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REVERSES OF THE JENSEN INEQUALITY IN TERMS OF FIRST DERIVATIVE AND APPLICATIONS

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Abstract. Two new reverses of the celebrated Jensen's integral inequality for differentiable convex functions with applications for means, the Hölder inequality and f-divergence measures in information theory are given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ − a.e.(almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x) |f(x)| d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) d\mu(x)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [13] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \int_\Omega w(\Phi \circ f) d\mu - \Phi\left(\int_\Omega w f d\mu\right) \\ &\leq \int_\Omega w(\Phi' \circ f) f d\mu - \int_\Omega w(\Phi' \circ f) d\mu \int_\Omega w f d\mu \\ &\leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \int_\Omega w \left| f - \int_\Omega w f d\mu \right| d\mu. \end{aligned} \tag{1.1}$$

For a generalization of the first inequality in (1.1) without the differentiability assumption and the derivative Φ' replaced with a selection φ from the subdifferential $\partial\Phi$, see the paper [26] by C.P. Niculescu.

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Remark 1. *If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the inequality:*

$$0 \le \frac{1}{\mu(\Omega)} \int_\Omega (\Phi \circ f) \, d\mu - \Phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f d\mu\right) \le \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) f d\mu - \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) \, d\mu \cdot \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \le \frac{1}{2} [\Phi'(M) - \Phi'(m)] \frac{1}{\mu(\Omega)} \int_\Omega \left| f - \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right| d\mu. \tag{1.2}$$

Corollary 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \ge 0 \ \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$0 \le \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \le \sum_{i=1}^n w_i \Phi'(x_i) x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \le \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|. \tag{1.3}$$

Remark 2. *We notice that the inequality between the first and the second term in (1.3) was proved in 1994 by Dragomir & Ionescu, see* [14].

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T_w(f, g) := \int_\Omega w f g d\mu - \int_\Omega w f d\mu \int_\Omega w g d\mu. \tag{1.4}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(f, g)| \le \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{1.5}$$

provided

$$-\infty < \gamma \le f(x) \le \Gamma < \infty, \quad -\infty < \delta \le g(x) \le \Delta < \infty \tag{1.6}$$

for $\mu -$ a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \le f(x) \le \Gamma < \infty$ for $\mu -$ a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_\Omega w \left| f - \int_\Omega w f d\mu \right| d\mu \le \left[\int_\Omega w f^2 d\mu - \left(\int_\Omega w f d\mu \right)^2 \right]^{\frac{1}{2}} \le \frac{1}{2} (\Gamma - \gamma). \tag{1.7}$$

On making use of the results (1.1) and (1.7), we can state the following string of reverse inequalities

$$
\begin{aligned}
0 &\leq \int_\Omega w\left(\Phi\circ f\right)d\mu-\Phi\left(\int_\Omega wfd\mu\right) \\
&\leq \int_\Omega w\left(\Phi'\circ f\right)fd\mu-\int_\Omega w\left(\Phi'\circ f\right)d\mu\int_\Omega wfd\mu \\
&\leq \frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu \\
&\leq \frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\left(M-m\right),
\end{aligned}
\tag{1.8}
$$

provided that $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$.

Remark 3. *We notice that the inequality between the first, second and last term from (1.8) was proved in the general case of positive linear functionals in 2001 by S.S. Dragomir in* [12].

Motivated by the above results, we establish in the current paper two new reverses of Jensen's integral inequality in terms of the first derivative of a convex function. Some natural application for inequalities between means, the Hölder inequality and for the f-divergence measures that play an important role in information theory are given as well.

2. Reverse Inequalities

The following reverse of the Jensen's inequality holds:

Theorem 2. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\leq f(x)\leq M<\infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

and such that $f,\Phi\circ f\in L_w(\Omega,\mu)$, then

$$
\begin{aligned}
0 &\leq \int_\Omega w(x)\,\Phi(f(x))\,d\mu(x)-\Phi\left(\bar{f}_{\Omega,w}\right) \\
&\leq \left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m} \\
&\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right],
\end{aligned}
\tag{2.1}
$$

where $\bar{f}_{\Omega,w}:=\int_\Omega w(x)\,f(x)\,d\mu(x)\in[m,M]$.

