Received 22/09/11

NEW REFINEMENTS OF THE NEUBERG-PEDOE INEQUALITY

JIAN LIU

ABSTRACT. In this note, by using elementary method, we prove new refinements of the famous Neuberg-Pedoe inequality involving two triangles.

1. **Introduction**

In 1891, J.Neuberg [1] found the first interesting inequality concerning with two triangles:

Let a, b, c denote the edge-lengths of the $\triangle ABC$ with area $\triangle$, and let a', b', c' denote the the edge-lengths of $\triangle A'B'C'$ with area $\triangle'$. Then

$$H \equiv a^2(b'^2 + c'^2 - a'^2) + b^2(c'^2 + a'^2 - b'^2) + c^2(a'^2 + b'^2 - c'^2) \geq 16\triangle\triangle', \quad (1.1)$$

with equality holds if and only if two triangles are similar.

In 1943, D.Pedoe [2] renewedly obtained inequality (1.1). Thereafter, many mathematicians has been interested in this inequality, and it is called Neuberg-Pedoe inequality. There exist a large number of research papers involving its new proofs, various generalizations, variations and applications, etc. Some related results with historical comments on the Neuberg-Pedoe inequality can be found in [1] to [15]. We recall here several refinements.

In 1983, K.S.Poh [7] proved the following refinement of (1.1):

$$H \geq E \geq 16\triangle\triangle'. \quad (1.2)$$

where

$$E = \sum a^2 \sum a'^2 - 2\left(\sum a^4 \sum a'^4\right)^{1/2},$$

and $\sum$ denotes the cyclic sum. The equality of $E \geq 16\triangle\triangle'$ holds if and only if $\sum \cot A = \sum \cot A'$, where A, A' denote the interior angles of $\triangle ABC, \triangle A'B'C'$ respectively, etc.

In 1984, Chia-Kuei Peng [8] established the following sharpening of the Neuberg-Pedoe inequality:

$$H \geq 8\left(\frac{a'^2 + b'^2 + c'^2}{a^2 + b^2 + c^2}\triangle^2 + \frac{a^2 + b^2 + c^2}{a'^2 + b'^2 + c'^2}\triangle'^2\right). \quad (1.3)$$

This result has also attracted much attention. For example, the authors of the monograph [14] gave its refinement:

$$H \geq E \geq 8\left(\frac{a'^2 + b'^2 + c'^2}{a^2 + b^2 + c^2}\triangle^2 + \frac{a^2 + b^2 + c^2}{a'^2 + b'^2 + c'^2}\triangle'^2\right). \quad (1.4)$$

2000 *Mathematics Subject Classification.* 51M16.
Key words and phrases. triangle, Neuberg-Pedoe inequality, inequality.

G.S. Leng and L.H. Tang [10] obtained the weighted generalization:

$$H \geq 8\left(\frac{xa'^2+yb'^2+zc'^2}{xa^2+yb^2+zc^2}\triangle^2+\frac{xa^2+yb^2+zc^2}{xa'^2+yb'^2+zc'^2}\triangle'^2\right), \tag{1.5}$$

where x, y, z are arbitrary non-negative real numbers.

The purpose of this note is to establish the following refinements of the Neuberg-Pedoe inequality:

Theorem 1. *Let R and r denote the radius of circumcircle and incenter of $\triangle ABC$ respectively, s the semi-perimeter. Let s' denote the semi-perimeter of $\triangle A'B'C'$. Other symbols are the same as above. Put*

$$H_1 = 32Rrs'^2 - 8s'\sum a'(s-b)(s-c),$$
$$H_2 = 16Rrs'^2 - 4\sum (s-b)(s-c)a'^2,$$
$$H_3 = 8\sum a(s-a)(s'-b')(s'-c'),$$

then

$$H \geq H_1 \geq H_2 \geq H_3 \geq 16\triangle\triangle', \tag{1.6}$$

with equalities if and only if $\triangle ABC$ and $\triangle A'B'C'$ are similar.

