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HERMITE-HADAMARD INEQUALITY FOR POINT-WISE CONVEX MAPS, APPLICATION TO CONVEX OPERATOR MAPS

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ABSTRACT. In the present paper, we give the Hermite-Hadamard type inequality for point-wise convex maps involving functional variables. A Jensen type inequality for the Legendre-Fenchel conjugacy is also established. In the quadratic case, we immediately obtain those of operator variables.

1. INTRODUCTION

Let Ω be a Lebesgue measurable set such that $mes\ \Omega > 0$. Let D be a nonempty convex subset of $\mathbb{R}$ and $f : D \longrightarrow \mathbb{R}$ be a convex function. Let $\phi \in L^1(\Omega)$ be such that $\phi(x) \in D$ almost everywhere and $f \circ \phi \in L^1(\Omega)$. Then, we have the integral Jensen inequality

$$f\left(\frac{1}{mes\ \Omega}\int_\Omega \phi(x)dx\right) \le \frac{1}{mes\ \Omega}\int_\Omega f(\phi(x)dx, \tag{1.1}$$

which provides a large utility in various mathematical contexts. As consequence, the following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

is known as Hermite-Hadamard inequality for convex mappings $f : [a,b] \longrightarrow \mathbb{R}$. Such inequality is very useful in many mathematical contexts and contributes as a tool for establishing some interesting estimations. For instance, several mean inequalities can be obtained via inequality (1.2). The following examples explain this latter situation.

Example 1.1. Choosing $f :]0,+\infty[\longrightarrow]0,+\infty[$ such that $f(x) = 1/x$, inequality (1.2) immediately gives

$$\left(\frac{a+b}{2}\right)^{-1} \le \frac{ln\ a - ln\ b}{a-b} \le \frac{a^{-1}+b^{-1}}{2},\quad a \ne b,$$

which, after taking the inverse of the three members, yields the known arithmetic-logarithmic-harmonic mean inequality.

Example 1.2. Let $f :]0,+\infty[\longrightarrow \mathbb{R}$ be such that $f(x) = -ln\ x$. By inequality (1.2) with a routine computation we obtain

$$\sqrt{ab} \le \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \le \frac{a+b}{2},\quad a \ne b,$$

which is nothing other than the standard arithmetic-identric-geometric mean inequality.

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As well known, inequality (1.2) has an extension for real-valued convex mapping with variables in a linear vector space E in the following sense: Let $C \subset E$ be a nonempty convex subset of E and $\phi : C \longrightarrow \mathbb{R}$ be a convex mapping, then for all $x, y \in C$ there holds

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f\left((1-t)x+ty\right)dt \leq \frac{f(x)+f(y)}{2}. \tag{1.3}$$

In particular, in every linear normed space $(E, \|.\|)$, we have

$$\|\frac{x+y}{2}\|^p \leq \int_0^1 \|(1-t)x+ty\|^p dt \leq \frac{\|x\|^p+\|y\|^p}{2}, \tag{1.4}$$

for all $x, y \in E$ and every real number $p \geq 1$.

In [1], the authors gave an iterative refinement of (1.3) as recited in the following.

Theorem 1.1. *Let C be a nonempty convex subset of a linear space E and $f : C \longrightarrow \mathbb{R}$ a convex function. For all $x, y \in C$, the sequences $(\Phi_n(x,y))_n$ and $(\Psi_n(x,y))_n$ defined by*

$$\begin{cases} \Phi_{n+1}(x,y) = \frac{1}{2}\Phi_n\left(x, \frac{x+y}{2}\right) + \frac{1}{2}\Phi_n\left(\frac{x+y}{2}, y\right), \ \Phi_0(x,y) = f\left(\frac{x+y}{2}\right), \\ \Psi_{n+1}(x,y) = \frac{1}{2}\Psi_n\left(x, \frac{x+y}{2}\right) + \frac{1}{2}\Psi_n\left(\frac{x+y}{2}, y\right), \ \Psi_0(x,y) = \frac{f(x)+f(y)}{2}, \end{cases} \tag{1.5}$$

are respectively monotonic increasing and decreasing and both converge to

$$m_f(x,y) := \int_0^1 f\left((1-t)x+ty\right)dt,$$

with the following estimation

$$0 \leq m_f(x,y) - \Phi_n(x,y) \leq \Psi_n(x,y) - m_f(x,y) \leq \frac{1}{2^n}\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right). \tag{1.6}$$