Proof. By the convexity of Φ we have that

$$\begin{aligned}
(2.2)\qquad & \int_{\Omega} w(x)\,\Phi(f(x))\,d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\
&= \int_{\Omega} w(x)\,\Phi\left[\frac{m(M-f(x))+M(f(x)-m)}{M-m}\right] d\mu(x) \\
&\quad - \Phi\left(\int_{\Omega} w(x)\left[\frac{m(M-f(x))+M(f(x)-m)}{M-m}\right] d\mu(x)\right) \\
&\leq \int_{\Omega} \frac{(M-f(x))\,\Phi(m)+(f(x)-m)\,\Phi(M)}{M-m} w(x)\,d\mu(x) \\
&\quad - \Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) \\
&= \frac{\left(M-\bar{f}_{\Omega,w}\right)\Phi(m)+\left(\bar{f}_{\Omega,w}-m\right)\Phi(M)}{M-m} \\
&\quad - \Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) := B
\end{aligned}$$

Then, by the convexity of Φ we have the gradient inequality

$$\Phi(t) - \Phi(M) \geq \Phi'_-(M)(t-b)$$

for any $t \in [m, M]$. If we multiply this inequality with $t - m \geq 0$, we deduce

$$(2.3)\qquad (t-m)\,\Phi(t) - (t-m)\,\Phi(b) \geq \Phi'_-(M)(t-M)(t-m), \quad t \in [m, M].$$

Similarly, we get

$$(2.4)\qquad (M-t)\,\Phi(t) - (M-t)\,\Phi(m) \geq \Phi'_+(m)(t-m)(M-t), \quad t \in [m, M].$$

Adding (2.3) to (2.4) and dividing by $M - m$, we deduce

$$\Phi(t) - \frac{(t-m)\,\Phi(M)+(M-t)\,\Phi(m)}{M-m} \geq \frac{(t-M)(t-m)}{M-m}\left[\Phi'_-(M)-\Phi'_+(m)\right],$$

for any $t \in [m, M]$.

By denoting

$$\Delta_\Phi(t; m, M) := \frac{(t-m)\,\Phi(M)+(M-t)\,\Phi(m)}{M-m} - \Phi(t), \quad t \in [m, M]$$

we then get the following inequality of interest (see also [10])

$$\begin{aligned}
(2.5)\qquad 0 \leq \Delta_\Phi(t; m, M) &\leq \frac{(M-t)(t-m)}{M-m}\left[\Phi'_-(M)-\Phi'_+(m)\right] \\
&\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]
\end{aligned}$$

for any $t \in [m, M]$.

Now, since with the above notations we have $B = \Delta_\Phi\left(\bar{f}_{\Omega,w}; m, M\right)$, then by (2.5) we have

$$\begin{aligned}
B &\leq \frac{\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\left[\Phi'_-(M)-\Phi'_+(m)\right] \\
&\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right],
\end{aligned}$$

and the proof is completed. □

Corollary 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $x_i \in I$ and $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then we have the inequality*

$$
\begin{aligned}
(2.6) \qquad 0 &\leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \\
&\leq (M - \bar{x}_p)(\bar{x}_p - m) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\
&\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right],
\end{aligned}
$$

where $\bar{x}_p := \sum_{i=1}^{n} p_i x_i \in I$.

Remark 4. *Define the weighted arithmetic mean of the positive n-tuple $x = (x_1, ..., x_n)$ with the nonnegative weights $w = (w_1, ..., w_n)$ by*

$$
A_n(w, x) := \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i
$$

where $W_n := \sum_{i=1}^{n} w_i > 0$ and the weighted geometric mean of the same n-tuple, by

$$
G_n(w, x) := \left(\prod_{i=1}^{n} x_i^{w_i}\right)^{1/W_n}.
$$

It is well know that the following arithmetic mean-geometric mean inequality holds true

$$
A_n(w, x) \geq G_n(w, x).
$$

Applying the inequality (2.6) for the convex function $\Phi(t) = -\ln t, t > 0$ we have

$$
\begin{aligned}
(2.7) \qquad 1 &\leq \frac{A_n(w, x)}{G_n(w, x)} \leq \exp\left[\frac{1}{Mm}(M - A_n(w, x))(A_n(w, x) - m)\right] \\
&\leq \exp\left[\frac{1}{4}\frac{(M - m)^2}{mM}\right],
\end{aligned}
$$

provided that $0 < m \leq x_i \leq M < \infty$ for $i \in \{1, ..., n\}$.

For the Lebesgue measurable function $g : [\alpha, \beta] \to \mathbb{R}$ we introduce the *Lebesgue p-norms* defined as

$$
\|g\|_{[\alpha,\beta],p} := \left(\int_{\alpha}^{\beta} |g(t)|^p \, dt\right)^{1/p} \quad \text{if } g \in L_p[\alpha, \beta],
$$

for $p \geq 1$ and

$$
\|g\|_{[\alpha,\beta],\infty} := ess \sup_{t \in [\alpha,\beta]} |g(t)| \text{ if } g \in L_\infty[\alpha, \beta],
$$

for $p = \infty$.