2. Two lemmas

Lemma 1. *Let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers, then the inequality:*

$$p_1x^2 + p_2y^2 + p_3z^2 \geqslant q_1yz + q_2zx + q_3xy \tag{2.1}$$

holds for arbitrary real numbers x, y, z if and only if

$$p_1 \geqslant 0, p_2 \geqslant 0, p_3 \geqslant 0, 4p_2p_3 - q_1^2 \geqslant 0, 4p_3p_1 - q_2^2 \geqslant 0, 4p_1p_2 - q_3^2 \geqslant 0,$$

and

$$M \equiv 4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geqslant 0. \tag{2.2}$$

If $p_1 > 0, p_2 > 0, p_3 > 0, q_1 > 0, q_2 > 0, q_3 > 0, 4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0$ and $M \geq 0$, then the equality in (2.1) holds if and only if $M = 0$ and $x : y : z = \sqrt{4p_2p_3 - q_1^2} : \sqrt{4p_3p_1 - q_2^2} : \sqrt{4p_1p_2 - q_3^2}$.

The above lemma is the decision theorem of ternary quadratic inequalities and it can be easily proved by using discriminant analysis method, see [16]. An important consequence which will be used below is the famous Wolstenholme inequality (see [14, p.421]):

$$x^2 + y^2 + z^2 \geq 2(yz\cos A + zx\cos B + xy\cos C), \tag{2.3}$$

where A, B, C are angles of $\triangle ABC$. Equality holds if and only if $x : y : z = a : b : c$. In fact, note that the following known identity in the triangle:

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1, \tag{2.4}$$

Wolstenholme inequality (2.3) follows from Lemma 1 at once.

Wolstenholme inequality has several equivalent forms. Now we give its new one:

Lemma 2. *For any $\triangle ABC$ and real numbers x, y, z, the following inequality holds:*

$$\sum x(y+z)(s-b)(s-c) \leq Rr\left(\sum x\right)^2 \tag{2.5}$$

with equality if and only if $x : y : z = a : b : c$.

Proof. We can easily check that

$$\sum x(y+z)(s-b)(s-c) = \sum a(s-a)yz. \tag{2.6}$$

Thus, inequality (2.5) is equivalent to

$$\sum a(s-a)yz \leq Rr\left(\sum x\right)^2. \tag{2.7}$$

Since $abc = 4Rrs$ and $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$, we have

$$\cos^2\frac{A}{2} = \frac{a(s-a)}{4Rr}. \tag{2.8}$$

Further, (2.5) is equivalent to

$$4\sum yz\cos^2\frac{A}{2} \leq \left(\sum x\right)^2, \tag{2.9}$$

which is equivalent with Wolstenholme (2.3). Clearly, the inequality condition in (2.5) is the same as (2.3). This completes the proof of Lemma 2. □

3. **Proof of the theorem**

Proof. We first prove the first inequality of (1.6):

$$H \geq H_1, \tag{3.1}$$

Namely,

$$\sum a^2(b'^2+c'^2-a'^2) - \left[32Rrs'^2 - 8s'\sum a'(s-b)(s-c)\right] \geq 0. \tag{3.2}$$

By the fact $abc = 4Rrs$ and the identity:

$$\sum a^2(b'^2+c'^2-a'^2) = \sum a'^2(b^2+c^2-a^2), \tag{3.3}$$

we need to prove that

$$s\sum a'^2(b^2+c^2-a^2) - \left[8abcs'^2 - 8ss'\sum a'(s-b)(s-c)\right] \geq 0. \tag{3.4}$$

To do so, we put $s'-a' = x, s'-b' = y, s'-c' = z$, then $a' = y+z, b' = z+x, c' = x+y, s' = \sum x$. Thus, (19) is equivalent to

$$\begin{aligned} &s\sum (y+z)^2(b^2+c^2-a^2) \\ &-\left[8abc\left(\sum x\right)^2 - 8s\sum x\sum (y+z)(s-b)(s-c)\right] \geq 0. \end{aligned}$$

