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BOUNDS FOR TWO MAPPINGS ASSOCIATED TO THE HERMITE-HADAMARD INEQUALITY

S.S. DRAGOMIR[1,2] AND I. GOMM[1]

Abstract. Some inequalities concerning two mappings associated to the celebrated Hermite-Hadamard integral inequality for convex function with applications for special means are given.

1. Introduction

The Hermite-Hadamard integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

In order to provide various refinements of this result, the first author introduced in 1991, see [2], the following associated mapping $H:[0,1]\to\mathbb{R}$ defined by

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

for a given convex function $f:[a,b]\to\mathbb{R}$.

The following theorem collects some of the main properties of H (see also [2], [3], [4] and [6]):

Theorem 1. *With the above assumptions, we have that the function H:*

(i) *is convex on* $[0,1]$;
(ii) *has the bounds:*

$$\inf_{t\in[0,1]}H(t)=H(0)=f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]}H(t)=H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx;$$

(iii) *increases monotonically on* $[0,1]$.

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(iv) *The following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \le \int_0^1 H(t)\,dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right]. \tag{1.1}$$

The corresponding double integral mapping in connection with the Hermite-Hadamard inequalities was considered first in [3] and is defined as

$$F:[0,1]\to\mathbb{R}, F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.$$

The following theorem provides some of the main results concerning this mapping [3] (see also [4]):

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be as above. Then*

(i) $F\left(\tau+\frac{1}{2}\right) = F\left(\frac{1}{2}-\tau\right)$ *for all* $\tau\in\left[0,\frac{1}{2}\right]$ *and* $F(t) = F(1-t)$ *for all* $t\in[0,1]$;

(ii) F *is convex on* $[0,1]$;

(iii) *We have the bounds:*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy;$$

(iv) *The following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le F\left(\frac{1}{2}\right);$$

(v) F *decreases monotonically on* $\left[0,\frac{1}{2}\right]$ *and increases monotonically on* $\left[\frac{1}{2},1\right]$;

(vi) *We have the inequality:*

$$H(t)\le F(t) \text{ for all } t\in[0,1].$$

For other related results, see for instance the research papers [1], [8], [9], [10], [12], [11], [13], [14], [15], the monograph online [7] and the references therein.

Motivated by the above results we establish in this paper some new bounds involving these two mappings. Applications for special means are also provided.

2. The Results

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned}(2.1)\qquad & 0\le 2\min\{t,1-t\}\\ &\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right]\\ &\le \frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)\,f\left(\frac{a+b}{2}\right)-H(t)\\ &\le 2\max\{t,1-t\}\\ &\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right]\end{aligned}$$

and

$$\begin{aligned}(2.2)\qquad & 0\le 2\min\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\right]\\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx-F(t)\\ &\le 2\max\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\right],\end{aligned}$$

for any $t\in[0,1]$.

Proof. Recall the following result obtained by the first author in [5] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}(2.3)\qquad & n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right]\\ &\le \frac{1}{P_n}\sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\\ &\le n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right],\end{aligned}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors in C and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n:=\sum_{i=1}^n p_i>0$.

For $n=2$ we deduce from (2.3) that

$$\begin{aligned}(2.4)\qquad & 2\min\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\ &\le t\Phi(x)+(1-t)\,\Phi(y)-\Phi(tx+(1-t)\,y)\\ &\le 2\max\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x,y\in C$ and $t\in[0,1]$.

On making use of the inequality (2.4) we can write for the convex function $f:[a,b]\to\mathbb{R}$ that

$$2\min\{t,1-t\}\left[\frac{f(x)+f\left(\frac{a+b}{2}\right)}{2}-f\left(\frac{x+\frac{a+b}{2}}{2}\right)\right] \tag{2.5}$$
$$\leq tf(x)+(1-t)f\left(\frac{a+b}{2}\right)-f\left(tx+(1-t)\frac{a+b}{2}\right)$$
$$\leq 2\max\{t,1-t\}\left[\frac{f(x)+f\left(\frac{a+b}{2}\right)}{2}-f\left(\frac{x+\frac{a+b}{2}}{2}\right)\right]$$

for any $x\in[a,b]$ and $t\in[0,1]$.

Integrating over $x\in[a,b]$ in (2.5) we get

$$2\min\{t,1-t\} \tag{2.6}$$
$$\times\left[\frac{1}{2}\left[\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)(b-a)\right]-\int_a^b f\left(\frac{x+\frac{a+b}{2}}{2}\right)dx\right]$$
$$\leq t\int_a^b f(x)\,dx+(1-t)f\left(\frac{a+b}{2}\right)(b-a)-H(t)(b-a)$$
$$\leq 2\max\{t,1-t\}$$
$$\times\left[\frac{1}{2}\left[\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)(b-a)\right]-\int_a^b f\left(\frac{x+\frac{a+b}{2}}{2}\right)dx\right]$$

and since

$$\int_a^b f\left(\frac{x+\frac{a+b}{2}}{2}\right)dx=2\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(s)\,ds$$

then from (2.6) we get (2.1).