The following result also holds:

Theorem 3. *With the assumptions in Theorem 2, we have the inequalities*

$$0 \leq \int_{\Omega} w(x)\Phi(f(x))\,d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \tag{2.8}$$
$$\leq \frac{\left(M-\bar{f}_{\Omega,w}\right)\int_m^{\bar{f}_{\Omega,w}}|\Phi'(t)|\,dt + \left(\bar{f}_{\Omega,w}-m\right)\int_{\bar{f}_{\Omega,w}}^M |\Phi'(t)|\,dt}{M-m}$$
$$:= \Lambda_\Phi\left(\bar{f}_{\Omega,w}; m, M\right),$$

where the integral in the second term of the inequality is taken in the Lebesgue sense.

We also have the bounds for $\Lambda_\Phi\left(\bar{f}_{\Omega,w}; m, M\right)$ *:*

$$\Lambda_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) \leq \begin{cases} \left[\frac{1}{2} + \frac{\left|\bar{f}_{\Omega,w} - \frac{m+M}{2}\right|}{M-m}\right]\int_m^M |\Phi'(t)|\,dt \\ \\ \left[\frac{1}{2}\int_m^M |\Phi'(t)|\,dt + \frac{1}{2}\left|\int_{\bar{f}_{\Omega,w}}^M |\Phi'(t)|\,dt - \int_m^{\bar{f}_{\Omega,w}} |\Phi'(t)|\,dt\right|\right], \end{cases} \tag{2.9}$$

and

$$\Lambda_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) \tag{2.10}$$
$$\leq \frac{\left(\bar{f}_{\Omega,w}-m\right)\left(M-\bar{f}_{\Omega,w}\right)}{M-m}\left[\|\Phi'\|_{\left[\bar{f}_{\Omega,w},M\right],\infty} + \|\Phi'\|_{\left[m,\bar{f}_{\Omega,w}\right],\infty}\right]$$
$$\leq \frac{1}{2}(M-m)\frac{\|\Phi'\|_{\left[\bar{f}_{\Omega,w},M\right],\infty} + \|\Phi'\|_{\left[m,\bar{f}_{\Omega,w}\right],\infty}}{2} \leq \frac{1}{2}(M-m)\|\Phi'\|_{[m,M],\infty}$$

and

$$\Lambda_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) \leq \frac{1}{M-m}\left[\left(\bar{f}_{\Omega,w}-m\right)\left(M-\bar{f}_{\Omega,w}\right)^{1/q}\|\Phi'\|_{\left[\bar{f}_{\Omega,w},M\right],p}\right. \tag{2.11}$$
$$\left. + \left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)^{1/q}\|\Phi'\|_{\left[m,\bar{f}_{\Omega,w}\right],p}\right]$$
$$\leq \frac{1}{M-m}\left[\left(\bar{f}_{\Omega,w}-m\right)^q\left(M-\bar{f}_{\Omega,w}\right)\right.$$
$$\left. + \left(M-\bar{f}_{\Omega,w}\right)^q\left(\bar{f}_{\Omega,w}-m\right)\right]^{1/q}\|\Phi'\|_{[m,M],p}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Observe that, with the above notations we have

$$\Lambda_\Phi(t; m, M) = \frac{(t-m)\Phi(M) + (M-t)\Phi(m)}{M-m} - \Phi(t) \tag{2.12}$$
$$= \frac{(t-m)\Phi(M) + (M-t)\Phi(m) - (M-m)\Phi(t)}{M-m}$$
$$= \frac{(t-m)\Phi(M) + (M-t)\Phi(m) - (M-t+t-m)\Phi(t)}{M-m}$$
$$= \frac{(t-m)[\Phi(M)-\Phi(t)] - (M-t)[\Phi(t)-\Phi(m)]}{M-m}$$

for any $t \in [m, M]$.

Taking the modulus on (2.12) and noticing that $\Lambda_{\Phi}(t;m,M) \geq 0$ for any $t \in [m,M]$, we have that

$$\Lambda_{\Phi}(t;m,M) \leq \frac{(t-m)|\Phi(M)-\Phi(t)|+(M-t)|\Phi(t)-\Phi(m)|}{M-m} = \frac{(t-m)\left|\int_t^M \Phi'(s)\,ds\right|+(M-t)\left|\int_m^t \Phi'(s)\,ds\right|}{M-m} \leq \frac{(t-m)\int_t^M |\Phi'(s)|\,ds+(M-t)\int_m^t |\Phi'(s)|\,ds}{M-m} \tag{2.13}$$

for any $t \in [m,M]$.

Finally, if we write the inequality (2.13) for $t = \bar{f}_{\Omega,w} \in [m,M]$ and utilize the inequality (2.2), we deduce the desired result (2.8).

