Received 22/09/11

ON A GEOMETRIC INEQUALITY OF OPPENHEIM

JIAN LIU

ABSTRACT. In this paper we give a simple proof of an Oppenheim's geometric inequality by using a new lemma, we also prove a refinement of the Oppenheim inequality. Some related conjectures which have been checked by the computer are put forward.

1. Introduction

Let P be an arbitrary interior point of the triangle ABC. Denote by R_1, R_2, R_3 the distances from P to the vertices A, B, C, and r_1, r_2, r_3 the distances from P to the sidelines BC, CA, AB respectively. Then

$$R_2R_3 + R_3R_1 + R_1R_2 \geq 4(r_2r_3 + r_3r_1 + r_1r_2). \tag{1.1}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center. This inequality was first published by J.M.Child [1]. In 1964, L.Carlitz [2] established the following stronger inequality:

$$R_2R_3 + R_3R_1 + R_1R_2 \geq 4(w_2w_3 + w_3w_1 + w_1w_2), \tag{1.2}$$

where w_1, w_2, w_3 are the internal angle-bisectors of $\angle BPC, \angle CPA, \angle APB$ respectively. The author [3] generalized further Carlitz's result in 1996:

$$\begin{aligned} &x^2(R_2R_3)^k + y^2(R_3R_1)^k + z^2(R_1R_2)^k \\ &\geq 2^k\left[yz(w_2w_3)^k + zx(w_3w_1)^k + xy(w_1w_2)^k\right]. \end{aligned} \tag{1.3}$$

where x, y, z are arbitrary real numbers and exponent k satisfies $0 < k \leq 1$. On the other hand, A.Oppenheim [4] considered the stronger version of (1.1) from another viewpoint as early as 1961, and he concluded that the following inequality holds without proof at the end of the reference:

Theorem 1.1. *For any arbitrary point P of $\triangle ABC$, we have*

$$\begin{aligned} &R_2R_3 + R_3R_1 + R_1R_2 \\ &\geq (r_1 + r_2)(r_3 + r_1) + (r_2 + r_3)(r_1 + r_2) + (r_3 + r_1)(r_2 + r_3), \end{aligned} \tag{1.4}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

Inequality (1.4) is equivalent to

$$R_2R_3 + R_3R_1 + R_1R_2 \geq r_1^2 + r_2^2 + r_3^2 + 3(r_2r_3 + r_3r_1 + r_1r_2). \tag{1.5}$$

It is easy to show that the combination coefficients of the right hand side is the best possible. In other words, (1.5) is the strongest in the following type inequality:

$$R_2R_3 + R_3R_1 + R_1R_2 \geq m(r_1^2 + r_2^2 + r_3^2) + n(r_2r_3 + r_3r_1 + r_1r_2), \tag{1.6}$$

2000 *Mathematics Subject Classification.* 51M16.

Key words and phrases. triangle, interior point, Oppenheim inequality, Erdös-Mordell inequality, conjecture.

where m, n are positive constants.

Oppenheim tried to prove his inequality in another paper [5] in the same year. His method is as follows: First suppose that $a \geq b \geq c$, then make use of the well known inequalities

$$aR_1 \geq cr_2 + br_3, \tag{1.7}$$

$$aR_1 \geq br_2 + cr_3, \tag{1.8}$$

(where $a = BC, b = CA, c = AB$)to prove respectively that the inequality holds in the following six cases:

(i) $r_1 \geq r_2 \geq r_3$; (ii) $r_1 \geq r_3 \geq r_2$; (iii) $r_2 \geq r_3 \geq r_1$;
(iv) $r_2 \geq r_1 \geq r_3$; (v) $r_3 \geq r_1 \geq r_2$; (vi) $r_3 \geq r_2 \geq r_1$.

However, he only discussed amply the two cases of (i) and (vi), and also pointed out the other four cases can be proved by the same way. The author thinks this proof is not faultless. Every case should be considered respectively. In a recent paper [6], J.M.Hamiton by using Oppenheim's method, finished the proof of all six cases. But his proof is very complicated.

The purpose of this note is to give a simple proof of the Oppenheim inequality (1.4), also prove the following refinement result:

Theorem 1.2. *For an arbitrary interior point P of the triangle ABC, we have*

$$\begin{aligned} &R_2R_3 + R_3R_1 + R_1R_2 \\ &\geq h_ar_1 + h_br_2 + h_cr_c + r_2r_3 + r_3r_1 + r_1r_2 \\ &\geq (r_1 + r_2)(r_3 + r_1) + (r_2 + r_3)(r_1 + r_2) + (r_3 + r_1)(r_2 + r_3), \end{aligned} \tag{1.9}$$

where h_a, h_b, h_c are three altitudes of the triangle ABC. Both equalities in (1.9) hold if and only if $\triangle ABC$ is equilateral and P is its center.

