f(x,d)]\,dx$$
$$-\frac{b-a}{2}\int_c^d[f(a,y)+f(b,y)]\,dy. \quad (2.27)$$

Now using the co-ordinated convexity of the first order partial derivatives and that of f, under the assumptions on p, the inequality

$$\begin{aligned}
&\Bigg[f(a+b-x,c+d-y)-f\left(a+b-x,s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\quad-f\left(t(a+b-x)+(1-t)\frac{a+b}{2},c+d-y\right)\\
&\quad+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad+\Bigg[f(a+b-x,y)-f\left(a+b-x,sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-f\left(t(a+b-x)+(1-t)\frac{a+b}{2},y\right)\\
&\quad+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad+\Bigg[f(x,c+d-y)-f\left(x,s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\quad-f\left(tx+(1-t)\frac{a+b}{2},c+d-y\right)\\
&\quad+f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\quad+\Bigg[f(x,y)-f\left(tx+(1-t)\frac{a+b}{2},y\right)-f\left(x,sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\Bigg]p(x,y)\\
&\leq(1-t)(1-s)\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\Bigg[\frac{\partial^2 f(a+b-x,c+d-y)}{\partial x\partial y}\\
&\quad-\frac{\partial^2 f(x,c+d-y)}{\partial x\partial y}-\frac{\partial^2 f(a+b-x,y)}{\partial x\partial y}+\frac{\partial^2 f(x,y)}{\partial x\partial y}\Bigg]\|p\|_\infty, \quad (2.28)
\end{aligned}$$

holds for all $(s,t)\in[0,1]^2$ and $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.
Integrating (2.28) over $x$on $\left[a,\frac{a+b}{2}\right]$ and over y on $\left[c,\frac{c+d}{2}\right]$, using (2.25)-(2.27), the inequality [4, Theorem 1] and the facts $H_p(t,1)\leq H_p(1,1)$, $H_p(1,s)\leq H_p(1,1)$, the second inequality in (2.23) holds. By [1, Theorem 2.2] the first inequality in (2.23) does hold.
By the co-ordinated convexity of first order partial derivatives and that of f, the following inequalities hold:

$$\frac{-f(a,c)+f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq-\frac{(b-a)(d-c)}{16}\frac{\partial^2 f(a,c)}{\partial y\partial x},$$

$$\frac{-f(a,d)+f\left(\frac{a+b}{2},d\right)+f\left(a,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq\frac{(b-a)(d-c)}{16}\frac{\partial^2 f(a,d)}{\partial y\partial x},$$

$$\frac{-f(b,c)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq\frac{(b-a)(d-c)}{16}\frac{\partial^2 f(b,c)}{\partial y\partial x},$$

and

$$\frac{-f(b,d)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{4}\leq -\frac{(b-a)(d-c)}{16}\frac{\partial^2 f(b,d)}{\partial y\partial x}.$$

Adding these results, multiplying the resulting inequality by $p(x,y)$ then integrating over $(x,y)\in[a,b]\times[c,d]$, the following holds:

$$-\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx - f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\times$$
$$\int_a^b\int_c^d p(x,y)dydx+\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\right]$$
$$\times\int_a^b\int_c^d p(x,y)dydx\leq\frac{(b-a)(d-c)}{16}\times$$
$$\left[\frac{\partial^2 f(a,d)}{\partial y\partial x}+\frac{\partial^2 f(b,c)}{\partial y\partial x}-\frac{\partial^2 f(b,d)}{\partial y\partial x}-\frac{\partial^2 f(a,c)}{\partial y\partial x}\right]\int_a^b\int_c^d p(x,y)dydx. \quad (2.29)$$

Since

$$\frac{1}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\right]\int_a^b\int_c^d p(x,y)dydx$$
$$\geq 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx \quad (2.30)$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx\leq H(t,s)\leq\int_a^b\int_c^d f(x,y)p(x,y)dydx. \quad (2.31)$$

