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APPROXIMATING THE STIELTJES INTEGRAL VIA A WEIGHTED TRAPEZOIDAL RULE WITH APPLICATIONS

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Abstract. In this paper we provide sharp error bounds in approximating the weighted Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,d\alpha(t)$ by the weighted trapezoidal rule $\frac{f(a)+f(b)}{2}\int_a^b g(t)\,d\alpha(t)$. Applications for continuous functions of selfadjoint operators in complex Hilbert spaces are given as well.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt \qquad ([18],\ [19]), \tag{1.1}$$

$$f(x)[u(b)-u(a)] \qquad ([11],\ [12]) \tag{1.2}$$

or with

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \qquad ([17]), \tag{1.3}$$

where $x\in[a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,s\in[a,b], \tag{1.4}$$

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [18],

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{1.5}$$

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The inequality (1.5) is sharp in the sense that the multiplicative constant $C = 1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants $m, M \in \mathbb{R}$ such that $m \leq f(t) \leq M$ for a.e. $t \in [a,b]$, then [18]

$$|D(f,u;a,b)| \leq \frac{1}{2} L (M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [19], where they showed that

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is sharp.

If we assume that f is $K-$Lipschitzian, then [19]

$$|D(f,u;a,b)| \leq \frac{1}{2} K (b-a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [16], [15], [14], and [8] and the references therein.

For the functional $\theta(f,u;a,b,x)$ we have the bound [11]:

$$\begin{aligned} &|\theta(f,u;a,b,x)| \\ &\leq H \left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f) \right] \\ &\leq H \times \begin{cases} \left[(x-a)^r + (b-x)^r\right] \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}} \left[\left(\bigvee_a^x (f) \right)^p + \left(\bigvee_x^b (f) \right)^p \right]^{\frac{1}{p}} \\ \qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (f), \end{cases} \end{aligned} \tag{1.9}$$

provided f is of bounded variation and u is of $r-H-$*Hölder type*, i.e.,

$$|u(t) - u(s)| \leq H |t-s|^r \qquad \text{for each } \ t,s \in [a,b], \tag{1.10}$$

with given $H > 0$ and $r \in (0,1]$.

If f is of $q-K-$Hölder type and u is of bounded variation, then [12]

$$|\theta(f,u;a,b,x)| \leq K \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^q \bigvee_a^b (u), \tag{1.11}$$

for any $x \in [a,b]$.

If u is monotonic nondecreasing and f of $q-K-$Hölder type, then the following refinement of (1.11) also holds [8]:

$$
\begin{aligned}
(1.12)\qquad |\theta(f,u;a,b,x)| &\leq K\Bigg[(b-x)^{q}u(b)-(x-a)^{q}u(a) \\
&\quad +q\left\{\int_a^x\frac{u(t)\,dt}{(x-t)^{1-q}}-\int_x^b\frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg] \\
&\leq K\left[(b-x)^{q}\left[u(b)-u(x)\right]+(x-a)^{q}\left[u(x)-u(a)\right]\right] \\
&\leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{q}\left[u(b)-u(a)\right],
\end{aligned}
$$

for any $x\in[a,b]$.

If f is monotonic nondecreasing and u is of $r-H-$Hölder type, then [8]:

$$
\begin{aligned}
(1.13)\qquad &|\theta(f,u;a,b,x)| \\
&\leq H\Bigg[\left[(x-a)^{r}-(b-x)^{r}\right]f(x) \\
&\quad +r\left\{\int_a^x\frac{f(t)\,dt}{(b-t)^{1-r}}-\int_x^b\frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg] \\
&\leq H\left\{(b-x)^{r}\left[f(b)-f(x)\right]+(x-a)^{r}\left[f(x)-f(a)\right]\right\} \\
&\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{r}\left[f(b)-f(a)\right],
\end{aligned}
$$

for any $x\in[a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [17], [8], [3] and [2] and the details are omitted.