In studying Oppenheim inequality, a lot of geometric inequalities were found by the author. We will state some related conjectures in the last section of this note.

2. Proofs of the theorems

The proofs of the two theorems are both need the following key lemma:

Lemma 2.1. *For any point P of $\triangle ABC$, we have*

$$R_2 + R_3 \geq 2r_1 + \frac{(r_2 + r_3)^2}{R_1}, \tag{2.1}$$

with equality if and only if $b = c$ and P is the circumcenter of $\triangle ABC$.

Proof. Inequality (1.9) is equivalent to

$$R_1(R_2 + R_3 - 2r_1) - (r_2 + r_3)^2 \geq 0.$$

Note that $R_2 + R_3 > 2r_1$, by inequality (1.7) and its two analogues $bR_2 \geq ar_3 + cr_1, cR_3 \geq br_1 + ar_2$, it is suffice to prove that

$$\frac{cr_2 + br_3}{a}\left(\frac{ar_3 + cr_1}{b} + \frac{br_1 + ar_2}{c} - 2r_1\right) \geq (r_2 + r_3)^2,$$

Namely,

$$\frac{(ar_2r_3 + br_3r_1 + cr_1r_2)(b - c)^2}{abc} \geq 0, \tag{2.2}$$

which is obviously true. We have known that if AO (O is the circumcenter of $\triangle ABC$) cuts BC at X then the equality in (1.7) holds if and only if P lies on the segment AX. According to this conclusion and (2.2), we conclude that the equality in (2) occurs if and only if $b = c$ and P is its circumcenter. This completes the proof of Lemma 2.1. □

Remark 2.1. By the arithmetic-geometric mean inequality, we can easily see that inequality (2) is stronger than the following inequality:

$$R_1(R_2 + R_3)^2 \geq 8r_1(r_2 + r_3)^2. \tag{2.3}$$

In fact, we have the more stronger inequality:

$$R_1(R_2 + R_3)^2 \geq 8w_1(w_2 + w_3)^2, \tag{2.4}$$

which was first posed by the author and proved by Zhi-Hua Zhang and Yu-Dong Wu [7].

Remark 2.2. The famous Erdös-Mordell inequality (see [8]-[20]):

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \tag{2.5}$$

can be deduced from Lemma 2.1 as follows: According to (1.9) and Cauchy inequality, we have that

$$\begin{aligned}
&2(R_1 + R_2 + R_3)\\
&\geq 2(r_1 + r_2 + r_3) + \frac{(r_2 + r_3)^2}{R_1} + \frac{(r_3 + r_1)^2}{R_2} + \frac{(r_1 + r_2)^2}{R_3}\\
&\geq 2(r_1 + r_2 + r_3) + \frac{4(r_1 + r_2 + r_3)^2}{R_1 + R_2 + R_3}.
\end{aligned}$$

Therefore

$$(R_1 + R_2 + R_3)^2 - (R_1 + R_2 + R_3)(r_1 + r_2 + r_3) - 2(r_1 + r_2 + r_3)^2 \geq 0.$$

Namely,

$$(R_1 + R_2 + R_3 + r_1 + r_2 + r_3)[R_1 + R_2 + R_3 - 2(r_1 + r_2 + r_3)] \geq 0,$$

So we have inequality (2.5).

The proofs about the Erdös-Mordell inequality has been giving constantly (see [10]-[20]). Recently, the author gave an alternative proof in [20].

2.1. **Proof of Theorem 1.1.**

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&R_1(R_2 + R_3) + R_2(R_3 + R_1) + R_3(R_1 + R_2)\\
&\geq 2(R_1r_1 + R_2r_2 + R_3r_3) + (r_2 + r_3)^2 + (r_3 + r_1)^2 + (r_1 + r_2)^2,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&R_2R_3 + R_3R_1 + R_1R_2 \geq R_1r_1 + R_2r_2 + R_3r_3\\
&+\frac{1}{2}\left[(r_2 + r_3)^2 + (r_3 + r_1)^2 + (r_1 + r_2)^2\right].
\end{aligned} \tag{2.6}$$

It is easy to see that the equality in (2.6) holds if and only if $a = b = c$ and P is the circumcenter.