From (2.29)-(2.31), (2.24) does hold and this completes the proof. □

Theorem 3. *Let f, p, H_p, G be defined as above, then*

$$H_p(t,s)\leq G(t,s)\int_a^b\int_c^d p(x,y)dydx \quad \text{for all} \quad (t,s)\in[0,1]^2. \quad (2.32)$$

$$4\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y)p\left(2x-\frac{a+b}{2},2y-\frac{c+d}{2}\right)dydx\leq\frac{1}{4}\left[f\left(\frac{3a+b}{4},\frac{3c+d}{4}\right)\right.$$
$$\left.+f\left(\frac{3a+b}{4},\frac{c+3d}{4}\right)+f\left(\frac{a+3b}{4},\frac{3c+d}{4}\right)+f\left(\frac{a+3b}{4},\frac{c+3d}{4}\right)\right]\int_a^b\int_c^d p(x,y)dydx$$
$$\leq(b-a)(d-c)\int_0^1\int_0^1 G(t,s)p\left((1-t)a+tb,(1-s)c+sd\right)dsdt$$
$$\leq\frac{1}{4}\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{2}\left\{f\left(a,\frac{c+d}{2}\right)\right.\right.$$
$$\left.\left.+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right\}\right]\int_a^b\int_c^d p(x,y)dydx. \quad (2.33)$$

Let the first order partial derivatives of f be co-ordinated convex and let p be bounded on $[a,b]\times[c,d]$. If f has continuous second order partial derivatives on $[a,b]\times[c,d]$

then

$$0 \geq f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx - H(t,s) \leq (b-a)(d-c)\left[H(t,s) + G(t,s) - 2f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right] \|p\|_\infty, \quad (2.34)$$

where

$$\|p\|_\infty = \sup_{(x,y)\in[a,b]\times[c,d]} |p(x,y)|.$$

Proof. Using the simple techniques of integration, under the assumptions on p, the following does hold:

$$\begin{aligned} G(t,s) \int_a^b \int_c^d p(x,y)dydx = \Big[& f\left(ta + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) \\ & + f\left(ta + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right) \\ & + f\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right) \\ & + f\left(tb + (1-t)\frac{a+b}{2}, sd + (1-s)\frac{c+d}{2}\right)\Big] \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} p(x,y)dydx, \end{aligned} \quad (2.35)$$

for $(s,t) \in [0,1]^2$.
By setting, $y_1 = ta + (1-t)\frac{a+b}{2}$, $x_1 = tx + (1-t)\frac{a+b}{2}$, $x_2 = t(a+b-x) + (1-t)\frac{a+b}{2}$, $y_2 = tb + (1-t)\frac{a+b}{2}$ in (2.1) for respective choices of $v = sy + (1-s)\frac{c+d}{2}$ and $v = s(c+d-y) + (1-s)\frac{c+d}{2}$, the followings hold true:

$$\begin{aligned} & f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \\ & \quad + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \\ & \quad \leq f\left(ta + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \\ & \quad\quad + f\left(tb + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) \end{aligned} \quad (2.36)$$

and

$$\begin{aligned} & f\left(tx + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2}\right) \\ & \quad + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2}\right) \\ & \quad \leq f\left(ta + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2}\right) \\ & \quad\quad + f\left(tb + (1-t)\frac{a+b}{2}, s(c+d-y) + (1-s)\frac{c+d}{2}\right), \end{aligned} \quad (2.37)$$

for $(x,y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$.

Adding (2.36) and (2.37), applying (2.2) for the particular choices of w_1, w_2, v_1 and v_2, multiplying the resulting inequality by $p(x,y)$, integrating over $(x,y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$ and using (2.35), we get (2.32).