Motivated by the above results, we consider in this paper the problem of providing sharp error bounds by approximating the weighted Riemann-Stieltjes integral $\int_a^b f(t)\,g(t)\,d\alpha(t)$ in terms of the weighted trapezoidal rule $\frac{f(a)+f(b)}{2}\int_a^b g(t)\,d\alpha(t)$. Applications for continuous functions of selfadjoint operators in complex Hilbert spaces are given as well.

2. The Results

The first main result is as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and let denote by $\bigvee_a^b(f)$ its total variation on $[a,b]$.*

(i) *If* $\alpha : [a,b] \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, $g : [a,b] \to \mathbb{C}$ *is continuous on* $[a,b]$ *and the Riemann-Stieltjes integral* $\int_a^b f(t) g(t) d\alpha(t)$ *exists, then*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(t) d\alpha(t) - \int_a^b f(t) g(t) d\alpha(t) \right| \leq \sup_{t\in[a,b]} \left[\max_{s\in[a,t]} |g(s)| \bigvee_a^t (\alpha) + \max_{s\in[t,b]} |g(s)| \bigvee_t^b (\alpha) \right] \bigvee_a^b (f) \leq \frac{1}{2} \max_{t\in[a,b]} |g(t)| \bigvee_a^b (\alpha) \bigvee_a^b (f). \tag{2.1}$$

The constant $\frac{1}{2}$ *is best possible in (2.1).*

(ii) *If* $\alpha : [a,b] \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on* $[a,b]$ *and* $g : [a,b] \to \mathbb{C}$ *is Riemann integrable on* $[a,b]$, *then and the Riemann-Stieltjes integral* $\int_a^b f(t) g(t) d\alpha(t)$ *exists and*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(t) d\alpha(t) - \int_a^b f(t) g(t) d\alpha(t) \right| \leq \frac{1}{2} L \int_a^b |g(t)| dt \bigvee_a^b (f). \tag{2.2}$$

The constant $\frac{1}{2}$ *is best possible in (2.2).*

(iii) *If* $\alpha : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$, $g : [a,b] \to \mathbb{C}$ *is continuous on* $[a,b]$ *and the Riemann-Stieltjes integral* $\int_a^b f(t) g(t) d\alpha(t)$ *exists, then*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(t) d\alpha(t) - \int_a^b f(t) g(t) d\alpha(t) \right| \leq \frac{1}{2} \int_a^b |g(t)| d\alpha(t) \bigvee_a^b (f). \tag{2.3}$$

The constant $\frac{1}{2}$ *is best possible in (2.3).*

The case when the function f is Lipschitzian is of interest and is incorporated in the following result.

Theorem 2. *Let* $f : [a,b] \to \mathbb{C}$ *be a Lipschitzian function with the constant* $K > 0$ *on* $[a,b]$.

(a) *If* $\alpha : [a,b] \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, $g : [a,b] \to \mathbb{C}$ *is continuous on* $[a,b]$, *then the Riemann-Stieltjes integral* $\int_a^b f(t) g(t) d\alpha(t)$ *exists and*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(t) d\alpha(t) - \int_a^b f(t) g(t) d\alpha(t) \right| \le \frac{1}{2} K \int_a^b \left[\max_{t\in[a,t]} |g(s)| \bigvee_a^t (\alpha) + \max_{t\in[t,b]} |g(s)| \bigvee_t^b (\alpha) \right] dt \le \frac{1}{2} K (b-a) \max_{t\in[a,b]} |g(s)| \bigvee_a^b (\alpha). \tag{2.4}$$

The constant $\frac{1}{2}$ *is best possible in (2.4).*

(aa) *If* $\alpha : [a,b] \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on* $[a,b]$ *and* $g : [a,b] \to \mathbb{C}$ *is Riemann integrable on* $[a,b]$, *then and the Riemann-Stieltjes integral* $\int_a^b f(t) g(t) d\alpha(t)$ *exists and*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(t) d\alpha(t) - \int_a^b f(t) g(t) d\alpha(t) \right| \le \frac{1}{2} K \int_a^b \left[\left| \int_a^t g(t) d\alpha(t) \right| + \left| \int_t^b g(t) d\alpha(t) \right| \right] dt \le \frac{1}{2} K L (b-a) \int_a^b |g(t)| dt. \tag{2.5}$$