Substituting $s = \frac{1}{2}(a+b+c)$ into the above, we have to prove

$$\begin{aligned} &\frac{1}{2}\sum a\sum (y+z)^2(b^2+c^2-a^2) - 8abc\left(\sum x\right)^2 \\ &+\sum a\sum x\sum (y+z)(c+a-b)(a+b-c) \geq 0. \end{aligned} \tag{3.5}$$

This can be rewritten as follows:

$$m_1x^2 + m_2y^2 + m_3z^2 - (n_1yz + n_2zx + n_3xy) \geq 0, \tag{3.6}$$

where

$$m_1 = a\left[a(b+c-a)+2(b-c)^2\right],$$
$$m_2 = b\left[b(c+a-b)+2(c-a)^2\right],$$
$$m_3 = c\left[c(a+b-c)+2(a-b)^2\right],$$
$$n_1 = a^3+b^3+c^3+8abc-3bc(b+c)-a^2(b+c)-a(b^2+c^2),$$
$$n_2 = a^3+b^3+c^3+8abc-3ca(c+a)-b^2(c+a)-b(c^2+a^2),$$
$$n_3 = a^3+b^3+c^3+8abc-3ab(a+b)-c^2(a+b)-c(a^2+b^2).$$

Then we can easily obtain

$$4m_1m_2m_3-(m_1m_2m_3+m_1n_1^2+m_2n_2^2+m_3n_3^2)=0, \tag{3.7}$$
$$4m_2m_3-n_1^2=(a+b+c)(c+a-b)(a+b-c)(b+c-a)^3>0, \tag{3.8}$$
$$4m_3m_1-n_2^2=(a+b+c)(a+b-c)(b+c-a)(c+a-b)^3>0, \tag{3.9}$$
$$4m_1m_2-n_3^2=(a+b+c)(b+c-a)(c+a-b)(a+b-c)^3>0. \tag{3.10}$$

According to Lemma 1, inequality (3.6) is proved and the equality in (3.6) holds if and only if $x:y:z=(b+c-a):(c+a-b):(a+b-c)$. Hence, the equality in (3.4) holds if and only if $(s'-a'):(s'-b'):(s'-c')=(s-a):(s-b):(s-c)$, namely $a':b':c'=a:b:c$. Therefore, the equality in (3.1) holds if and only if $\triangle ABC \sim \triangle A'B'C'$.

Secondly, we prove the second of inequality chain (1.6):

$$H_1 \geq H_2. \tag{3.11}$$

It is easy to check that

$$H_1-H_2=4\left[4Rrs'^2-\sum a'(b'+c')(s-b)(s-c)\right]. \tag{3.12}$$

So we need to prove that

$$\sum a'(b'+c')(s-b)(s-c) \leq 4Rrs'^2. \tag{3.13}$$

which is an evident consequence of Lemma 2. Obviously, the equalities in (3.13) and (3.11) hold if and only if $\triangle ABC \sim \triangle A'B'C'$.

Next, we prove the third inequality of (1.6):

$$H_2 \geq H_3, \tag{3.14}$$

Namely,

$$16Rrs'^2-4\sum(s-b)(s-c)a'^2 \geq 8\sum a(s-a)(s'-b')(s'-c'). \tag{3.15}$$

As the way to prove inequality (3.4), we put $s'-a'=x, s'-b'=y, s'-c'=z$, then inequality (3.15) is equivalent to

$$4Rr\left(\sum x\right)^2-4\sum(s-b)(s-c)(y+z)^2-8\sum a(s-a)yz \geq 0,$$

Multiplying both sides by $2s$ then using $abc=4Rrs, s=(a+b+c)/2$, the inequality becomes

$$8abc\left(\sum x\right)^2-(a+b+c)\sum(c+a-b)(a+b-c)(y+z)^2$$
$$-4(a+b+c)\sum a(b+c-a)yz \geq 0,$$

That is

$$2\sum a(c+a-b)(a+b-c)x^2-2(b+c-a)(c+a-b)(a+b-c)\sum yz \geq 0, \quad (3.16)$$

or

$$\sum (b+c-a)\left[(a+b-c)y-(c+a-b)z\right]^2 \geq 0, \quad (3.17)$$

which is obviously true.