By simple techniques of integration, under the assumptions on p, the following identities hold:

$$4\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\int_{\frac{3c+d}{4}}^{\frac{c+3d}{4}} f(x,y)p\left(2x-\frac{a+b}{2}, 2y-\frac{c+d}{2}\right)dydx$$
$$=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4}, \frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{x}{2}+\frac{a+b}{4}, \frac{3(c+d)}{4}-\frac{y}{2}\right)+\right.$$
$$\left.f\left(\frac{3(a+b)}{4}-\frac{x}{2}, \frac{y}{2}+\frac{c+d}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}, \frac{3(c+d)}{4}-\frac{y}{2}\right)\right]p(x,y)dydx, \tag{2.38}$$

$$\left[f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right)+f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right)+f\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right)\right.$$
$$\left.+f\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right)\right]\int_a^b\int_c^d p(x,y)dydx = 4\left[f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right)+f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right)\right.$$
$$\left.+f\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right)+f\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right)\right]\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} p(x,y)dydx, \tag{2.39}$$

$$(b-a)(d-c)\int_0^1\int_0^1 G(t,s)p((1-t)a+tb,(1-s)c+sd)\,dsdt = \frac{(b-a)(d-c)}{4}\times$$
$$\left[\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)p(ta+(1-t)b, sc+(1-s)d)\,dsdt\right.$$
$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)p((1-t)a+tb, sc+(1-s)d)\,dsdt$$
$$+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)p(ta+(1-t)b,(1-s)c+sd)\,dsdt$$
$$+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)p((1-t)a+tb,(1-s)c+sd)\,dsdt$$
$$+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)p(ta+(1-t)b, sc+(1-s)d)\,dsdt$$
$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)p((1-t)a+tb, sc+(1-s)d)\,dsdt$$
$$+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)p(ta+(1-t)b,(1-s)c+sd)\,dsdt$$
$$+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)p((1-t)a+tb,(1-s)c+sd)\,dsdt$$

$$+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, sc+(1-s)d\right) dsdt$$
$$+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, sc+(1-s)d\right) dsdt$$
$$+\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, (1-s)c+sd\right) dsdt$$
$$+\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, (1-s)c+sd\right) dsdt$$
$$+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, sc+(1-s)d\right) dsdt$$
$$+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, sc+(1-s)d\right) dsdt$$
$$+\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left(ta+(1-t)b, (1-s)c+sd\right) dsdt$$
$$\left.+\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) p\left((1-t)a+tb, (1-s)c+sd\right) dsdt\right]$$
$$=\frac{1}{4}\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}\left[f\left(\frac{b+x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+y}{2}\right)\right.$$
$$+f\left(\frac{a+x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{b+x}{2},\frac{d+y}{2}\right)$$
$$+f\left(\frac{2a+b-x}{2},\frac{d+y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+x}{2},\frac{d+y}{2}\right)$$
$$+f\left(\frac{a+x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{a+x}{2},\frac{c+y}{2}\right)$$
$$+f\left(\frac{a+2b-x}{2},\frac{d+y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+y}{2}\right)+f\left(\frac{b+x}{2},\frac{c+y}{2}\right)$$
$$\left.+f\left(\frac{b+x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{2c+d-y}{2}\right)\right]p(x,y)dydx$$

and

$$\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{2}\left\{f\left(a,\frac{c+d}{2}\right)\right.\right.$$
$$\left.\left.+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right\}\right]\int_{a}^{b}\int_{c}^{d}p(x,y)dydx$$
$$=\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{2}\left\{f\left(a,\frac{c+d}{2}\right)\right.\right.$$
$$\left.\left.+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right\}\right]\int_{a}^{\frac{a+b}{2}}\int_{c}^{\frac{c+d}{2}}p(x,y)dydx. \quad (2.40)$$