Now, let H be a real or complex Hilbert space with its inner product $< ., . >$ and the associated norm $\|.\|$. A real function F defined on an interval J is said to be operator monotone increasing if for all bounded self-adjoint operators A and B, defined from H into itself, with spectra in J,

$$A \leq B \Longleftrightarrow F(A) \leq F(B),$$

for the Löwner partial ordering: $A \leq B$ if and only if $B - A$ is positive semi-definite. The operator convexity and concavity of F are defined in a similar way.
With this, analogue of inequality (1.3) from scalar convex mapping to operator convex one can be obtained for particular cases. Let us observe the following example.

Example 1.3. Let A and B be as in the above. Inequality (1.4) with $x = Au, y = Bu, u \in H$ and $p = 2$ yields

$$\|\frac{Au+Bu}{2}\|^2 \leq \int_0^1 \|(1-t)Au+tBu\|^2 dt \leq \frac{\|Au\|^2+\|Bu\|^2}{2}.$$

This, with the fact that $\|u\|^2 = \langle u, u\rangle$ for all $u \in H$, can be written as

$$\left\langle \left(\frac{A+B}{2}\right)^2 u, u\right\rangle \leq \int_0^1 \left\langle \left((1-t)A+tB\right)^2 u, u\right\rangle dt \leq \left\langle \left(\frac{A^2+B^2}{2}\right) u, u\right\rangle,$$

or in terms of Löwner operator order

$$\left(\frac{A+B}{2}\right)^2 \leq \int_0^1 ((1-t)A+tB)^2\,dt \leq \frac{A^2+B^2}{2},$$

which is an analogue of inequality (1.3) for the convex operator map $A \longmapsto A^2$.

2. Point-Wise Convexity

We preserve the same notations as in the previous section. The notation $\overline{\mathbb{R}}^H$ refers to the space of all functions defined from H into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ equipped with the point-wise partial ordering,

$$\forall\, f,\ g \in \overline{\mathbb{R}}^H, \quad f \leq g \iff \forall u \in H \quad f(u) \leq g(u),$$

where we extend the structure of $\mathbb{R}$ on $\overline{\mathbb{R}}$ by setting

$$\forall\ x \in \overline{\mathbb{R}}, \quad -\infty \leq x \leq +\infty, \qquad (+\infty) + x = +\infty, \quad 0.\infty = \infty.$$

With this, let $\mathcal{C}$ be a nonempty convex subset of $\overline{\mathbb{R}}^H$ and $\Phi : \mathcal{C} \longrightarrow \overline{\mathbb{R}}^H$ be a map. We say that Φ is point-wise convex if for all $f, g \in \mathcal{C}$ and all real $t \in]0,1[$ there holds

$$\Phi\left((1-t)f + tg\right) \leq (1-t)\Phi(f) + t\Phi(g).$$

The point-wise concavity and point-wise monotonicity notions can be defined in a similar manner.

Let $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and $f : H \longrightarrow \widetilde{\mathbb{R}}$, the notation f^* stands for the Legendre-Fenchel conjugate of the functional f defined by

$$\forall u^* \in H \qquad f^*(u^*) := \sup_{u \in H}\{Re < u, u^* > -f(u)\} = \sup_{u \in dom\ f}\{Re < u, u^* > -f(u)\},$$

where *dom* f refers to the effective domain of f defined by

$$dom\ f := \{u \in H,\ f(u) < +\infty\}.$$

The bi-conjugate duality $f \longmapsto f^{**} := (f^*)^*$ is defined similarly. It is well known that $f = f^{**}$ if and only if $f \in \Gamma_0(H)$, where the notation $\Gamma_0(H)$ refers to the cone of all convex lower semi-continuous functionals that are not identically equal to $+\infty$. We denote by $\sigma = (1/2)\|.\|^2$ the unique self-conjugate convex functional.