Now, we observe that

$$\frac{(t-m)\int_t^M |\Phi'(s)|\,ds+(M-t)\int_m^t |\Phi'(s)|\,ds}{M-m} \tag{2.14}$$
$$\leq \begin{cases} \max\{t-m,M-t\}\int_m^M |\Phi'(t)|\,dt \\ \max\left\{\int_t^M |\Phi'(s)|\,ds,\int_m^t |\Phi'(s)|\,ds\right\}(M-m) \end{cases}$$
$$= \begin{cases} \left[\frac{1}{2}(M-m)+\left|t-\frac{m+M}{2}\right|\right]\int_m^M |\Phi'(t)|\,dt \\ \left[\frac{1}{2}\int_m^M |\Phi'(s)|\,ds+\frac{1}{2}\left|\int_t^M |\Phi'(s)|\,ds-\int_m^t |\Phi'(s)|\,ds\right|\right](M-m) \end{cases}$$

for any $t \in [m,M]$. This proves the inequality (2.9).

By the Hölder's inequality we have

$$\int_t^M |\Phi'(s)|\,ds \leq \begin{cases} (M-t)\|\Phi'\|_{[t,M],\infty} \\ (M-t)^{1/q}\|\Phi'\|_{[t,M],p} \quad \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

and

$$\int_m^t |\Phi'(s)|\,ds \leq \begin{cases} (t-m)\|\Phi'\|_{[m,t],\infty} \\ (t-m)^{1/q}\|\Phi'\|_{[m,t],p} \quad \text{if } p>1, \frac{1}{p}+\frac{1}{q}=1 \end{cases}$$

which give that

$$\frac{(t-m)\int_t^M |\Phi'(s)|\,ds+(M-t)\int_m^t |\Phi'(s)|\,ds}{M-m} \tag{2.15}$$
$$\leq \frac{(t-m)(M-t)\|\Phi'\|_{[t,M],\infty}+(M-t)(t-m)\|\Phi'\|_{[m,t],\infty}}{M-m}$$
$$= \frac{(t-m)(M-t)}{M-m}\left[\|\Phi'\|_{[t,M],\infty}+\|\Phi'\|_{[m,t],\infty}\right]$$
$$\leq \frac{1}{2}(M-m)\frac{\|\Phi'\|_{[t,M],\infty}+\|\Phi'\|_{[m,t],\infty}}{2}$$
$$\leq \frac{1}{2}(M-m)\max\left\{\|\Phi'\|_{[t,M],\infty},\|\Phi'\|_{[m,t],\infty}\right\} = \frac{1}{2}(M-m)\|\Phi'\|_{[m,M],\infty}$$

and

$$\begin{aligned}
(2.16)\qquad & \frac{(t-m)\int_t^M |\Phi'(s)|\,ds + (M-t)\int_m^t |\Phi'(s)|\,ds}{M-m} \\
&\le \frac{(t-m)(M-t)^{1/q}\|\Phi'\|_{[t,M],p} + (M-t)(t-m)^{1/q}\|\Phi'\|_{[m,t],p}}{M-m} \\
&\le \frac{1}{M-m}\left[\left((t-m)(M-t)^{1/q}\right)^q + \left((M-t)(t-m)^{1/q}\right)^q\right]^{1/q} \\
&\times \left[\|\Phi'\|_{[t,M],p}^p + \|\Phi'\|_{[m,t],p}^p\right]^{1/p} \\
&= \frac{1}{M-m}\left[(t-m)^q(M-t) + (M-t)^q(t-m)\right]^{1/q}\|\Phi'\|_{[m,M],p}
\end{aligned}$$

for any $t\in[m,M]$.

These prove the desired inequalities (2.10) and (2.11). □

The discrete case is as follows:

Corollary 3. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m,M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $x_i \in I$ and $p_i \ge 0$ for $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$, then we have the inequality*

$$\begin{aligned}
(2.17)\qquad 0 &\le \sum_{i=1}^n p_i\Phi(x_i) - \Phi(\bar{x}_p) \\
&\le \frac{(M-\bar{x}_p)\int_m^{\bar{x}_p}|\Phi'(t)|\,dt + (\bar{x}_p - m)\int_{\bar{x}_p}^M |\Phi'(t)|\,dt}{M-m} \\
&:= \Lambda_\Phi(\bar{x}_p; m, M),
\end{aligned}$$