By choosing, $y_1 = \frac{3a+b}{4}$, $x_1 = \frac{x}{2} + \frac{a+b}{4}$, $x_2 = \frac{3(a+b)}{4} - \frac{x}{2}$, $y_2 = \frac{a+3b}{4}$ in (2.1) for respective settings $v = \frac{y}{2} + \frac{c+d}{4}$ and $v = \frac{3(c+d)}{4} - \frac{y}{2}$, the following hold true:

$$f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{y}{2} + \frac{c+d}{4}\right) \leq f\left(\frac{3a+b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) + f\left(\frac{a+3b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) \tag{2.41}$$

and

$$f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{3(c+d)}{4} - \frac{y}{2}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{3(c+d)}{4} - \frac{y}{2}\right) \leq f\left(\frac{3a+b}{4}, \frac{3(c+d)}{4} - \frac{y}{2}\right) + f\left(\frac{a+3b}{4}, \frac{3(c+d)}{4} - \frac{y}{2}\right), \tag{2.42}$$

for $(x, y) \in \left[a, \frac{a+b}{2}\right] \times \left[c, \frac{c+d}{2}\right]$.

Adding (2.41) and (2.42), applying (2.2) for the appropriate choices of w_1, w_2, v_1 and v_2, the following holds:

$$f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{y}{2} + \frac{c+d}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{y}{2} + \frac{c+d}{4}\right) + f\left(\frac{x}{2} + \frac{a+b}{4}, \frac{3(c+d)}{4} - \frac{y}{2}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}, \frac{3(c+d)}{4} - \frac{y}{2}\right) \leq f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) + f\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right) + f\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right) \tag{2.43}$$

Choosing $y_1 = \frac{x+a}{2}$, $x_1 = x_2 = \frac{3a+b}{4}$, $y_2 = \frac{2a+b-x}{2}$ in (2.1) for respective settings $v = \frac{3c+d}{4}$ and $v = \frac{c+3d}{4}$, the following hold true:

$$f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) \leq \frac{1}{2}\left[f\left(\frac{x+a}{2}, \frac{3c+d}{4}\right) + f\left(\frac{2a+b-x}{2}, \frac{3c+d}{4}\right)\right] \tag{2.44}$$

and

$$f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) \leq \frac{1}{2}\left[f\left(\frac{x+a}{2}, \frac{c+3d}{4}\right) + f\left(\frac{2a+b-x}{2}, \frac{c+3d}{4}\right)\right]. \tag{2.45}$$

Adding (2.44) and (2.45), applying (2.2) for the suitable choices of w_1, w_2, v_1 and v_2 the following holds:

$$f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + f\left(\frac{3a+b}{4}, \frac{c+3d}{4}\right) \leq \frac{1}{4}\left[f\left(\frac{2a+b-x}{2}, \frac{2c+d-y}{2}\right) + f\left(\frac{x+a}{2}, \frac{2c+d-y}{2}\right) + f\left(\frac{2a+b-x}{2}, \frac{y+c}{2}\right) + f\left(\frac{2a+b-x}{2}, \frac{c+2d-y}{2}\right) + f\left(\frac{x+a}{2}, \frac{y+d}{2}\right) + f\left(\frac{2a+b-x}{2}, \frac{y+d}{2}\right) + f\left(\frac{x+a}{2}, \frac{y+c}{2}\right) + f\left(\frac{x+a}{2}, \frac{c+2d-y}{2}\right)\right]. \tag{2.46}$$

Analogously

$$f\left(\frac{a+3b}{4},\frac{3c+d}{4}\right)+f\left(\frac{a+3b}{4},\frac{c+3d}{4}\right)$$
$$\le\frac{1}{4}\left[f\left(\frac{a+2b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{y+d}{2}\right)\right.$$
$$+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{c+2d-y}{2}\right)$$
$$\left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)\right]. \tag{2.47}$$