(aaa) *If* $\alpha : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$, $g : [a,b] \to \mathbb{C}$ *is continuous on* $[a,b]$, *then the Riemann-Stieltjes integral* $\int_a^b f(t) g(t) d\alpha(t)$ *exists and*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(t) d\alpha(t) - \int_a^b f(t) g(t) d\alpha(t) \right| \le \frac{1}{2} K \int_a^b \left[\left| \int_a^t g(t) d\alpha(t) \right| + \left| \int_t^b g(t) d\alpha(t) \right| \right] dt \le \frac{1}{2} K (b-a) \int_a^b |g(t)| d\alpha(t). \tag{2.6}$$

The constant $\frac{1}{2}$ *is best possible in (2.6).*

Remark 1. *It is an open problem for the authors wether or not the constant* $\frac{1}{2}$ *in (2.5) is best possible.*

3. Proofs

We need the following lemma that is interesting in itself as well:

Lemma 1. *Assume that the functions* $f, g, \alpha : [a,b] \to \mathbb{C}$ *are such that the Riemann-Stieltjes integrals* $\int_a^b f(t) g(t) d\alpha(t)$ *and* $\int_a^b g(t) d\alpha(t)$ *exist. Then we have the* *equality*

$$\frac{f(a)+f(b)}{2}\int_a^b g(t)\,d\alpha(t)-\int_a^b f(t)\,g(t)\,d\alpha(t) = \frac{1}{2}\int_a^b\left(\int_a^t g(s)\,d\alpha(s)-\int_t^b g(s)\,d\alpha(s)\right)df(t). \tag{3.1}$$

Proof. Observe that

$$\begin{aligned} &\frac{1}{2}\int_a^b\left(\int_a^t g(s)\,d\alpha(s)-\int_t^b g(s)\,d\alpha(s)\right)df(t) \\ &=\frac{1}{2}\int_a^b\left(\int_a^t g(s)\,d\alpha(s)-\int_a^b g(s)\,d\alpha(s)+\int_a^t g(s)\,d\alpha(s)\right)df(t) \\ &=\int_a^b\left(\int_a^t g(s)\,d\alpha(s)-\frac{1}{2}\int_a^b g(s)\,d\alpha(s)\right)df(t). \end{aligned} \tag{3.2}$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\int_a^b\left(\int_a^t g(s)\,d\alpha(s)-\frac{1}{2}\int_a^b g(s)\,d\alpha(s)\right)df(t) \\ &=\left(\int_a^t g(s)\,d\alpha(s)-\frac{1}{2}\int_a^b g(s)\,d\alpha(s)\right)f(t)\Bigg|_a^b \\ &-\int_a^b f(t)\,d\left(\int_a^t g(s)\,d\alpha(s)-\frac{1}{2}\int_a^b g(s)\,d\alpha(s)\right) \\ &=\left(\int_a^b g(s)\,d\alpha(s)-\frac{1}{2}\int_a^b g(s)\,d\alpha(s)\right)f(b) \\ &+\left(\frac{1}{2}\int_a^b g(s)\,d\alpha(s)\right)f(a)-\int_a^b f(t)\,d\left(\int_a^t g(s)\,d\alpha(s)\right). \end{aligned} \tag{3.3}$$

On applying the well known property of the Riemann-Stieltjes integral with integrators that are expressed by an integral (see for instance [1, p. 158-p. 159]) we have

$$\int_a^b f(t)\,d\left(\int_a^t g(s)\,d\alpha(s)\right)=\int_a^b f(t)\,g(t)\,d\alpha(t)$$

and by (3.2) and (3.3) we deduce the desired representation (3.1).