There is equality in (3.17) only when $x:y:z=(s-a):(s-b):(s-c)$. Hence, equality in (3.14) occurs only if $(s-a):(s-a')=(s-b):(s'-b')=(s-c):(s'-c')$, this means $\triangle ABC \sim \triangle A'B'C'$.

Finally, we prove the following inequality:

$$H_3 \geq 16\triangle\triangle', \quad (3.18)$$

or

$$\sum a(s-a)(s'-b')(s'-c') \geq 2\triangle\triangle', \quad (3.19)$$

which is given in [9] by Zhen-Ping An. His proof is as follows:

In the equivalent form (2.9) of Wolstenholme inequality, we take

$$x=\cos^2\frac{A}{2}\tan\frac{A'}{2}, y=\cos^2\frac{B}{2}\tan\frac{A'}{2}, z=\cos^2\frac{C}{2}\tan\frac{C'}{2},$$

and then make use of the well-known identity in $\triangle A'B'C'$:

$$\sum \tan\frac{B'}{2}\tan\frac{C'}{2}=1, \quad (3.20)$$

we get trigonometric inequality

$$\sum \cos^2\frac{A}{2}\tan\frac{A'}{2} \geq 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}. \quad (3.21)$$

Further, using (2.8), $\triangle = rs$, $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{4R}$ and $\tan\frac{A'}{2}=\frac{1}{\triangle'}(s'-b')(s'-c')$, we then get inequality (3.19). It is not difficult to know equality in (3.18) is as (1.1).

This completes the proof of Theorem 1. □

Remark 3.1. *It is well known that* $\sqrt{a(s-a)}, \sqrt{b(s-b)}, \sqrt{c(s-c)}$ *form a triangle* $A_0B_0C_0$ *with area* $\frac{1}{2}S$ *(see* [14]*). If we substitute* $\triangle ABC$ *in Neuberg-Pedoe inequality (1.1) by* $\triangle A_0B_0C_0$ *and substitute* $\triangle A'B'C'$ *by triangle* $A'_0B'_0C'_0$ *whose sides are* $\sqrt{a'(s'-a')}, \sqrt{b'(s'-b')}, \sqrt{c'(s'-c')}$ *and its area is* $\frac{1}{2}S'$*. After simple calculations we get the inequality (3.19). Therefore, inequality (3.19) actually is a consequence of Neuberg-Pedoe inequality.*

Remark 3.2. *The equality* $H_2 \geq 16\triangle\triangle'$ *is actually equivalent to*

$$\frac{a'^2}{s-a}+\frac{b'^2}{s-b}+\frac{c'^2}{s-c} \leq 4\left(\frac{R}{s}-\frac{r'}{s'}\right)\frac{s'^2}{r}. \quad (3.22)$$

It seems to be difficult to prove this equality directly.

