Received 26/08/11

HERMITE-HADAMARD INEQUALITES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE $P-$CONVEX

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Abstract. In this paper we extend some estimates of the right hand side of a Hermite- Hadamard type inequalities for functions whose second derivatives absolute values are $P-$convex. Applications to some special means are considered.

Keywords: Hermite-Hadamard inequality, $P-$convex functions, special means

1. Introduction

Let $f : I \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as the Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the function f. Both inequalities hold in the reversed direction if f is concave (see [14]).

It is well known that the Hermite-Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [1-12]) and the references therein. In [11] Dragomir and Agarwal established the following results connected with the right hand-side of (1) as well as to apply them for some elementary inequalities for real numbers and numerical integration.

Theorem 1.1. *Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{2}$$

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Theorem 1.2. *Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{1/p}}.\left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \quad (3)$$

In [14] Pearce and Pečarić proved the following theorem:

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, for $q \ge 1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}. \quad (4)$$

Recall that the function $f : [a, b] \to \mathbb{R}$ is said to be quasiconvex if for every $x, y \in I$ we have

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\}, \text{ for all } t \in [0, 1].$$

The generalizations of the Theorems 1.1, 1.2 are introduced by Ion in [12] for quasiconvex functions. Then, Alomari et al. in [1] improved the results in [12] and 1.3, for twice differentiable quasiconvex functions.

On the other hand, S.S. Dragomir et al. in [9] defined the following class of functions:

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ (or $P-$convex)if it is nonnegative and, for all $x, y \in I$ and $\lambda \in [0, 1]$, satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y). \quad (5)$$

Note that $P(I)$ contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category, see [2, 10, 15] and references therein. Özdmir and Yildeiz in [13] proved the folloving results

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $|f''| \in P(I)$, $a, b \in I^\circ$ with $a < b$. If f'' is a $P-$convex, $0 \le \lambda \le 1$, then the following inequality holds:*

$$\begin{aligned}&\left|(1-\lambda)f(\frac{a+b}{2}) - \lambda\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \begin{cases} \frac{(b-a)^2}{24}(8\lambda^3 - 3\lambda + 1)\{|f''(a)| + |f''(b)|\}, & 0 \le \lambda \le \frac{1}{2}, \\ \frac{(b-a)^2}{24}(3\lambda - 1)\{|f''(a)| + |f''(b)|\}, & \frac{1}{2} \le \lambda \le 1. \end{cases}\end{aligned} \quad (6)$$

Corollary 1.1. *If in Theorem 1.4 we choose $\lambda = 1$, we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{12}\left\{\left|f''(a)\right| + \left|f''(b)\right|\right\}. \quad (7)$$

Theorem 1.5. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is $P-$convex, $0 \le \lambda \le 1$ and $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}&\left|(1-\lambda)f(\frac{a+b}{2}) - \lambda\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \begin{cases} \frac{(b-a)^2}{48}(8\lambda^3 - 3\lambda + 1)\left(\{|f''(a)|^q + |f''(b)|^q\}\right)^{\frac{1}{q}}, & 0 \le \lambda \le \frac{1}{2}, \\ \frac{(b-a)^2}{48}(3\lambda - 1)\left(\{|f''(a)|^q + |f''(b)|^q\}\right)^{\frac{1}{q}}, & \frac{1}{2} \le \lambda \le 1. \end{cases}\end{aligned} \quad (8)$$

Corollary 1.2. *If in Theorem 1.5 we choose $\lambda = 1$, we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{24}\left(\left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{9}$$

The main purpose of this paper is to establish new estimations and refinements of the Hermite-Hadamard inequality (1) for functions whose second derivatives in absolute value are $P-$convex, which are better than results in [13]. Applications for special means are considered.

2. Main results

In order to prove our main theorems throughout this paper, we need the following Lemma in [4].

Lemma 2.1. *Suppose that $f: I \to \mathbb{R}$ is a twice differentiable function on I°, the interior of I. Assume that $a, b \in I^\circ$, with $a < b$ and f'' is integrable on $[a, b]$. Then, the following equality holds,*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\\ &= \frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left(f''(\frac{1+t}{2}a + \frac{1-t}{2}b) + f''(\frac{1-t}{2}a + \frac{1+t}{2}b)\right)dt.\end{aligned} \tag{10}$$

In the following theorem, we shall propose some new upper bound for the right-hand side of (1) for $P-$convex functions, which is better than the inequality had done in [13].