Again by replacing $y_1=a$, $x_1=\frac{x+a}{2}$, $x_2=\frac{2a+b-x}{4}$, $y_2=\frac{a+b}{2}$ in (2.1) for respective settings $v=\frac{2c+d-y}{2}$, $v=\frac{y+c}{2}$, $v=\frac{y+d}{2}$ and $v=\frac{c+2d-y}{2}$, the followings hold:

$$f\left(\frac{x+a}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{2c+d-y}{2}\right)$$
$$\le f\left(a,\frac{2c+d-y}{2}\right)+f\left(\frac{a+b}{2},\frac{2c+d-y}{2}\right), \tag{2.48}$$

$$f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{y+c}{2}\right)$$
$$\le f\left(a,\frac{c+y}{2}\right)+f\left(\frac{a+b}{2},\frac{c+y}{2}\right), \tag{2.49}$$

$$f\left(\frac{x+a}{2},\frac{y+d}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{y+d}{2}\right)$$
$$\le f\left(a,\frac{y+d}{2}\right)+f\left(\frac{a+b}{2},\frac{y+d}{2}\right) \tag{2.50}$$

and

$$f\left(\frac{x+a}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+2d-y}{2}\right)$$
$$\le f\left(a,\frac{c+2d-y}{2}\right)+f\left(\frac{a+b}{2},\frac{c+2d-y}{2}\right). \tag{2.51}$$

Adding (2.48)-(2.51), applying (2.2) for the particular choices of w_1, w_2, v_1 and v_2 to obtain:

$$f\left(\frac{x+a}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)$$
$$+f\left(\frac{2a+b-x}{2},\frac{y+d}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+c}{2}\right)$$
$$+f\left(\frac{x+a}{2},\frac{c+2d-y}{2}\right)+f\left(\frac{2a+b-x}{2},\frac{c+2d-y}{2}\right)\le 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$+f(a,c)+f(a,d)+2f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right). \tag{2.52}$$

Analogously

$$\begin{aligned}
&f\left(\frac{a+2b-x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{2c+d-y}{2}\right)\\
&\quad+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{a+2b-x}{2},\frac{c+2d-y}{2}\right)\\
&\quad+f\left(\frac{a+2b-x}{2},\frac{y+d}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\\
&\quad+f\left(\frac{a+2b-x}{2},\frac{2c+d-y}{2}\right)+f\left(\frac{x+b}{2},\frac{c+2d-y}{2}\right)\\
&\leq 2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f(b,c)+f(b,d)\\
&\quad+2f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right). \qquad (2.53)
\end{aligned}$$

From the inequalities (2.43), (2.46), (2.47), (2.52), (2.53), under the assumptions on p and the identities (2.38)-(2.40), we drive (2.33).

By integration by parts, we have

$$\begin{aligned}
&st\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\\
&\quad\times\left[\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\\
&\quad-\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\\
&\left.+\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]dydx\\
&=st\int_a^b\int_c^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\\
&\quad\times\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dydx\\
&=(b-a)(d-c)\left[G(t,s)+H(t,s)\right]\\
&\quad-\frac{b-a}{2}\int_c^d\left\{f\left(ta+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad\left.+f\left(tb+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right\}dy\\
&\quad-\frac{d-c}{2}\int_a^b\left\{f\left(tx+(1-t)\frac{a+b}{2},sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad\left.+f\left(tx+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right)\right\}dx. \qquad (2.54)
\end{aligned}$$

By using the co-ordinated convexity of the first order partial derivatives and that of f, under the assumptions on p, the following inequality holds:

$$\begin{aligned}
&\left[f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
&\qquad\qquad \left.-f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad -\left[f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad +\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right. \\
&\qquad\qquad \left.-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad -\left[f\left(\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right. \\
&\qquad\qquad \left.-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad +\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
&\qquad\qquad \left.-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad -\left[f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad +\left[f\left(tx+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right. \\
&\qquad\qquad \left.-f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\quad -\left[f\left(\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right. \\
&\qquad\qquad \left.-f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]p(x,y) \\
&\leq st\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\left[\frac{\partial^2}{\partial x\partial y}f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right. \\
&\quad -\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) \\
&\quad -\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right) \\
&\quad \left.+\frac{\partial^2}{\partial x\partial y}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]\|p\|_\infty,
\end{aligned}$$

for all $(t,s)\in[0,1]^2$ and $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$.