An important and typical example of $\widetilde{\mathbb{R}}^H-$functional having good properties is f_A defined by

$$\forall u \in H \qquad f_A(u) = (1/2) Re < Au, u >,$$

where A is a bounded linear operator defined from H into itself. The functional f_A is quadratic in the sense $f_A(\lambda u) = |\lambda|^2 f_A(u)$ for all $u \in H$ and all complex number λ. It is easy to see that the conjugate operation preserves the quadratic character, i.e. if f is quadratic so is f^* and every continuous quadratic functional can be written in the form f_A.

Now, we will state some examples of mappings involving functional variables and having some interesting properties.

Example 2.1. The first important example of mapping with functional variable is the conjugate duality map $f \longrightarrow f^*$. It is well-known that if A is a positive definite operator then, f_A^* takes the explicit form,

$$\forall u^* \in H \qquad f_A^*(u^*) = (1/2) < A^{-1}u^*, u^* > .$$

In another way, the conjugate operation can be interpreted as an inverse in the sense

$$(f_A)^* = f_{A^{-1}},$$

for every positive definite operator A defined from H into itself. Further, it is well-known that the map $f \longmapsto f^*$ is point-wise convex.

Example 2.2. For $f \in \Gamma_0(H)$, we set

$$\mathcal{L}(f) = \int_0^1 \frac{\sigma - ((1-t)\sigma + tf)^*}{t} dt,$$

which is the Logarithm of the functional f in convex analysis introduced in [3]. This functional logarithm extends the logarithm of a positive definite operator A in the sense that

$$\mathcal{L}(f_A) = f_{Log\ A}.$$

As studied in [3], the map $f \longmapsto \mathcal{L}(f)$ has analogue properties of that $A \longmapsto Log\ A$. In particular, $f \longmapsto \mathcal{L}(f)$ is point-wise concave and $\mathcal{L}(f^*) = -\mathcal{L}(f)$. This justifies again that the conjugate operation can be interpreted as an inverse in some sense.

Example 2.3. Let $f \in \Gamma_0(H)$, $0 < m < 1$, and define

$$f^{(m)} = \frac{sin m\pi}{\pi} \int_0^{+\infty} \frac{t^{m-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{1+t} f^* \right)^* dt, \tag{2.1}$$

which is the convex better m-iterate of f introduced in [4]. When $m = 1/n$, $n \geq 2$ integer, $f^{(1/n)}$ is said the convex n-th root of f. As shown in [4], the functional $f^{(m)}$ is a reasonable extension of A^m from the case that the variable A is a positive operator to the case that the variable is a convex functional in the following sense

$$(f_A)^{(m)} = f_{A^m}.$$

Further, $f \longmapsto f^{(m)}$ is point-wise concave.

Now, we will observe the following question: what should be the analogue of (1.1) and (1.2) when the scalar-valued convex function f is a functional-valued point-wise convex map Φ and the scalar variables a and b are functional variables f and g.

The fundamental goal of this paper is to give an analogue of inequality (1.1) for the point-wise convex duality map and analogue of (1.2) for a point-wise convex map. As consequence we deduce that of $f \longmapsto f^*$, $f \longmapsto \mathcal{L}(f)$ and $f \longmapsto f^{(m)}$. In the quadratic case, we immediately obtain those of operators. At the end, we give an analogue of Theorem 1.1 for a point-wise convex map.

3. Hermite-Hadamard Inequality for Point-Wise Convex Maps

Our first main result is recited in the following.