Theorem 2.1. *Let $f: I \to \mathbb{R}$ be a twice differentiable function on I° such that $|f''|$ is a $P-$convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L_1[a, b]$. Then, the following inequality holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{(b-a)^2}{24}\left[\left|f''(a)\right| + 2\left|f''(\frac{a+b}{2})\right| + \left|f''(b)\right|\right].\end{aligned} \tag{11}$$

Proof. Since $|f''|$ is a $P-$convex function, by using Lemma 2.1 we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|\\ &= \left|\frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left(f''(\frac{1+t}{2}a + \frac{1-t}{2}b) + f''(\frac{1-t}{2}a + \frac{1+t}{2}b)\right)dt.\right|\\ &\le \frac{(b-a)^2}{16}\int_0^1 |1-t^2|\left(\left|f''(\frac{1+t}{2}a + \frac{1-t}{2}b)\right| + \left|f''(\frac{1-t}{2}a + \frac{1+t}{2}b)\right|\right)dt\\ &= \frac{(b-a)^2}{24}\left[\left|f''(a)\right| + 2\left|f''(\frac{a+b}{2})\right| + \left|f''(b)\right|\right].\end{aligned} \tag{12}$$

□

An immediate consequence of Theorem 2.1 is as follows.

Corollary 2.1. *Let f as in Theorem 2.1, if in addition*
(i) $f''(\frac{a+b}{2}) = 0$, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{24}\left[\left|f''(b)\right|+\left|f''(b)\right|\right]. \tag{13}$$

(ii) $f''(a)=f''(b)=0$, *then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left|f''(\frac{a+b}{2})\right|. \tag{14}$$

The corresponding version for powers of the absolute value of the second derivative is incorporated in the following theorem.

Theorem 2.2. *Let* $f: I \to \mathbb{R}$ *be a differentiable function on* I°. *Assume that* $p \in \mathbb{R}$, $p > 1$ *such that* $|f''|^{p/p-1}$ *is a* $P-$*convex function on* I. *Suppose that* $a, b \in I^\circ$ *with* $a < b$ *and* $f'' \in L_1[a,b]$. *Then, the following inequality holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{24}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left[\left(\left|f''(a)\right|^q+\left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right. \\
&\left.+\left(\left|f''(b)\right|^q+\left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right],
\end{aligned} \tag{15}$$

where $1/p + 1/q = 1$.

Proof. By assumption, Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \left|\frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left(f''(\frac{1+t}{2}a+\frac{1-t}{2}b)+f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right)dt\right| \\
&\leq \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p\,dt\right)^{1/p}\left[\left(\int_0^1\left|f''(\frac{1+t}{2}a+\frac{1-t}{2}b)\right|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1\left|f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right|^q dt\right)^{1/q}\right] dt \\
&\leq \frac{(b-a)^2}{24}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left[\left(\left|f''(a)\right|^q+\left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right. \\
&\left.+\left(\left|f''(b)\right|^q+\left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right].
\end{aligned}$$

where $1/p + 1/q = 1$. We note that, the Beta and Gamma functions are defined respectively, as follows

$$\Gamma(x)=\int_0^1 e^{-x}t^{x-1}dt, \qquad x>0,$$

and

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad x>0,\ y>0,$$

are used to evaluate the integral $\int_0^1 \left(1-t^2\right)^p dt$. Indeed, by setting $t^2 = u$, we get

$$dt = \frac{1}{2}u^{-1/2}du,$$

and using property

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

of Beta function, we obtain

$$\begin{aligned}\int_0^1 \left(1-t^2\right)^p dt &= \frac{1}{2}\int_0^1 u^{-1/2}\left(1-u\right)^p du = \frac{1}{2}\beta(\frac{1}{2}, p+1) \\ &= 2^{-1}\frac{\Gamma(\frac{1}{2})\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} = 2^{-1}\frac{\sqrt{\pi}\,\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \\ &= \left(\frac{\sqrt{\pi}}{2}\right)\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)},\end{aligned}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and proof is completed. □

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.2. *Let f as in Theorem 2.2, if in addition*
(i) $f''(\frac{a+b}{2}) = 0$, then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{24}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left(\left|f''(b)\right| + \left|f''(b)\right|\right). \tag{16}$$

(ii) $f''(a) = f''(b) = 0$, then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{12}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p}\left|f''(\frac{a+b}{2})\right|. \tag{17}$$

Another similar result may be extended in the following theorem.