Integrating the above inequality over $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$, under the assumptions on p, using the facts $H(0,s)\leq H(t,s)$, $H(t,0)\leq H(t,s)$ and [1, Theorem 2.2],we get

$$\begin{aligned}
0 \geq f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx - H(t,s) \\
\leq (b-a)(d-c)\left[H(t,s)+G(t,s)\right]\|p\|_\infty \\
-\left[\frac{b-a}{2}\int_c^d \left\{f\left(ta+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right.\right. \\
+ f\left(tb+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) dy \\
+\frac{d-c}{2}\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) \\
\left.\left. +f\left(tx+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right\} dx\right]\|p\|_\infty . \quad (2.55)
\end{aligned}$$

Since f is convex on the co-ordinates, by Jensen's inequality for integrals the followings hold:

$$\begin{aligned}
&\frac{b-a}{2}\int_c^d f\left(ta+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) dy \\
&+\frac{b-a}{2}\int_c^d f\left(tb+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) dy \\
&\geq (b-a)(d-c)\,G(t,0) \geq (b-a)(d-c)\,G(0,0), \quad (2.56)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d-c}{2}\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right) dx \\
&+\frac{d-c}{2}\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right) dx \\
&\geq (b-a)(d-c)\,G(0,s) \geq (b-a)(d-c)\,G(0,0). \quad (2.57)
\end{aligned}$$

By (2.55)-(2.57) we get (2.34).
This completes the proof of the theorem. □

Theorem 4. *Let f, p, G, H_p, L_p be defined as above, then L_p is co-ordinated convex on $[0,1]^2$ and we have the following inequalities:*

$$\begin{aligned}
G(t,s)\int_a^b \int_c^d p(x,y)dydx \leq L_p(t,s) \leq (1-t)(1-s)\int_a^b \int_c^d f(x,y)p(x,y)dydx \\
+ts\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b \int_c^d p(x,y)dydx + \frac{1}{2}t(1-s)\times \\
\int_a^b \int_c^d \left[f(a,y)+f(b,y)\right]p(x,y)dydx + \frac{s(1-t)}{2}\int_a^b \int_c^d \left[f(x,c)+f(x,d)\right]p(x,y)\times \\
dydx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}, \quad (2.58)
\end{aligned}$$

$$H_p(1-t,1-s) \leq L_p(t,s) \quad (2.59)$$

and

$$\frac{H_p(1-t,1-s)+H_p(t,s)}{2} \le L_p(t,s). \tag{2.60}$$

Moreover, the following bound is true:

$$\sup_{(t,s)\in[0,1]^2} L_p(t,s) = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{2.61}$$

Proof. Co-ordinated convexity of L_p directly follows from co-ordinated convexity of f.
By simple techniques of integration, under the assumption on p, the following does hold:

$$\begin{aligned} L_p(t,s) = \frac{1}{4}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} &[f\left(ta+(1-t)x, sc+(1-s)y\right) \\ &+ f\left(ta+(1-t)x, sc+(1-s)(c+d-y)\right) \\ &+ f\left(ta+(1-t)(a+b-x), sc+(1-s)y\right) \\ &f\left(ta+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right) \\ &+ f\left(ta+(1-t)(a+b-x), sd+(1-s)y\right) \\ &+ f\left(ta+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right) \\ &+ f\left(ta+(1-t)x, sd+(1-s)y\right) + f\left(ta+(1-t)x, sd+(1-s)(c+d-y)\right) \\ &+ f\left(tb+(1-t)x, sc+(1-s)y\right) + f\left(tb+(1-t)x, sc+(1-s)(c+d-y)\right) \\ &+ f\left(tb+(1-t)(a+b-x), sc+(1-s)y\right) \\ &+ f\left(tb+(1-t)(a+b-x), sc+(1-s)(c+d-y)\right) \\ &+ f\left(tb+(1-t)(a+b-x), sd+(1-s)y\right) \\ &+ f\left(tb+(1-t)(a+b-x), sd+(1-s)(c+d-y)\right) \\ &+ f\left(tb+(1-t)x, sd+(1-s)(c+d-y)\right) \\ &+ f\left(tb+(1-t)x, sd+(1-s)y\right)]\, p(x,y)dydx, \end{aligned} \tag{2.62}$$