Now observe that

$$\begin{aligned}&R_1r_1+R_2r_2+R_3r_3+\frac{1}{2}\left[(r_2+r_3)^2+(r_3+r_1)^2+(r_1+r_2)^2\right]\\&-\left[(r_1+r_2)(r_3+r_1)+(r_2+r_3)(r_1+r_2)+(r_3+r_1)(r_2+r_3)\right]\\&=R_1r_1+R_2r_2+R_3r_3-2(r_2r_3+r_3r_1+r_1r_2)\end{aligned}$$

and the well known result (see [4],[8],[9]):

$$R_1r_1+R_2r_2+R_3r_3\geq 2(r_2r_3+r_3r_1+r_1r_2), \tag{2.7}$$

Oppenheim inequality (1.4) follows from (2.6) at once. Clearly, the equality in (1.4) holds is the same as (2.6). Theorem 1.1 is proved. □

2.2. **Proof of Theorem 1.2.**

Proof. We first prove the first inequality of (1.9)

$$R_2R_3+R_3R_1+R_1R_2\geq h_ar_1+h_br_2+h_cr_c+r_2r_3+r_3r_1+r_1r_2. \tag{2.8}$$

According to the known inequality (1.8) and its two analogues, we get

$$\begin{aligned}&R_1r_1+R_2r_2+R_3r_3+\frac{1}{2}\left[(r_2+r_3)^2+(r_3+r_1)^2+(r_1+r_2)^2\right]\\&\geq\frac{(br_2+cr_3)r_1}{a}+\frac{(cr_3+ar_1)r_2}{b}+\frac{(ar_1+br_2)r_3}{c}\\&\quad+\frac{1}{2}\left[(r_2+r_3)^2+(r_3+r_1)^2+(r_1+r_2)^2\right]\\&=r_2r_3+r_3r_1+r_1r_2\\&\quad+r_1\left(r_1+\frac{br_2+cr_3}{a}\right)+r_2\left(r_2+\frac{cr_3+ar_1}{b}\right)+r_3\left(r_3+\frac{ar_1+br_2}{c}\right)\\&=r_2r_3+r_3r_1+r_1r_2+(ar_1+br_2+cr_3)\left(\frac{r_1}{a}+\frac{r_2}{b}+\frac{r_3}{c}\right)\\&=h_ar_1+h_br_2+h_cr_3+r_2r_3+r_3r_1+r_1r_2,\end{aligned}$$

where we used the identity $ar_1+br_2+cr_3=2S=ah_a=bh_b=ch_c$($S$ is the area of $\triangle ABC$). Therefore, inequality (2.8) follows from (2.6). Clearly, the equality condition in (2.8) is the same as (2.6).

It is easy to check that the second inequality of (1.9) is equivalent to

$$h_ar_1+h_br_2+h_cr_3\geq(r_1+r_2+r_3)^2, \tag{2.9}$$

which follows from Cauchy inequality and the simple identity:

$$\frac{r_1}{h_a}+\frac{r_2}{h_b}+\frac{r_3}{h_c}=1. \tag{2.10}$$

The proof of Theorem 1.2 is completed. □

Remark 2.3. In [5], A.Oppenheim pointed out a set of inequalities equivalent to (1.4) by using various geometric transformations (see [4],[9],[21]). If we apply these transformations to the stronger inequality (2.8), then we can get some new results. For example, applying reciprocation transformation to (2.8), one obtain the following inequality:

$$\begin{aligned}&\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}-\frac{1}{R_2R_3}-\frac{1}{R_3R_1}-\frac{1}{R_1R_2}\\&\geq\frac{h_a}{r_1R_1^2}+\frac{h_b}{r_2R_2^2}+\frac{h_c}{r_3R_3^2}.\end{aligned} \tag{2.11}$$

3. Some related conjectures

In this section we will give some related conjectures which all have been checked by the computer.

First considering the stronger inequality of Lemma 2.1, we propose the following conjecture:

Conjecture 3.1. *For any interior point P of $\triangle ABC$, we have*

$$R_2 + R_3 \geq 2w_1 + \frac{(w_2 + w_3)^2}{R_1}. \tag{3.1}$$

If the above equality is valid, using the same way to deduce Erdös-Mordell inequality, we can prove Barrow's inequality:

$$R_1 + R_2 + R_3 \geq 2(w_1 + w_2 + w_3). \tag{3.2}$$

In addition, if (3.1) holds true, then using it we can easily prove that

$$4R_1^2 + (R_2 + R_3)^2 \geq 8R_1 w_1 + 4(w_2 + w_3)^2. \tag{3.3}$$

This inequality inspires the author to put forward the following inequality:

Conjecture 3.2. *For any interior point P of $\triangle ABC$, we have*

$$R_1^2 + R_2 R_3 \geq 2(R_1 w_1 + 2w_2 w_3). \tag{3.4}$$

When the author considered the proof of Conjecture 3.1, we conjectured the following:

Conjecture 3.3. *For any interior point P of $\triangle ABC$, we have*

$$\frac{a^2}{(w_2 + w_3)^2} - \frac{(w_2 + w_3)^2}{R_1^2} \geq \frac{4w_1}{R_1}. \tag{3.5}$$

In deed, inequality (3.5) is stronger than (2.10). The following similar inequality has not yet been proved:

Conjecture 3.4. *For any interior point P of $\triangle ABC$, we have*

$$\frac{a^2}{(r_2 + r_3)^2} - \frac{(r_2 + r_3)^2}{R_1^2} \geq \frac{4r_1}{R_1}. \tag{3.6}$$

In [22], the author obtained the inequality:

$$R_2 R_3 + R_3 R_1 + R_1 R_2 \geq \left(\frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3}\right)^2. \tag{3.7}$$

This inequality prompts the author again to pose the following stronger inversion of Oppenheim inequality (1.5):

Conjecture 3.5. *For any interior point P of $\triangle ABC$, we have*

$$\left(\frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3}\right)^2 \geq r_1^2 + r_2^2 + r_3^2 + 3(r_2 r_3 + r_3 r_1 + r_1 r_2). \tag{3.8}$$

In addition, it is possible that the Oppenheim inequality (1.4) has a stronger version:

$$\begin{aligned} R_2 R_3 + R_3 R_1 + R_1 R_2 \geq{} & (w_1 + w_2)(w_3 + w_1) + (w_2 + w_3)(w_1 + w_2) \\ & + (w_3 + w_1)(w_2 + w_3). \end{aligned} \tag{3.9}$$

We also think it has the following exponential generalization:

Conjecture 3.6. *If real number k satisfies $0 < k \le 2$, then we have*

$$(R_2R_3)^k + (R_3R_1)^k + (R_1R_2)^k \ge (w_1 + w_2)^k(w_3 + w_1)^k$$
$$+(w_2 + w_3)^k(w_1 + w_2)^k + (w_3 + w_1)^k(w_2 + w_3)^k. \tag{3.10}$$

When $-0.35 \le k < 0$ the inequality is reverse.

On the other hand, we also suppose that the equivalent form of (3.9) can be generalized to the case involving two points:

Conjecture 3.7. *For any interior point P of $\triangle ABC$ and arbitrary point Q, we have*

$$(R_2 + R_3)D_1 + (R_3 + R_1)D_2 + (R_1 + R_2)D_3$$
$$\ge 2(w_1^2 + w_2^2 + w_3^2) + 6(w_2w_3 + w_3w_1 + w_1w_2), \tag{3.11}$$

where D_1, D_2, D_3 denote distances from Q to the vertices A, B, C respectively.

It seems to be very hard even to prove the following much weaker inequality:

$$(R_2 + R_3)D_1 + (R_3 + R_1)D_2 + (R_1 + R_2)D_3 \ge 8(r_2r_3 + r_3r_1 + r_1r_2), \tag{3.12}$$

which is similar the following inequality proved by the authors of [23]:

$$R_1D_1 + R_2D_2 + R_3D_3 \ge 4(r_2r_3 + r_3r_1 + r_1r_2). \tag{3.13}$$

When the author studied inequality (3.9), the following interesting inequality was found:

Conjecture 3.8. *For any interior point P of $\triangle ABC$, we have*

$$R_1^2 + 2R_2R_3 \ge w_1^2 + w_2^2 + w_3^2 + 3(w_2w_3 + w_3w_1 + w_1w_2), \tag{3.14}$$

Note that inequality (2.8), we also propose the conjecture:

Conjecture 3.9. *For any interior point P of $\triangle ABC$ holds:*

$$R_1^2 + 2R_2R_3 \ge h_ar_1 + h_br_2 + h_cr_3 + r_2r_3 + r_3r_1 + r_1r_2. \tag{3.15}$$

For inequality (2.8), we pose the following stronger inequality:

Conjecture 3.10. *For any interior point P of $\triangle ABC$ holds:*

$$R_2R_3 + R_3R_1 + R_1R_2 \ge w_aw_1 + w_bw_2 + w_cw_3 + w_2w_3 + w_3w_1 + w_1w_2. \tag{3.16}$$

where w_a, w_b, w_c are the angle bisectors of the $\triangle ABC$.

Comparing with (3.15) and (1.2), we suggest that again:

Conjecture 3.11. *For any interior point P of $\triangle ABC$ holds:*

$$w_aw_1 + w_bw_2 + w_cw_3 \ge 3(w_2w_3 + w_3w_1 + w_1w_2). \tag{3.17}$$

If this inequality holds, then it shows that (3.16) is stronger than (1.2).