This concludes the proof of the lemma. □

It is well know that, if the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists, where $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$ and $p:[a,b]\to\mathbb{C}$ is bounded on $[a,b]$, then we have the inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v). \tag{3.4}$$

Now, since $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$, then by (3.1) and utilizing the property (3.4), we have

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(t)\,d\alpha(t)-\int_a^b f(t)\,g(t)\,d\alpha(t)\right| \le \frac{1}{2}\sup_{t\in[a,b]}\left|\int_a^t g(s)\,d\alpha(s)-\int_t^b g(s)\,d\alpha(s)\right|\bigvee_a^b(f), \tag{3.5}$$

which is an inequality of interest in itself.

(i) Since $\alpha:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$, then by the property (3.4) we have

$$\begin{aligned}\left|\int_a^t g(s)\,d\alpha(s)-\int_t^b g(s)\,d\alpha(s)\right| &\le \left|\int_a^t g(s)\,d\alpha(s)\right|+\left|\int_t^b g(s)\,d\alpha(s)\right| \\ &\le \max_{s\in[a,t]}|g(s)|\bigvee_a^t(\alpha)+\max_{s\in[t,b]}|g(s)|\bigvee_t^b(\alpha) \\ &\le \max_{s\in[a,b]}|g(s)|\bigvee_a^b(\alpha)\end{aligned} \tag{3.6}$$

for any $t\in[a,b]$.

Taking the supremum over $t\in[a,b]$ in (3.6) and making use of the inequality (3.5), we deduce the desired result (2.1).

(ii) It is well known that, if $p:[a,b]\to\mathbb{C}$ is Riemann integrable and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and we have the inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b|p(t)|\,dt. \tag{3.7}$$

Now, on utilizing this property, we have that

$$\begin{aligned}\left|\int_a^t g(s)\,d\alpha(s)-\int_t^b g(s)\,d\alpha(s)\right| &\le \left|\int_a^t g(s)\,d\alpha(s)\right|+\left|\int_t^b g(s)\,d\alpha(s)\right| \\ &\le L\int_a^t|g(s)|\,ds+L\int_t^b|g(s)|\,ds \\ &= L\int_a^b|g(s)|\,ds\end{aligned} \tag{3.8}$$

for any $t\in[a,b]$.

Taking the supremum over $t\in[a,b]$ in (3.8) and making use of the inequality (3.5), we deduce the desired result (2.2).

(iii) It is well known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and we have the inequality

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t). \tag{3.9}$$

Now, on utilizing this property, we have that

$$\left|\int_a^t g(s)\,d\alpha(s) - \int_t^b g(s)\,d\alpha(s)\right| \leq \left|\int_a^t g(s)\,d\alpha(s)\right| + \left|\int_t^b g(s)\,d\alpha(s)\right| \tag{3.10}$$
$$\leq \int_a^t |g(s)|\,d\alpha(s) + \int_t^b |g(s)|\,d\alpha(s)$$
$$= L\int_a^b |g(s)|\,d\alpha(s)$$

for any $t \in [a,b]$.

Taking the supremum over $t \in [a,b]$ in (3.10) and making use of the inequality (3.5), we deduce the desired result (2.3).

Now, for the best constants.

If we choose $\alpha : [a,b] \to \mathbb{R}$, $\alpha(t) = t$, $g : [a,b] \to \mathbb{R}$, $g(t) = 1$ and $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, then the assumptions in $(i), (ii)$ and (iii) of Theorem 1 are satisfied and the inequalities (2.1), (2.2) and (2.3) become

$$\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt\right| \leq \frac{1}{2}(b-a)\bigvee_a^b(f), \tag{3.11}$$

that holds for any function $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$.

Assume that (3.11) is valid with a constant $C > 0$ instead of $\frac{1}{2}$, i.e., we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt\right| \leq C(b-a)\bigvee_a^b(f), \tag{3.12}$$

for any function $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$.