where $\Lambda_\Phi(\bar{x}_p; m, M)$ satisfies the bounds

$$\begin{aligned}
(2.18)\qquad & \Lambda_\Phi(\bar{x}_p; m, M) \\
&\le \begin{cases} \left[\frac{1}{2} + \frac{\left|\bar{x}_p - \frac{m+M}{2}\right|}{M-m}\right]\int_m^M |\Phi'(t)|\,dt \\[2ex] \left[\frac{1}{2}\int_m^M |\Phi'(t)|\,dt + \frac{1}{2}\left|\int_{\bar{x}_p}^M |\Phi'(t)|\,dt - \int_m^{\bar{x}_p}|\Phi'(t)|\,dt\right|\right], \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(2.19)\qquad & \Lambda_\Phi(\bar{x}_p; m, M) \\
&\le \frac{(\bar{x}_p - m)(M-\bar{x}_p)}{M-m}\left[\|\Phi'\|_{[\bar{x}_p,M],\infty} + \|\Phi'\|_{[m,\bar{x}_p],\infty}\right] \\
&\le \frac{1}{2}(M-m)\frac{\|\Phi'\|_{[\bar{x}_p,M],\infty} + \|\Phi'\|_{[m,\bar{x}_p],\infty}}{2} \le \frac{1}{2}(M-m)\|\Phi'\|_{[m,M],\infty}
\end{aligned}$$

and

$$\Lambda_{\Phi}\left(\bar{x}_p;m,M\right)\leq\frac{1}{M-m}\Big[\left(\bar{x}_p-m\right)\left(M-\bar{x}_p\right)^{1/q}\left\|\Phi'\right\|_{[\bar{x}_p,M],p}+\left(M-\bar{x}_p\right)\left(\bar{x}_p-m\right)^{1/q}\left\|\Phi'\right\|_{[m,\bar{x}_p],p}\Big]\leq\frac{1}{M-m}\left[\left(\bar{x}_p-m\right)^q\left(M-\bar{x}_p\right)+\left(M-\bar{x}_p\right)^q\left(\bar{x}_p-m\right)\right]^{1/q}\left\|\Phi'\right\|_{[m,M],p}. \tag{2.20}$$

Remark 5. *Under the assumptions of Remark 4 , on applying the inequality (2.17) for the convex function $\Phi(t)=-\ln t$, we have the following reverse of the arithmetic mean-geometric mean inequality*

$$1\leq\frac{A_n(w,x)}{G_n(w,x)}\leq\left(\frac{A_n(w,x)}{m}\right)^{M-A_n(w,x)}\left(\frac{M}{A_n(w,x)}\right)^{A_n(w,x)-m}. \tag{2.21}$$

3. Applications for the Hölder Inequality

It is well known that if $f\in L_p(\Omega,\mu)$, $p>1$, where the Lebesgue space $L_p(\Omega,\mu)$ is defined by

$$L_p(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_\Omega|f(x)|^p\,d\mu(x)<\infty\}$$

and $g\in L_q(\Omega,\mu)$ with $\frac{1}{p}+\frac{1}{q}=1$ then $fg\in L(\Omega,\mu):=L_1(\Omega,\mu)$ and the *Hölder inequality* holds true

$$\int_\Omega|fg|\,d\mu\leq\left(\int_\Omega|f|^p\,d\mu\right)^{1/p}\left(\int_\Omega|g|^p\,d\mu\right)^{1/q}.$$

Assume that $p>1$. If $h:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds

$$-\infty<m\leq|h(x)|\leq M<\infty\text{ for }\mu\text{-a.e. }x\in\Omega$$

and is such that $h,|h|^p\in L_w(\Omega,\mu)$, for a $\mu-$measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq0$ for $\mu-$ a.e. $x\in\Omega$ and $\int_\Omega wd\mu>0$, then from (2.1) we have

$$0\leq\frac{\int_\Omega|h|^p\,wd\mu}{\int_\Omega wd\mu}-\left(\frac{\int_\Omega|h|\,wd\mu}{\int_\Omega wd\mu}\right)^p\leq p\frac{M^{p-1}-m^{p-1}}{M-m}\left(M-\overline{|h|}_{\Omega,w}\right)\left(\overline{|h|}_{\Omega,w}-m\right)\leq\frac{1}{4}p\left(M-m\right)\left(M^{p-1}-m^{p-1}\right), \tag{3.1}$$

where $\overline{|h|}_{\Omega,w}:=\frac{\int_\Omega|h|wd\mu}{\int_\Omega wd\mu}\in[m,M]$.