Remark 3.3. *It is easy to show that equality* $H \geq H_3$ *is equivalent to the following trigonometric inequality:*

$$\begin{aligned}&(\cot B+\cot C)\cot A'+(\cot C+\cot A)\cot B'+(\cot A+\cot B)\cot C'\\ &\geq \left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\tan\frac{A'}{2}+\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\tan\frac{B'}{2}\\ &+\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\tan\frac{C'}{2}. \quad (3.23)\end{aligned}$$

Remark 3.4. *For the left hand side of (3.21), the author* [17] *has established the following interesting extension:*

$$\sum \sin^2 \frac{A'}{2} \cot \frac{A}{2} \geq \sum \cos^2 \frac{A}{2} \tan \frac{A'}{2}, \tag{3.24}$$

with equality if and only if the two triangles are similar. In fact, inequality (3.24) is equivalent to the following weighted useful inequality with arbitrary positive real numbers x, y, z (see [17],[18]*):*

$$\frac{s-a}{x} + \frac{s-b}{y} + \frac{s-c}{z} \geq \frac{s(xa+yb+zc)}{yza+zxb+xyc}. \tag{3.25}$$

with equality if and only if $x = y = z$.

References

[1] J. Neuberg, *Sur les projections et contre-projections d'un triangle fixe*, Acad. Roy. de Belgique 44(1891), pp. 31–33.

[2] D. Pedoe, *An Inequality for Two triangles*, Proc. Cambridge Philos. Soc. 38(1943), pp. 397–398.

[3] A. Oppenheim, *Some inequalities for triangles*, Univ.Beograd. Publ.Elektrotehn. Fak.Ser.Mat.Fiz. No.357-380(1971), pp. 21–28.

[4] O. Bottema, M.S. Klamkin *Joint Triangle Inequalities*. Simon Stevin 48(1974), pp. 3–8.

[5] L. Yang and J.Z. Zhang, *A generalization to several dimensions of the Neuberg-Pedoe inequality with applications*, Bull. Australian Math. Soc., 27 (1983), pp. 203–214.

[6] G.Z. Chang, *Proving Pedoe's inequality by complex number computation*, Amer.Math.Monthly 89(1982), pp. 692.

[7] K.S. Poh, *A short note on a Pedoe's therorem about two triangles*, Math.Medley (Singapore) 11(1983), pp. 57–61.

[8] C.K. Peng, *Sharpening the Neuberg-Pedoe Inequality*, Crux.Math.10(1984), pp. 68–69.

[9] Z.P. An, *An inequality involving two triangles*, Bull Math, 12(1987), pp. 1–3(in Chinese).

[10] G.S. Leng and L.H. Tang, *The weighted generalizations of the Peng Chia-Kuei's and Chang Geng Zhe's inequality*, Hunan Bull Math, 4(1993), pp. 39–41(in Chinese).

[11] G.S. Leng and L.H. Tang, *Some generalizations to several dimensional of the Pedoe inequality with applications*, Acta Math. Sinica, 40 (1997), pp. 14–21(in Chinese).

[12] G.S. Leng, *Inequalities for edge lengths and volume of two simplices*, Geometric Dedicate, 68(1997), pp. 43-48.

[13] X.Y. Li and Y.Zhang, *The new k-n type Neuberg-Pedoe inequalities*, Taiwanese Journal of Mathematics, 5(2008), pp. 1177–1189.

[14] D.S. Mitrinović, J.E. Pečarić and V.Volenec, *Recent Advances in Geometric Inequalities, Kluwer Academic Publishers*, Dordrecht, Boston, London, 1989.

[15] S.H. Wu, *A new sharpened version of the NeubergCPedoe inequality*, Appl.Math.Lett., 21(6)(2008), pp. 558–562.

[16] J. Liu, *Two theorems of ternary quadrtic inequalities and its applications*, Journal of High School Mathemtics, 5(1996), pp. 16–19(in Chinese).

[17] J. Liu, *Another inequality involving sides of a triangle and its applications*, Journal of High School Mathemtics, 11(1992), pp. 11–14(in Chinese).

[18] J. Liu, *On the Inequality $R_p < R$ of the Pedal Triangle*, RGMIA Research Report Collection, 14(2011), Article 46, 12 pp. http://rgmia.org/v14.php.

East China Jiaotong University, Jiangxi province Nanchang City, 330013, China
E-mail address: china99jian@163.com