Theorem 3.1. *Let $\mathcal{C}$ be a nonempty convex subset of $\widetilde{\mathbb{R}}^H$ and $\Phi : \mathcal{C} \longrightarrow \widetilde{\mathbb{R}}^H$ be a point-wise convex map. Then the functional double inequality*

$$\Phi\left(\frac{f+g}{2}\right) \leq \int_0^1 \Phi\left((1-t)f + tg\right) dt \leq \frac{\Phi(f) + \Phi(g)}{2} \tag{3.1}$$

holds for all $f, g \in \mathcal{C}$.

If Φ is point-wise concave then the above functional inequalities are reversed.

Proof. First, we notice that the functional variables f and g of the map Φ can take the value $+\infty$ and so the functional equalities $f - f = 0$ and $f - g = -(g - f)$ are not always true. For the same, the functional inequalities $f \leq g$ and $f - g \leq 0$ are not, in general, equivalent. With this, it is easy to verify that

$$f + g = ((1-t)f + tg) + (tf + (1-t)g),$$

for all $f, g \in \widetilde{\mathbb{R}}^H$ and $t \in]0,1[$. By the point-wise convexity of Φ we then have

$$\Phi\left(\frac{f+g}{2}\right) \leq \frac{1}{2}\left(\Phi((1-t)f+tg) + \Phi(tf+(1-t)g)\right) \leq \frac{\Phi(f)+\Phi(g)}{2} \tag{3.2}$$

Integrating the three sides of (3.2) and remarking that

$$\int_0^1 \Phi\left((1-t)f+tg\right) dt = \int_0^1 \Phi\left(tf+(1-t)g\right) dt,$$

we deduce the desired result. □

If we apply Theorem 3.1 for the point-wise convex map $f \longmapsto f^*$ on $\widetilde{\mathbb{R}}^H$ we obtain the following result.

Corollary 3.2. *Let $f, g \in \widetilde{\mathbb{R}}^H$ then there holds*

$$\left(\frac{f+g}{2}\right)^* \leq \int_0^1 \left((1-t)f+tg\right)^* dt \leq \frac{f^*+g^*}{2}. \tag{3.3}$$

Now, applying successively the above theorem for the point-wise concave maps $f \longmapsto \mathcal{L}(f)$ and $f \longmapsto f^{(m)}$ on $\Gamma_0(H)$, we immediately obtain the following.

Corollary 3.3. *Let $f, g \in \Gamma_0(H)$ and $0 < m < 1$ be a real number. Then we have the following*

$$\frac{\mathcal{L}(f)+\mathcal{L}(g)}{2} \leq \int_0^1 \mathcal{L}\left((1-t)f+tg\right) dt \leq \mathcal{L}\left(\frac{f+g}{2}\right).$$

$$\frac{f^{(m)}+g^{(m)}}{2} \leq \int_0^1 \left((1-t)f+tg\right)^{(m)} dt \leq \left(\frac{f+g}{2}\right)^{(m)}.$$

Remark 3.1. The double functional inequality (3.3) implies that

$$\left(\frac{f^*+g^*}{2}\right)^* \leq \left(\int_0^1 \left((1-t)f+tg\right)^* dt\right)^* \leq \frac{f+g}{2}.$$

Setting

$$\mathcal{H}(f,g) := \left(\frac{f^*+g^*}{2}\right)^*, \quad \mathcal{L}(f,g) := \left(\int_0^1 \left((1-t)f+tg\right)^* dt\right)^*,$$

which are respectively the harmonic and logarithmic functional means of f and g, we obtain the arithmetic-logarithmic-harmonic functional mean inequality, [5]

$$\mathcal{H}(f,g) \leq \mathcal{L}(f,g) \leq \mathcal{A}(f,g) := \frac{f+g}{2}.$$

In particular, choosing $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = (1/2)ax^2$, $g(x) = (1/2)bx^2$ with $a > 0, b > 0$, we find again the statement of Example 1.1.

As application for bounded linear operators, Theorem 3.1 when combined with Examples 2.1, 2.2, 2.3 yields the following result.