Consider the function $f_0 : [a,b] \to \mathbb{R}$ given by

$$f_0(t) := \begin{cases} 1 & \text{if} \quad t = a \\ 0 & \text{if} \quad t \in (a,b) \\ 1 & \text{if} \quad t = b. \end{cases}$$

This function is of bounded variation with $\int_a^b f_0(t)\,dt = 0$ and $\bigvee_a^b(f_0) = 2$. Replacing these values in (3.12) give $b - a \leq 2C(b-a)$ which implies that $C \geq \frac{1}{2}$.

To prove Theorem 2, we observe that, since $f : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then by the identity (3.1) and the property (3.7) we have the following inequality

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(t)\,d\alpha(t) - \int_a^b f(t)\,g(t)\,d\alpha(t)\right| \tag{3.13}$$
$$\leq \frac{1}{2}K\int_a^b \left|\int_a^t g(s)\,d\alpha(s) - \int_t^b g(s)\,d\alpha(s)\right| dt,$$

which is an inequality of interest in itself.

(a) Since $\alpha : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ is continuous on $[a,b]$, then by the property (3.4) we have the inequality (3.6), which by integration on $[a,b]$ and utilizing (3.13) produces the desired result (2.4).

The statements (aa) and (aaa) follow in a similar manner and the details are left to the reader.

In order to prove the sharpness of the constant $\frac{1}{2}$ in (2.4) and (2.6) we consider the function $f : [a,b] \to \mathbb{R}$, $f(t) := \left|t - \frac{a+b}{2}\right|$, which is Lipschitzian with the constant $K = 1$. If we take $g(t) = 1, t \in [a,b]$ then for any function $\alpha : [a,b] \to \mathbb{C}$ of bounded variation we have

$$
\begin{aligned}
(3.14)\qquad I &:= \frac{f(a)+f(b)}{2}\int_a^b g(t)\,d\alpha(t) - \int_a^b f(t)\,g(t)\,d\alpha(t) \\
&= \frac{b-a}{2}\int_a^b d\alpha(t) - \int_a^b \left|t - \frac{a+b}{2}\right| d\alpha(t) \\
&= \frac{b-a}{2}\left[\alpha(b) - \alpha(a)\right] \\
&- \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) d\alpha(t) - \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) d\alpha(t).
\end{aligned}
$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$
\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) d\alpha(t) = -\frac{b-a}{2}\alpha(a) + \int_a^{\frac{a+b}{2}} \alpha(t)\,dt
$$

and

$$
\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) d\alpha(t) = \frac{b-a}{2}\alpha(b) - \int_{\frac{a+b}{2}}^b \alpha(t)\,dt.
$$

Inserting these values in (3.14) we get

$$
I = \int_{\frac{a+b}{2}}^b \alpha(t)\,dt - \int_a^{\frac{a+b}{2}} \alpha(t)\,dt.
$$

If we take now the function $\alpha : [a,b] \to \mathbb{R}$, $\alpha(t) = sgn\left(t - \frac{a+b}{2}\right)$, then this function is monotonic nondecreasing and we have $I = b - a$, $\bigvee_a^b(\alpha) = \alpha(b) - \alpha(a) = 2$ and

$$
\frac{1}{2}K(b-a)\max_{t\in[a,b]}|g(s)|\bigvee_a^b(\alpha) = b - a
$$

and

$$
\frac{1}{2}K(b-a)\int_a^b |g(t)|\,d\alpha(t) = b - a
$$

which shows that the constant $\frac{1}{2}$ is best possible in both inequalities (2.4) and (2.6).

4. Applications for Selfadjoint Operators in Hilbert Spaces

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. It is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$U = \int_{m-0}^{M} \lambda dE_\lambda, \tag{4.1}$$

which in terms of vectors can be written as

$$\langle Ux, y\rangle = \int_{m-0}^{M} \lambda d \langle E_\lambda x, y\rangle, \tag{4.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

It is also known that for any continuous function $f : [m, M] \to \mathbb{R}$, we have the following *spectral representation*:

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda) d(\langle E_\lambda x, y\rangle), \tag{4.3}$$

for any $x, y \in H$.