Proposition 1. *If $f\in L_p(\Omega,\mu)$, $g\in L_q(\Omega,\mu)$ with $p>1,\frac{1}{p}+\frac{1}{q}=1$ and there exists the constants $\gamma,\Gamma>0$ and such that*

$$\gamma\leq\frac{|f|}{|g|^{q-1}}\leq\Gamma\ \mu\text{-a.e on }\Omega$$

then we have

$$0 \le \frac{\int_\Omega |f|^p \, d\mu}{\int_\Omega |g|^q \, d\mu} - \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^p \tag{3.2}$$
$$\le p\frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma}\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)$$
$$\le \frac{1}{4}p(\Gamma - \gamma)\left(\Gamma^{p-1} - \gamma^{p-1}\right).$$

Proof. The inequalities (3.2) follow from (3.1) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q.$$

The details are omitted. □

Remark 6. *We observe that for $p = q = 2$ we have from the first inequality in (3.2) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$0 \le \int_\Omega |g|^2 \, d\mu \int_\Omega |f|^2 \, d\mu - \left(\int_\Omega |fg| \, d\mu\right)^2 \tag{3.3}$$
$$\le \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu} - \gamma\right)\left(\int_\Omega |g|^2 \, d\mu\right)^2$$
$$\le \frac{1}{4}(\Gamma - \gamma)^2\left(\int_\Omega |g|^2 \, d\mu\right)^2,$$

provided that $f, g \in L_2(\Omega, \mu)$ and there exists the constants $\gamma, \Gamma > 0$ such that

$$\gamma \le \frac{|f|}{|g|} \le \Gamma \ \mu\text{-}a.e \ on \ \Omega.$$

Corollary 4. *With the assumptions of Proposition 1 we have the following additive reverses of the Hölder inequality:*

(3.4)

$$0 \le \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}\left(\int_\Omega |g|^q \, d\mu\right)^{1/q} - \int_\Omega |fg| \, d\mu$$
$$\le p^{1/p}\left(\frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma}\right)^{\frac{1}{p}}\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^{\frac{1}{p}}\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)^{\frac{1}{p}}\int_\Omega |g|^q \, d\mu$$
$$\le \frac{1}{4^{1/p}}p^{1/p}(\Gamma - \gamma)^{1/p}\left(\Gamma^{p-1} - \gamma^{p-1}\right)^{1/p}\int_\Omega |g|^q \, d\mu$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By multiplying in (3.2) with $\left(\int_\Omega |g|^q \, d\mu\right)^p$ we have

$$\int_\Omega |f|^p \, d\mu \left(\int_\Omega |g|^q \, d\mu\right)^{p-1} - \left(\int_\Omega |fg| \, d\mu\right)^p$$
$$\le p\frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma}\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)\left(\int_\Omega |g|^q \, d\mu\right)^p$$
$$\le \frac{1}{4}p(\Gamma - \gamma)\left(\Gamma^{p-1} - \gamma^{p-1}\right)\left(\int_\Omega |g|^q \, d\mu\right)^p,$$

which is equivalent with

$$
\begin{aligned}
(3.5)\qquad & \int_\Omega |f|^p \, d\mu \left(\int_\Omega |g|^q \, d\mu\right)^{p-1} \\
& \leq \left(\int_\Omega |fg| \, d\mu\right)^p + p\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right) \\
& \times \left(\int_\Omega |g|^q \, d\mu\right)^p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \\
& \leq \left(\int_\Omega |fg| \, d\mu\right)^p + \frac{1}{4} p\left(\Gamma - \gamma\right)\left(\Gamma^{p-1} - \gamma^{p-1}\right)\left(\int_\Omega |g|^q \, d\mu\right)^p .
\end{aligned}
$$

Taking the power $1/p$ with $p > 1$ and employing the following elementary inequality that state that for $p > 1$ and $\alpha, \beta > 0$,

$$
(\alpha + \beta)^{1/p} \leq \alpha^{1/p} + \beta^{1/p}
$$

we have from the first part of (3.5) that

$$
\begin{aligned}
(3.6)\quad & \int_\Omega |f|^p \, d\mu \left(\int_\Omega |g|^q \, d\mu\right)^{1-\frac{1}{p}} \\
& \leq \int_\Omega |fg| \, d\mu \\
& + \left[p\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)\left(\int_\Omega |g|^q \, d\mu\right)^p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma}\right]^{1/p} .
\end{aligned}
$$

Since $1 - \frac{1}{p} = \frac{1}{q}$, we get from (3.6) the first inequality in (3.4). The rest is obvious. □

If $h : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds

$$
-\infty < m \leq |h(x)| \leq M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega
$$

and is such that $h, |h|^p \in L_w(\Omega, \mu)$, for a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ − a.e. $x \in \Omega$ and $\int_\Omega w d\mu > 0$, then from Theorem 3 we have amongst other the following inequality

$$
\begin{aligned}
(3.7)\qquad 0 & \leq \frac{\int_\Omega |h|^p \, w d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega |h| \, w d\mu}{\int_\Omega w d\mu}\right)^p \\
& \leq (M^p - m^p)\left[\frac{1}{2} + \frac{1}{M - m}\left|\frac{\int_\Omega |h| \, w d\mu}{\int_\Omega w d\mu} - \frac{m + M}{2}\right|\right].
\end{aligned}
$$