Corollary 3.4. *Let A and B be two positive definite operators defined from H into itself. Then one has*

$$\left(\frac{A+B}{2}\right)^{-1} \leq \int_0^1 ((1-t)A+tB)^{-1}\,dt \leq \frac{A^{-1}+B^{-1}}{2}. \tag{3.4}$$

$$\frac{log\ A + log\ B}{2} \leq \int_0^1 log\,((1-t)A+tB)\,dt \leq log\left(\frac{A+B}{2}\right). \tag{3.5}$$

$$\frac{A^m+B^m}{2} \leq \int_0^1 ((1-t)A+tB)^m\,dt \leq \left(\frac{A+B}{2}\right)^m.$$

Various operator mean inequalities can be deduced from the above. As examples, we may state the following.

Example 3.1. Since the map $X \longmapsto X^{-1}$ is operator decreasing (on the convex cone of positive invertible operators) then inequality (3.4) implies that

$$\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \leq \left(\int_0^1 ((1-t)A+tB)^{-1}\,dt\right)^{-1} \leq \frac{A+B}{2},$$

Setting

$$\mathcal{H}(A,B) := \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} = 2A(A+B)^{-1}B,$$

$$\mathcal{L}(A,B) := \left(\int_0^1 ((1-t)A+tB)^{-1}\,dt\right)^{-1},$$

$$\mathcal{A}(A,B) := \frac{A+B}{2},$$

which are, respectively the harmonic, logarithmic and arithmetic monotone operator means, we obtain

$$\mathcal{H}(A,B) \leq \mathcal{L}(A,B) \leq \mathcal{A}(A,B),$$

called the arithmetic-logarithmic-harmonic operator mean inequality, see [5]. In the scalar case, the above double inequality is reduced to

$$\frac{2ab}{a+b} \leq \frac{b-a}{ln\ b - ln\ a} \leq \frac{a+b}{2}, \quad a \neq b.$$

Example 3.2. The double operator inequality (3.5) is equivalent to

$$exp\frac{log\ A + log\ B}{2} \preceq exp\int_0^1 log\,((1-t)A+tB)\,dt \preceq \frac{A+B}{2},$$

where $\preceq$ is the chaotic partial ordering defined by, $A \preceq B$ if and only if $log\ A \leq log\ B$. In terms of operator means we obtain

$$\mathcal{CG}(A,B) \preceq \mathcal{CI}(A,B) \preceq \mathcal{A}(A,B),$$

which is the arithmetic-chaotic identric-chaotic geometric operator mean inequality, with

$$\mathcal{CG}(A,B) := exp\left(\frac{1}{2}log\ A + \frac{1}{2}log\ B\right)$$

is the chaotic geometric operator mean of A and B, [2], and

$$\mathcal{CI}(A,B) := exp\int_0^1 log\,((1-t)A+tB)\,dt$$

is the chaotic identric operator mean of A and B, [6, 7].

We end this section by stating another application for bounded linear operators. In what previous we have seen that if A is a positive definite operator then $(f_A)^* = f_{A^{-1}}$. In the case where the operator A is only positive semi-definite, i.e. not necessary invertible, and $A^{1/2}$ denotes the square root of A (i.e. the unique positive semi-definite operator X such that $X^2 = A$) then it is well known that

$$(f_A)^*(u) = (1/2)\|(A^{1/2})^+u\|^2 \text{ if } u \in ran\ A^{1/2}, \quad (f_A)^*(u) = +\infty \text{ else,} \tag{3.6}$$

where $(A^{1/2})^+$ refers to the pseudo-inverse of $A^{1/2}$. It follows that, $A \leq B$ if and only if $ran\ A^{1/2} \subset ran\ B^{1/2}$ and $\|(B^{1/2})^+u\| \leq \|(A^{1/2})^+u\|$ for all $u \in ran\ A^{1/2}$. This when combined with Theorem 3.1 yields the following result.