Theorem 3. *Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$ and $g : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$, then we have*

$$\begin{aligned} &\left|\langle f(A) g(A) x, y\rangle - \frac{f(m)+f(M)}{2}\langle g(A)x, y\rangle\right| \\ &\leq \frac{1}{2}\max_{t\in[m,M]}|g(t)| \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \bigvee_m^M (f) \\ &\leq \frac{1}{2}\max_{t\in[m,M]}|g(t)| \|x\| \|y\| \bigvee_m^M (f) \end{aligned} \tag{4.4}$$

for any $x, y \in H$ and

$$\begin{aligned} &\left|\langle f(A) g(A) x, x\rangle - \frac{f(m)+f(M)}{2}\langle g(A)x, x\rangle\right| \\ &\leq \frac{1}{2}\langle |g(A)| x, x\rangle \bigvee_m^M (f) \end{aligned} \tag{4.5}$$

for any $x \in H$.

Proof. If we use the inequality (2.1) we can write that

$$\begin{aligned} &\left|\frac{f(m)+f(M)}{2}\int_{m-0}^{M} g(t) d\langle E_\lambda x, y\rangle - \int_{m-0}^{M} f(t) g(t) d\langle E_\lambda x, y\rangle\right| \\ &\leq \frac{1}{2}\max_{t\in[m,M]}|g(t)| \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \bigvee_m^M (f) \end{aligned} \tag{4.6}$$

for any $x, y \in H$.

Since, by the spectral representation (4.3) we have

$$\int_{m-0}^{M} g(t)\, d\langle E_\lambda x, y\rangle = \langle g(A)\, x, y\rangle$$

and

$$\int_{m-0}^{M} f(t)\, g(t)\, d\langle E_\lambda x, y\rangle = \langle f(A)\, g(A)\, x, y\rangle,$$

for any $x, y \in H$, then by (4.6) we deduce the first inequality in (4.4).

To prove last part of (4.4), we first notice that if P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle$$

for any $x, y \in H$.

Further, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators that

$$\begin{aligned}
&\bigvee_m^M \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\
&= \sup_d \left\{ \sum_{i=0}^{n-1} \left|\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y\right\rangle\right| \right\} \\
&\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle^{1/2} \right] \right\} := I.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle \right]^{1/2} \right\} \\
&\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle \right]^{1/2} \right\} \\
&= \left[\bigvee_m^M \left(\left\langle E_{(\cdot)} x, x\right\rangle\right) \right]^{1/2} \left[\bigvee_m^M \left(\left\langle E_{(\cdot)} y, y\right\rangle\right) \right]^{1/2} = \|x\| \, \|y\|
\end{aligned}$$

for any $x, y \in H$. These prove the last part of (4.6).

Now, on utilizing the inequality (2.3), we also have

$$\begin{aligned}
&\left| \frac{f(m) + f(M)}{2} \int_{m-0}^{M} g(t)\, d\langle E_\lambda x, x\rangle - \int_{m-0}^{M} f(t)\, g(t)\, d\langle E_\lambda x, x\rangle \right| \\
&\leq \frac{1}{2} \int_{m-0}^{M} |g(t)|\, d\langle E_\lambda x, x\rangle \bigvee_m^M (f),
\end{aligned} \tag{4.7}$$

for any $x \in H$.