for $(s,t)\in[0,1]^2$.
By setting, $y_1 = ta+(1-t)x$, $x_1 = x_2 = ta+(1-t)\frac{a+b}{2}$, $y_2 = ta+(1-t)(a+b-x)$ in (2.1) for respective settings $v = sc+(1-s)\frac{c+d}{2}$ and $v = sd+(1-s)\frac{c+d}{2}$, the following hold true:

$$\begin{aligned} 2f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-c)\frac{c+d}{2}\right) &\le f\left(ta+(1-t)x, sc+(1-c)\frac{c+d}{2}\right) \\ &+ f\left(ta+(1-t)(a+b-x), sc+(1-c)\frac{c+d}{2}\right) \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} 2f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-c)\frac{c+d}{2}\right) &\le f\left(ta+(1-t)x, sd+(1-c)\frac{c+d}{2}\right) \\ &+ f\left(ta+(1-t)(a+b-x), sd+(1-c)\frac{c+d}{2}\right), \end{aligned} \tag{2.64}$$

for $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$. □

Multiplying (2.63) and (2.64) by 2, then adding and using (2.2) for the particular choices of w_1, w_2, v_1 and v_2, we get

$$\begin{aligned}
&4\left[f\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.+f\left(ta+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\leq f(ta+(1-t)x, sc+(1-s)y)+f(ta+(1-t)x, sc+(1-s)(c+d-y))\\
&\qquad+f(ta+(1-t)(a+b-x), sc+(1-s)y)\\
&\qquad+f(ta+(1-t)(a+b-x), sc+(1-s)(c+d-y))\\
&+f(ta+(1-t)x, sd+(1-s)y)+f(ta+(1-t)x, sd+(1-s)(c+d-y))\\
&\qquad+f(ta+(1-t)(a+b-x), sd+(1-s)y)\\
&\qquad+f(ta+(1-t)(a+b-x), sd+(1-s)(c+d-y)). \quad (2.65)
\end{aligned}$$

Analogously

$$\begin{aligned}
&4\left[f\left(tb+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\right.\\
&\qquad\left.+f\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right]\\
&\leq f(tb+(1-t)x, sc+(1-s)y)+f(tb+(1-t)x, sc+(1-s)(c+d-y))\\
&\qquad+f(tb+(1-t)(a+b-x), sc+(1-s)y)\\
&\qquad+f(tb+(1-t)(a+b-x), sc+(1-s)(c+d-y))\\
&+f(tb+(1-t)x, sd+(1-s)y)+f(tb+(1-t)x, sd+(1-s)(c+d-y))\\
&\qquad+f(tb+(1-t)(a+b-x), sd+(1-s)y)\\
&\qquad+f(tb+(1-t)(a+b-x), sd+(1-s)(c+d-y)). \quad (2.66)
\end{aligned}$$

Multiplying the inequalities (2.65) and (2.66) by $p(x,y)$, integrating the resulting over $(x,y)\in\left[a,\frac{a+b}{2}\right]\times\left[c,\frac{c+d}{2}\right]$ and making use of identities (2.35) and (2.62), the first inequality of (2.58) holds true.
By using the co-ordinated convexity of f and the inequalities:

$$\int_a^b\int_c^d f(a,y)p(x,y)dydx \leq \frac{f(a,c)+f(a,d)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(b,y)p(x,y)dydx \leq \frac{f(b,c)+f(b,d)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(x,c)p(x,y)dydx \leq \frac{f(a,c)+f(b,c)}{2}\int_a^b\int_c^d p(x,y)dydx,$$