Finally, we are going to put forward a conjecture involving six segments R_1, R_2, R_3 and r_1, r_2, r_3. It is easy to prove that the following inequality which is similar to the preceding inequality (2.3):

$$R_1(R_2 + R_3) > 2r_1(r_2 + r_3), \tag{3.18}$$

where the constant 2 of the right hand side is the best possible. It is equivalent to

$$\frac{R_2 + R_3}{r_2 + r_3} - \frac{2r_1}{R_1} > 0. \tag{3.19}$$

The above strict inequality motivates us to find the following stronger conjecture:

Conjecture 3.12. *For any interior point P of $\triangle ABC$ holds:*

$$\frac{R_2+R_3}{r_2+r_3}-\frac{2r_1}{R_1}\geq 1. \tag{3.20}$$

It is very interesting that the equality in (3.20) seems to be special. We conjectured that the equality holds if and only if $b=c$ and P coincide with a fixed point of the altitude drawn from vertex A to the side BC. But we do not know what the fixed point is.

References

[1] J.M.Child, Inequalities connected with a triagnle, Math Gazette, 23(1939), 138-143.
[2] L.Carlitz, Some inequalities for a triangle, Amer. Math. Monthly, 71(1964), 881-885.
[3] J.Liu, A generalization of a weighted trigonometric inequality and its equivalent form, Secondary Education(Math),4(1996), 29-32.(In Chinese)
[4] A.Oppenheim, The Erdös-Mordell Inequality and and other inequalities for a triangle, Amer.Math.Monthly, 68(1961), 226-230.
[5] A.Oppenheim, New inequalities for a triangle and an internal point, Ann. Math. Pura Appl. (4)53(1961), 157-164.
[6] J.M.Hamilton, An exploration of the the Erdös-Mordell inequality (Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in the Mathematics Program), 160pp. http://etd.ohiolink.edu/search.cgi?q=inequality.
[7] Z.H.Zhang and Y.D.Wu, On a trigonometric inequality and its applications, RGMIA Research Report Collection, **7(3)**(2004), Article 8,4 pp.
[8] O.Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, Geometric Inequalities, Groningen, 1969.
[9] D.Mitrinovió, J. E. Peŏarić and V. Volenec, Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, Dordrecht,Boston,London, 1989.
[10] L.J.Mordell and D.F.Barrow, Solution of Problem 3740, Amer.Math.Monthly, 44(1937), 252-254.
[11] G.R.Veldkamp, The Erdös-Mordell Inequality, Nieuw Tijdschr.Wisk, 45(1957/1958),193-196.
[12] V.Komornik, A short proof of the Erdös-Mordell theorem, Amer.Math.Monthly, 104(1997), 57-60.
[13] D.K.Kazarinoff, A simple proof of the Erdös-Mordell inequality for triangles, Michigan Mathematical Journal, 4(1957), 97-98.
[14] L.Bankoff, An elementary proof of the Erdöos-Mordell theorem, Amer.Math.Monthly, 65(1958), 521.
[15] A.Avez, A short proof of the Erdös and Mordell theorem, Amer.Math.Monthly, 100(1993), 60-62.
[16] H.Lee, Another proof of the Erdös-Mordell theorem, Forum Geom, 1(2001), 7-8.
[17] N.Dergiades, Signed distances and the Erdös-Mordell inequality, Forum Geom, 4(2004), 67-68.
[18] C.Alsina, R.B.Nelsen, A visual proof of the Erdös-Mordell inequality, Forum Geom, 7(2001), 99-102.
[19] J.Liu, A sharpening of the Erdös-Mordell and its applications, Journal of Chongqing Normal University (Natural Science Edition), 22(2)(2005), 12-14(in Chinese).
[20] J.Liu, A new proof of the Erdös-Mordell inequality, RGMIA Research Report Collection, 14(2011), Article 13, 6 pp. http://rgmia.org/v14.php.
[21] J.Liu, Several new inequalities for the triangle, Mathematics Competition, Hunan Education Press, **15**(1992), 80-100(in Chinese).
[22] J.Liu, A weighted geometric inequality and its applications, J.Inequal.Pure Appl.Math, 9(2),art.58, 2008.
[23] Y.D.Wu, C.L.Yu, Z.H.Zhang, A Geometric Inequality of the Generalized Erdös-Mordell Type, J.Inequal.Pure Appl.Math, 10(4),art.106, 2009.

East China Jiaotong University, Jiangxi province Nanchang City, 330013, China
E-mail address: china99jian@163.com