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°. Assume that $q \ge 1$ such that $|f''|^q$ is a $P-$convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L_1[a,b]$. Then, the following inequality holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{(b-a)^2}{24}\left[\left(\left|f''(a)\right|^q + \left|f''(\frac{a+b}{2})\right|^q\right)^{1/q} + \left(\left|f''(b)\right|^q + \left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right],\end{aligned} \tag{18}$$

where $1/p + 1/q = 1$.

Proof. Suppose that $a, b \in I^\circ$. From Lemma 2.1 and using well known power mean inequality we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \left|\frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left(f''(\frac{1+t}{2}a+\frac{1-t}{2}b) + f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right)dt\right| \\
&\le \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)\,dt\right)^{1-1/q}\left[\left(\int_0^1 (1-t^2)\left|f''(\frac{1+t}{2}a+\frac{1-t}{2}b)\right|^q dt\right)^{1/q}\right. \\
&+ \left.\left(\int_0^1 (1-t^2)\left|f''(\frac{1-t}{2}a+\frac{1+t}{2}b)\right|^q dt\right)^{1/q}\right] dt \\
&\le \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-1/q}\left[\left(\frac{2}{3}\left\{\left|f''(a)\right|^q + \left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right. \\
&+ \left.\left(\frac{2}{3}\left\{\left|f''(b)\right|^q + \left|f''(\frac{a+b}{2})\right|^q\right\}\right)^{1/q}\right] \\
&= \frac{(b-a)^2}{24}\left[\left(\left|f''(a)\right|^q + \left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right. \\
&+ \left.\left(\left|f''(b)\right|^q + \left|f''(\frac{a+b}{2})\right|^q\right)^{1/q}\right],
\end{aligned}$$

which completes the proof. □

Corollary 2.3. *Let f as in Theorem 2.3, if in addition*
(i) $f''(\frac{a+b}{2}) = 0$, then (13) holds.
(ii) $f''(a) = f''(b) = 0$, (14) holds.

3. Applications to special means

Now, we consider the applications of our Theorems to the special means. We consider the means for arbitrary real numbers $\alpha, \beta (\alpha \neq \beta)$. We take
(1) Arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \qquad \alpha,\beta \in \mathbb{R}.$$

(2) Logarithmic mean:

$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}.$$

(3) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{N},\ \alpha,\beta \in \mathbb{R}, \alpha \neq \beta.$$

Now, using the results of Section 2, we give some applications for special means of real numbers.

Proposition 3.1. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, we have*

$$|L_n^n(a,b) - A(a^n, b^n)| \leq \frac{n(n-1)}{24}(b-a)^2 \left[|a|^{n-2} + 2\left|\frac{a+b}{2}\right|^{n-2} + |b|^{n-2} \right].$$

Proof. The assertion follows from Theorem 2.1 applied to the $P-$convex function $f(x) = x^n$, $x \in \mathbb{R}$. □

Proposition 3.2. *Let $a, b \in \mathbb{R}$, $a < b$ and $0 \notin [0,1]$. Then, for all $p > 1$ we have*

$$\left|L^{-1}(a,b) - A(a^{-1}, b^{-1})\right| \leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{1/p} \left[\left(|a|^{-3q} + \left|\frac{a+b}{2}\right|^{-3q}\right)^{1/q} + \left(|b|^{-3q} + \left|\frac{a+b}{2}\right|^{-3q}\right)^{1/q}\right],$$

where $1/p + 1/q = 1$.

Proof. The assertion follows from Theorem 2.2 applied to the $P-$convex function $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

Proposition 3.3. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, for all $q \geq 1$ we have*

$$|L_n^n(a,b) - A(a^n, b^n)| \leq \frac{n(n-1)}{24}(b-a)^2 \left[\left(|a|^{(n-2)q} + \left|\frac{a+b}{2}\right|^{(n-2)q}\right)^{1/q} + \left(|b|^{(n-2)q} + \left|\frac{a+b}{2}\right|^{(n-2)q}\right)^{1/q}\right].$$

Proof. The assertion follows from Theorem 2.3 applied to the $P-$convex function $f(x) = x^n$, $x \in \mathbb{R}$. □

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