$$\int_a^b\int_c^d f(x,d)p(x,y)dydx \leq \frac{f(a,d)+f(b,d)}{2}\int_a^b\int_c^d p(x,y)dydx$$

and

$$\int_a^b\int_c^d f(x,y)p(x,y)dydx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx,$$

we get second inequalities of (2.58).
Again using the co-ordinated convexity of f

$$H_p(1-t,1-s)=\int_a^b\int_c^d f\left((1-t)\,x+t\frac{a+b}{2},(1-s)\,y+s\frac{c+d}{2}\right)p(x,y)dydx$$
$$=\int_a^b\int_c^d f\left(\frac{ta+(1-t)\,x}{2}+\frac{tb+(1-t)\,x}{2},\frac{sc+(1-s)\,y}{2}+\frac{sd+(1-s)\,y}{2}\right)\times$$
$$p(x,y)dydx\le L_p(t,s).$$

This proves 2.59.
From (2.32), (2.58) and (2.59), we get (2.60). Using (2.58), we get (2.61). This completes the proof.

Remark 1. *If $p\,(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$, then $H_p\,(t,s)=H\,(t,s)$ and $L_p\,(t,s)=L\,(t,s)$ for all $(t,s)\in[0,1]^2$ and hence from all the above Theorems we get the inequalities related to the mappings H, G and L.*

References

[1] M. Alomari and M. Darus, Fejér inequality for double integrals, Facta Universitatis (NIŠ), Ser. Math. Inform. 24 (2009), 15.28.

[2] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000. Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html].

[3] S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. Appl.*, **245** (2000), 489–501.

[4] S. S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics*, **4** (2001), 775-788.

[5] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.*, **24** (1906), 369–390.(In Hungarian).

[6] Ming-In Ho, Fejer inequalities for Wright-convex functions, *JIPAM. J. Inequal. Pure Appl. Math.* **8** (1) (2007), article 9.

[7] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une function considérée par Riemann *J. Math. Pures and Appl.*, **58** (1983), 171-215.

[8] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis **3** (1883) 82.

[9] M. A. Latif, On some Fejér-type inequalities for double integrals, to appear.

[10] K. L. Tseng, S. R. Hwang and S. S. Dragomir, On some new inequalities of Hermite-Hadamard-Fejér type involving convex functions, *Demonstratio Math.*, **XL**(1) (2007), 51–64.

[11] K. L. Tseng, S. R. Hwang and S. S. Dragomir, Fejér-type Inequalities (I), (Submitted) Preprint RGMIA *Res. Rep. Coll.* **12**(2009), Supplemet, (4) Article 5. [Online http://www.staff.vu.edu.au/RGMIA/v12n4.asp.].

[12] K. L. Tseng, S. R. Hwang and S. S. Dragomir, Fejér-type Inequalities (II), (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 16, pp.1-12. [Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp.].

[13] K. L. Tseng, S. R. Hwang and S. S. Dragomir, Some companions of Fejér's inequality for convex functions, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 19, pp.1-12. [Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp.].

[14] K. L. Tseng, S. R. Hwang and S. S. Dragomir, Refinements of Fejér's inequality for convex functions, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 20, pp.1-11. [Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp.].

[15] K. L. Tseng, S. R. Hwang and S. S. Dragomir, On some weighted integral inequalities for convex functions related to Fejér's results, (Submitted) Preprint *RGMIA Res. Rep. Coll.* **12**(2009), Supplement, Article 21, pp.1-20. [Online http://www.staff.vu.edu.au/RGMIA/v12(E).asp.]

[16] D. Y. Hwang, K. L. Tseng and G. S. Yang , Some Hadamard's inequalities for co–ordinated convex functions in a rectangle from the plane, Taiwanese Journal ofMathematics, 11 (2007), 63–73.

[17] G. S. Yang and K. L. Tseng, Inequalities of Hermite-Hadamard-Fejér type for convex functions and Lipschitzian functions, *Taiwanese J. Math.*, **7**(3) (2003), 433–440.

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