From this inequality we can state that:

Proposition 2. *With the assumptions of Proposition 1 we have*

$$
\begin{aligned}
(3.8)\qquad 0 & \leq \frac{\int_\Omega |f|^p \, d\mu}{\int_\Omega |g|^q \, d\mu} - \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^p \\
& \leq (\Gamma^p - \gamma^p)\left[\frac{1}{2} + \frac{1}{\Gamma - \gamma}\left|\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \frac{\gamma + \Gamma}{2}\right|\right].
\end{aligned}
$$

Finally, the following additive reverse of the Hölder inequality can be stated as well:

Corollary 5. *With the assumptions of Proposition 1 we have*

$$\left(\int_{\Omega}|f|^{p}\,d\mu\right)^{1/p}\left(\int_{\Omega}|g|^{q}\,d\mu\right)^{1/q}-\int_{\Omega}|fg|\,d\mu \le \left(\Gamma^{p}-\gamma^{p}\right)^{1/p}\left[\frac{1}{2}+\frac{1}{\Gamma-\gamma}\left|\frac{\int_{\Omega}|fg|\,d\mu}{\int_{\Omega}|g|^{q}\,d\mu}-\frac{\gamma+\Gamma}{2}\right|\right]^{1/p}\int_{\Omega}|g|^{q}\,d\mu. \tag{3.9}$$

Remark 7. *We observe that for $p=q=2$ we have from the first inequality in (3.8) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$\int_{\Omega}|g|^{2}\,d\mu\int_{\Omega}|f|^{2}\,d\mu-\left(\int_{\Omega}|fg|\,d\mu\right)^{2} \le \left(\Gamma^{2}-\gamma^{2}\right)\left[\frac{1}{2}+\frac{1}{\Gamma-\gamma}\left|\frac{\int_{\Omega}|fg|\,d\mu}{\int_{\Omega}|g|^{2}\,d\mu}-\frac{\gamma+\Gamma}{2}\right|\right]\left(\int_{\Omega}|g|^{2}\,d\mu\right)^{2} \tag{3.10}$$

provided that $f,g\in L_{2}(\Omega,\mu)$ and there exists the constants $\gamma,\Gamma>0$ such that

$$\gamma\le\frac{|f|}{|g|}\le\Gamma \ \mu\text{-a.e on } \Omega.$$

One can easily observe that the bound provided by (3.10) is not as good as the one given by (3.3). The details are omitted.

4. Applications for f-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [18], Kullback and Leibler [23], Rényi [29], Havrda and Charvat [16], Kapur [21], Sharma and Mittal [31], Burbea and Rao [4], Rao [28], Lin [24], Csiszár [7], Ali and Silvey [1], Vajda [37], Shioya and Da-te [32] and others (see for example [25] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [28], genetics [25], finance, economics, and political science [30], [35], [36], biology [27], the analysis of contingency tables [15], approximation of probability distributions [6], [22], signal processing [19], [20] and pattern recognition [3], [5]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P}:=\left\{p|p:\Omega\to\mathbb{R},\,p(x)\ge0,\,\int_{\Omega}p(x)\,d\mu(x)=1\right\}$. The Kullback-Leibler divergence [23] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q):=\int_{\Omega}p(x)\ln\left[\frac{p(x)}{q(x)}\right]d\mu(x),\quad p,q\in\mathcal{P}, \tag{4.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [17], χ^2-*divergence* D_{χ^2}, $\alpha-$*divergence* D_α, *Bhattacharyya* *distance* D_B [2], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [18], *triangular discrimination* D_Δ [34], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(x) - q(x)|\, d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.2}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(x)} - \sqrt{q(x)}\right| d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.3}$$

$$D_{\chi^2}(p,q) := \int_\Omega p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x)\right], \quad p,q \in \mathcal{P}; \tag{4.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(x)\,q(x)}\, d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(x)\,q(x)}{p(x)+q(x)} d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.7}$$

$$D_J(p,q) := \int_\Omega [p(x) - q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \mathcal{P}; \tag{4.8}$$

$$D_\Delta(p,q) := \int_\Omega \frac{[p(x) - q(x)]^2}{p(x)+q(x)} d\mu(x), \quad p,q \in \mathcal{P}. \tag{4.9}$$

For other divergence measures, see the paper [21] by Kapur or the book on line [33] by Taneja.