Since

$$\int_{m-0}^{M} g(t)\, d\langle E_\lambda x, x\rangle = \langle g(A)\, x, x\rangle$$

and

$$\int_{m-0}^{M} f(t)\, g(t)\, d\langle E_\lambda x, x\rangle = \langle f(A)\, g(A)\, x, x\rangle ,$$

for any $x \in H$, then (4.7) implies the desired inequality (4.5). □

The case when f is Lipschitzian is incorporated in the following result:

Theorem 4. *Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $K > 0$ on $[m, M]$ and $g : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$, then we have*

$$\left|\langle f(A)\, g(A)\, x, y\rangle - \frac{f(m) + f(M)}{2} \langle g(A)\, x, y\rangle\right| \tag{4.8}$$
$$\leq \frac{1}{2} K (M - m) \max_{t\in[m,M]} |g(t)| \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right)$$
$$\leq \frac{1}{2} K (M - m) \max_{t\in[m,M]} |g(t)| \, \|x\| \, \|y\|$$

for any $x, y \in H$ and

$$\left|\langle f(A)\, g(A)\, x, x\rangle - \frac{f(m) + f(M)}{2} \langle g(A)\, x, x\rangle\right| \tag{4.9}$$
$$\leq \frac{1}{2} K (M - m) \langle |g(A)|\, x, x\rangle$$

for any $x \in H$.

The proof follows by the statements (a) and (aaa) of Theorem 2 and the details are omitted.

The previous results can be used to provide inequalities for the quantity $\langle h(A)\, x, y\rangle$ when the function h can be decomposed in a product of two functions f and g as those considered above. A simple example of such a function is the "entropy function" $h : (0, \infty) \to \mathbb{R}$, $h(t) = t \ln t$.

Let A be a positive definite operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.

1. Now, if we apply Theorem 3 for the choice $f(t) = t$ and $g(t) = \ln t$, $t > 0$, then we have

$$\left|\langle A \ln Ax, y\rangle - \frac{m + M}{2} \langle \ln Ax, y\rangle\right| \tag{4.10}$$
$$\leq \frac{1}{2} (M - m) \max\{|\ln m|, |\ln M|\} \bigvee_{m}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right)$$
$$\leq \frac{1}{2} (M - m) \max\{|\ln m|, |\ln M|\} \, \|x\| \, \|y\|$$

for any $x, y \in H$ and

$$\left| \langle A \ln Ax, x \rangle - \frac{m+M}{2} \langle \ln Ax, x \rangle \right| \leq \frac{1}{2} (M-m) \langle |\ln A| \, x, x \rangle \tag{4.11}$$

for any $x \in H$.

Theorem 4 provides for the choice $f(t) = t$ and $g(t) = \ln t$, $t > 0$ the same inequalities (4.10) and (4.11).

2. Now, if we apply Theorem 3 for the dual choice $f(t) = \ln t$ and $g(t) = t$, $t > 0$, then we have

$$\begin{aligned} \left| \langle A \ln Ax, y \rangle - \langle Ax, y \rangle \ln \sqrt{mM} \right| &\leq \frac{1}{2} M \ln \left(\frac{M}{m} \right) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\leq \frac{1}{2} M \ln \left(\frac{M}{m} \right) \|x\| \, \|y\| \end{aligned} \tag{4.12}$$

for any $x, y \in H$ and

$$\left| \langle A \ln Ax, x \rangle - \langle Ax, x \rangle \ln \sqrt{mM} \right| \leq \frac{1}{2} \langle Ax, x \rangle \ln \left(\frac{M}{m} \right) \tag{4.13}$$

for any $x \in H$.

Theorem 4 provides for the choice $f(t) = \ln t$ and $g(t) = t$, $t > 0$, the inequalities

$$\begin{aligned} \left| \langle A \ln Ax, y \rangle - \langle Ax, y \rangle \ln \sqrt{mM} \right| &\leq \frac{1}{2} \frac{M}{m} (M-m) \bigvee_m^M \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\leq \frac{1}{2} \frac{M}{m} (M-m) \|x\| \, \|y\| \end{aligned} \tag{4.14}$$

for any $x, y \in H$ and

$$\left| \langle A \ln Ax, x \rangle - \langle Ax, x \rangle \ln \sqrt{mM} \right| \leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \langle Ax, x \rangle \tag{4.15}$$

for any $x \in H$.

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