Theorem 3.5. *Let A and B be two positive semi-definite operators defined from H into itself. Then the following statements are met*

$$ran\ A^{1/2} \cap ran\ B^{1/2} \subseteq ran\left((1-t)A + tB\right)^{1/2} \subseteq ran\left(\frac{A+B}{2}\right)^{1/2}, \tag{3.7}$$

for all $t \in [0,1]$ almost everywhere, and

$$\|\left(\left(\frac{A+B}{2}\right)^{1/2}\right)^+ u\|^2 \leq \int_0^1 \|\left(((1-t)A + tB)^{1/2}\right)^+ u\|^2 dt$$
$$\leq \frac{1}{2}\|(A^{1/2})^+u\|^2 + \frac{1}{2}\|(B^{1/2})^+u\|^2,$$

for all $u \in ran\ A^{1/2} \cap ran\ B^{1/2}$.

The above theorem has many interesting consequences. For instance, the double inclusion (3.7) gives us new good information. More precisely, the following result holds.

Corollary 3.6. *Let A and B be two positive semi-definite operators defined from H into itself. Then one has*

$$\max_{0 \leq t \leq 1} ran\left((1-t)A + tB\right)^{1/2} = ran\left(\frac{A+B}{2}\right)^{1/2},$$

where the maximum is taken with respect to the set-inclusion partial ordering, i.e. $M \subset H, N \subset H$, $M \leq N$ if and only if $M \subset N$.

4. Jensen Inequality for Legendre-Fenchel Duality

Let T be a nonempty set and H a Hilbert space. For fixed $t \in T$, let $F(t,.) : H \longrightarrow \widetilde{I\!R}$ which we will briefly write F_t. The functional $F_t^* : H \longrightarrow \widetilde{I\!R}$ denotes the conjugate of F_t for fixed $t \in T$. Now, we are in a position to state the integral Jensen type inequality, recited in the following.

Theorem 4.1. *Let $d\nu(t)$ be a probability measure on T and $(F_t)_{t\in T}$ be a family of measurable functions with respect to $d\nu(t)$. Then, the following inequality holds*

$$\left(\int_T F_t d\nu(t)\right)^* \leq \int_T F_t^* d\nu(t).$$

Proof. By definition, we can write for all $u^* \in H$

$$\left(\int_T F_t d\nu(t)\right)^*(u^*) = \sup_{u\in H} \int_T \left(Re < u^*, u > - F_t(u)\right) d\nu(t),$$

or again

$$\left(\int_{\mathcal{T}} F_t d\nu(t)\right)^* (u^*) \leq \int_{\mathcal{T}} \sup_{u\in H}(Re < u^*, u > -F_t(u)) d\nu(t).$$

The desired inequality follows, thus completes the proof. □

Theorem 4.1 has many consequences involving interesting functional inequalities. For instance, the Hermite-Hadamard type inequalities proved in the above section can be again deduced from the above theorem by setting

$$F_t = (1-t)f + tg, \quad \mathcal{T} = [0,1], \ d\nu(t) = dt.$$

Further, we have also the following results.

Corollary 4.2. *Let $(F_t)_{t\in\mathcal{T}}$ be a family of $\Gamma_0(H)$-functionals. Then, we have the following inequality*

$$\mathcal{L}\left(\int_{\mathcal{T}} F_t d\nu(t)\right) \geq \int_{\mathcal{T}} \mathcal{L}(F_t) d\nu(t).$$

Proof. For $s \in]0,1[$ we can write

$$\sigma - \left((1-s)\sigma + s\int_{\mathcal{T}} F_t d\nu(t)\right)^* = \sigma - \left(\int_{\mathcal{T}} ((1-s)\sigma + sF_t)\, d\nu(t)\right)^*.$$

According to Theorem 4.1, we have

$$\sigma - \left((1-s)\sigma + s\int_{\mathcal{T}} F_t d\nu(t)\right)^* \geq \int_{\mathcal{T}} \left(\sigma - ((1-s)\sigma + sF_t)^*\right) d\nu(t).$$

By Example 2.2 and Fubini Theorem we deduce the desired result. □

Corollary 4.3. *Let $(F_t)_{t\in\mathcal{T}}$ be as in the above, then the following inequality holds*

$$\left(\int_{\mathcal{T}} F_t d\nu(t)\right)^{(m)} \geq \int_{\mathcal{T}} F_t^{(m)} d\nu(t).$$

Proof. Similar to that of the above corollary. We omit the routine details. □

As application for linear operators, Theorem 4.1 combined respectively with Examples 2.1, 2.2, 2.3 gives the next results.