Csiszár $f-$divergence is defined as follows [8]

$$I_f(p,q) := \int_\Omega p(x)\, f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p,q \in \mathcal{P}, \tag{4.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (4.1) – (4.9), are particular instances of Csiszár $f-$divergence. There are also many others which are not in this class (see for example [33]). For the basic properties of Csiszár $f-$divergence see [8], [9] and [37].

The following result holds:

Proposition 3. *Let* $f : (0,\infty) \to \mathbb{R}$ *be a convex function with the property that* $f(1) = 0$. *Assume that* $p,q \in \mathcal{P}$ *and there exists the constants* $0 < r < 1 < R < \infty$ *such that*

$$r \le \frac{q(x)}{p(x)} \le R \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{4.11}$$

Then we have the inequalities

$$\begin{aligned} 0 \le I_f(p,q) &\le (R-1)(1-r)\frac{f'_-(R) - f'_+(r)}{R-r} \\ &\le \frac{1}{4}(R-r)\left[f'_-(R) - f'_+(r)\right]. \end{aligned} \tag{4.12}$$

Proof. Utilising Theorem 2 we can write that

$$\begin{aligned}(4.13)\qquad &\int_{\Omega} p(x) f\left(\frac{q(x)}{p(x)}\right) d\mu(x) - f\left(\int_{\Omega} q(x)\, d\mu(x)\right) \\ &\leq \left(R - \int_{\Omega} q(x)\, d\mu(x)\right)\left(\int_{\Omega} q(x)\, d\mu(x) - r\right) \frac{f'_{-}(R) - f'_{+}(r)}{R-r} \\ &\leq \frac{1}{4}(R-r)\left[f'_{-}(R) - f'_{+}(r)\right],\end{aligned}$$

for $p, q \in \mathcal{P}$ satisfying (4.11) and since $f\left(\int_{\Omega} q(x)\, d\mu(x)\right) = f(1) = 0$ we get from (4.13) the desired result (4.12). □

By the use of Theorem 3 we can also state the following result:

Proposition 4. *With the assumptions in Proposition 3, we have the inequalities*

$$(4.14)\qquad 0 \leq I_f(p,q) \leq B_f(r,R)$$

where

$$(4.15)\qquad B_f(r,R) := \frac{(R-1)\int_r^1 |f'(t)|\, dt + (1-r)\int_1^R |f'(t)|\, dt}{R-r}.$$

Moreover, we have the following bounds for $B_f(r,R)$

$$\begin{aligned}(4.16)\qquad &B_f(r,R) \\ &\leq \begin{cases} \left[\frac{1}{2} + \frac{\left|1 - \frac{r+R}{2}\right|}{R-r}\right]\int_r^R |f'(t)|\, dt \\ \left[\frac{1}{2}\int_r^R |f'(t)|\, dt + \frac{1}{2}\left|\int_1^R |f'(t)|\, dt - \int_r^1 |f'(t)|\, dt\right|\right], \end{cases}\end{aligned}$$

and

$$\begin{aligned}(4.17)\qquad &B_f(r,R) \\ &\leq \frac{(1-r)(R-1)}{R-r}\left[\|f'\|_{[1,R],\infty} + \|f'\|_{[r,1],\infty}\right] \\ &\leq \frac{1}{2}(R-r)\frac{\|f'\|_{[1,R],\infty} + \|f'\|_{[r,1],\infty}}{2} \leq \frac{1}{2}(R-r)\|f'\|_{[r,R],\infty}\end{aligned}$$

and

$$\begin{aligned}(4.18)\qquad &B_f(r,R) \\ &\leq \frac{1}{R-r}\left[(1-r)(R-1)^{1/q}\|f'\|_{[1,R],p} + (R-1)(1-r)^{1/q}\|f'\|_{[r,1],p}\right] \\ &\leq \frac{1}{R-r}\left[(1-r)^q(R-1) + (R-1)^q(1-r)\right]^{1/q}\|f'\|_{[r,R],p}\end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The above results can be utilized to obtain various inequalities for the divergence measures in information theory that are particular instances of f-divergence.

Consider, for example, the Kullback-Leibler divergence measure

$$D_{KL}(p,q) := \int_{\Omega} p(x) \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p, q \in \mathcal{P},$$

which is an f-divergence for the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$.

If $p, q \in \mathcal{P}$ such that there exists the constants $0 < r < 1 < R < \infty$ with

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x \in \Omega, \tag{4.19}$$

then we get from (4.12) that

$$D_{KL}(p, q) \leq \frac{(R-1)(1-r)}{rR} \tag{4.20}$$

and from (4.14) that

$$D_{KL}(p, q) \leq \ln \left(\frac{R^{1-r}}{r^{R-1}} \right)^{\frac{1}{R-r}}.$$

The interested reader can obtain similar results for other divergence measures as listed above. However, the details are omitted.

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