Corollary 4.4. *Let $(\mathcal{A}_t)_{t\in\mathcal{T}}$ be a family of positive definite operators defined from H into itself. Then the following operator inequality holds*

$$\left(\int_{\mathcal{T}} \mathcal{A}_t d\nu(t)\right)^{-1} \leq \int_{\mathcal{T}} \mathcal{A}_t^{-1} d\nu(t).$$

$$log\left(\int_{\mathcal{T}} \mathcal{A}_t d\nu(t)\right) \geq \int_{\mathcal{T}} log\mathcal{A}_t d\nu(t).$$

$$\left(\int_{\mathcal{T}} \mathcal{A}_t d\nu(t)\right)^{m} \geq \int_{\mathcal{T}} \mathcal{A}_t^{m} d\nu(t).$$

Remark 4.1. With some precautions, the above notions and their related results can be extended from the case of a Hilbert space to the case of a locally convex linear space.

5. Refinement of H-H inequality for Point-Wise Convex Maps

The result in Theorem 3.1 can be improved as follows:

Theorem 5.1. *Let* $\Phi : \widetilde{\mathbb{R}}^H \longrightarrow \widetilde{\mathbb{R}}^H$ *and* $f, g \in \widetilde{\mathbb{R}}^H$ *then*

$$\Phi\left(\frac{f+g}{2}\right) \leq \frac{1}{2}\left(\Phi\left(\frac{3f+g}{4}\right) + \Phi\left(\frac{f+3g}{4}\right)\right) \leq \int_0^1 \Phi\left((1-t)f+tg\right)dt \tag{5.1}$$
$$\leq \frac{1}{2}\left(\Phi\left(\frac{f+g}{2}\right) + \frac{\Phi(f)+\Phi(g)}{2}\right) \leq \frac{\Phi(f)+\Phi(g)}{2}$$

Proof. On making the change of variable $u = 2t$, we have

$$\int_0^{1/2} \Phi\left((1-t)f+tg\right)dt = \frac{1}{2}\int_0^1 \Phi\left((1-u)f + u\frac{f+g}{2}\right)du$$

while for the change of variable $u = 2t - 1$ we have

$$\int_{1/2}^1 \Phi\left((1-t)f+tg\right)dt = \frac{1}{2}\int_0^1 \Phi\left((1-u)\frac{f+g}{2} + ug\right)du.$$

Now, applying the inequality 3.1, we have

$$\Phi\left(\frac{3f+g}{4}\right) \leq \int_0^1 \Phi\left((1-u)f + u\frac{f+g}{2}\right)du \leq \frac{1}{2}\left[\Phi(f) + \Phi\left(\frac{f+g}{2}\right)\right]$$

and

$$\Phi\left(\frac{f+3g}{4}\right) \leq \int_0^1 \Phi\left((1-u)\frac{f+g}{2} + ug\right)du \leq \frac{1}{2}\left[\Phi\left(\frac{f+g}{2}\right) + \Phi(g)\right].$$

If we divide both inequalities with 2 and add the obtained results we deduce the desired double inequality. □

Our aim in this section is to state an analogue of Theorem 1.1 for point-wise convex maps. Precisely, putting

$$M_\Phi(f,g) = \int_0^1 \Phi\left((1-t)f+tg\right)dt,$$

we have the following result.

Theorem 5.2. *Let $\mathcal{C}$ be a nonempty convex subset of $\widetilde{\mathbb{R}}^H$ and $\Phi : \mathcal{C} \longrightarrow \widetilde{\mathbb{R}}^H$ a point-wise convex map. For all $f, g \in \mathcal{C}$, define the functional sequences $(\Lambda_n(f,g))_n$ and $(\Delta_n(f,g))_n$ by*

$$\begin{cases} \Lambda_{n+1}(f,g) = \frac{1}{2}\Lambda_n\left(f, \frac{f+g}{2}\right) + \frac{1}{2}\Lambda_n\left(\frac{f+g}{2}, g\right), \ \Lambda_0(f,g) = \Phi\left(\frac{f+g}{2}\right), \\ \Delta_{n+1}(f,g) = \frac{1}{2}\Delta_n\left(f, \frac{f+g}{2}\right) + \frac{1}{2}\Delta_n\left(\frac{f+g}{2}, y\right), \ \Delta_0(f,g) = \frac{\Phi(f)+\Phi(g)}{2}. \end{cases} \tag{5.2}$$

Then $(\Lambda_n(f,g))_n$ is point-wisely increasing and $(\Delta_n(f,g))_n$ is point-wisely decreasing. If moreover the following condition

$$dom\ \Phi\left(\frac{f+g}{2}\right) = dom\ \Phi(f) \cap dom\ \Phi(g) \tag{5.3}$$

holds then the functionals sequences $(\Lambda_n(f,g))_n$ *and* $(\Delta_n(f,g))_n$ *both converge point-wisely to* $M_\Phi(f,g)$ *with the following estimation*

$$0 \leq M_\Phi(f,g) - \Lambda_n(f,g) \leq \Delta_n(f,g) - M_\Phi(f,g) \leq \frac{1}{2^n}\left(\frac{\Phi(f)+\Phi(g)}{2} - \Phi\left(\frac{f+g}{2}\right)\right).$$

Proof. Reiterating the same idea as in the proof of Theorem 5.1, the proof here is similar to that of Theorem 1.1, [1], with some precautions to the fact that we consider here the point-wise partial ordering and convergence. Further, our functionals can take the value $+\infty$ and so the condition (5.3) appears to ensure the conclusion. We omit the details for the reader. □

We leave to the reader the routine task of formulating the relevant corollaries when we combine Theorem 5.2 with Examples 2.1, 2.2, 2.3. Further, a quadratic version of the above theorem can be immediately stated. As example, we may cite the following.

Corollary 5.3. *Let* A *and* B *be two positive definite operators defined from* H *into itself and define the operator sequences*

$$\begin{cases} \Upsilon_{n+1}(A,B) = \frac{1}{2}\Upsilon_n\left(A, \frac{A+B}{2}\right) + \frac{1}{2}\Upsilon_n\left(\frac{A+B}{2}, B\right), \ \Upsilon_0(A,B) = \left(\frac{A+B}{2}\right)^{-1}, \\ \Theta_{n+1}(A,B) = \frac{1}{2}\Theta_n\left(A, \frac{A+B}{2}\right) + \frac{1}{2}\Theta_n\left(\frac{A+B}{2}, B\right), \ \Theta_0(A,B) = \frac{A^{-1}+B^{-1}}{2}. \end{cases} \tag{5.4}$$

Then $(\Upsilon_n(A,B))_n^{-1}$ *and* $(\Theta_n(A,B))_n^{-1}$ *are respectively operator decreasing and increasing and both converge strongly to the logarithm operator mean* $\mathcal{L}(A,B)$ *of* A *and* B*, with the following operator inequalities*

$$\mathcal{H}(A,B) \leq ... \leq (\Theta_n(A,B))^{-1} \leq \mathcal{L}(A,B) \leq (\Upsilon_n(A,B))^{-1} \leq ... \leq \mathcal{A}(A,B) \tag{5.5}$$

The above iterative inequalities (5.5) give some refinements of the arithmetic-logarithmic-harmonic operator mean inequality that proves the interest of this work and the generality of our approach.

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