Now, if we write the inequality (2.4) for the convex function f and integrate over x and y on $[a,b]$, we get

$$2\min\{t,1-t\} \tag{2.7}$$
$$\times\left[\int_a^b\int_a^b\left[\frac{f(x)+f(y)}{2}\right]dxdy-\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right]$$
$$\leq\int_a^b\int_a^b[tf(x)+(1-t)f(y)]\,dxdy-\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy$$
$$\leq 2\max\{t,1-t\}$$
$$\times\left[\int_a^b\int_a^b\left[\frac{f(x)+f(y)}{2}\right]dxdy-\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\right],$$

for any $t\in[0,1]$.

Since

$$\int_a^b\int_a^b\left[\frac{f(x)+f(y)}{2}\right]dxdy=(b-a)\int_a^b f(x)\,dx,$$
$$\int_a^b\int_a^b[tf(x)+(1-t)f(y)]\,dxdy=(b-a)\int_a^b f(x)\,dx$$

and

$$\int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy = F\left(\frac{1}{2}\right)(b-a)^2,$$

then we deduce from (2.7) the desired result (2.2). □

Corollary 1. *With the above assumptions we have*

$$\begin{aligned} 0 &\leq 2\min\{t,1-t\} \\ &\times \left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\right] - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right) - \frac{1}{2}\left[H(t)+H(1-t)\right] \\ &\leq 2\max\{t,1-t\} \\ &\times \left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\right] - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right], \end{aligned} \tag{2.8}$$

for any $t \in [0,1]$.

Proof. Follows from the inequality (2.1) written for $1-t$ instead of t, by adding the obtained two inequalities and dividing the sum by 2. □

3. Applications for L_p-means

Let us consider the convex mapping $f:(0,\infty)\to\mathbb{R},\ f(x)=x^p,\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the mapping

$$H_p(t) := \frac{1}{b-a}\int_a^b (tx+(1-t)A(a,b))^p\,dx,\ t\in[0,1].$$

It is obvious that $H_p(0) = A^p(a,b),\ H_p(1) = L_p^p(a,b)$ where, we recall that $A(a,b) = \frac{a+b}{2}$,

$$L_p^p(a,b) := \frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a},\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$$

and for $t\in(0,1)$ we have

$$\begin{aligned} H_p(t) &= \frac{1}{[tb+(1-t)A(a,b)]-[ta+(1-t)A(a,b)]}\int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} y^p\,dy \\ &= L_p^p(ta+(1-t)A(a,b),\,tb+(1-t)A(a,b)). \end{aligned} \tag{3.1}$$

The following proposition holds, via Theorem 1, applied for the convex function $f(x)=x^p$.

Proposition 1. *With the above assumptions, we have for the function* H_p*:*

(*i*) *is convex on* $[0,1]$;
(*ii*) *has the bounds:*

$$\inf_{t\in[0,1]} H_p(t) = A^p(a,b),\ \sup_{t\in[0,1]} H_p(t) = L_p^p(a,b);$$

(*iii*) *increases monotonically on* $[0,1]$.

(iv) The following inequalities hold

$$A^{p}(a,b) \leq L_{p}^{p}(A(a,A(a,b)),A(b,A(a,b))) \leq \int_{0}^{1} H_{p}(t)\,dt \leq A\left(A^{p}(a,b),L_{p}^{p}(a,b)\right). \tag{3.2}$$

Now, on making use of Theorem 3 we can state the following result as well:

Proposition 2. *With the above assumptions, we have*

$$\begin{aligned} 0 &\leq 2\min\{t,1-t\} \\ &\times\left[\frac{1}{2}\left[L_{p}^{p}(a,b)+A^{p}(a,b)\right]-L_{p}^{p}\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)\right] \\ &\leq tL_{p}^{p}(a,b)+(1-t)A^{p}(a,b)-H_{p}(t) \\ &\leq 2\max\{t,1-t\} \\ &\times\left[\frac{1}{2}\left[L_{p}^{p}(a,b)+A^{p}(a,b)\right]-L_{p}^{p}\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)\right] \end{aligned} \tag{3.3}$$

for any $t\in[0,1]$.

Now, consider the function

$$F_{p}(t):=\frac{1}{(b-a)^{2}}\int_{a}^{b}\int_{a}^{b}(tx+(1-t)y)^{p}\,dxdy.$$

We observe that $F_{p}(1)=F_{p}(0)=L_{p}^{p}(a,b)$ and for $t\in(0,1)$ we have

$$\begin{aligned} F_{p}(t) &= \frac{1}{b-a}\int_{a}^{b}\left(\frac{1}{b-a}\int_{a}^{b}(tx+(1-t)y)^{p}\,dx\right)dy \\ &= \frac{1}{b-a}\int_{a}^{b}\left(\frac{1}{[tb+(1-t)y]-[ta+(1-t)y]}\int_{ta+(1-t)y}^{tb+(1-t)y}s^{p}ds\right)dy \\ &= \frac{1}{b-a}\int_{a}^{b}L_{p}^{p}(ta+(1-t)y,tb+(1-t)y)\,dy. \end{aligned} \tag{3.4}$$

Utilising Theorem 2 we can state the following results:

Proposition 3. *We have the following properties:*

(i) $F_{p}\left(\tau+\frac{1}{2}\right)=F_{p}\left(\frac{1}{2}-\tau\right)$ *for all* $\tau\in\left[0,\frac{1}{2}\right]$ *and* $F_{p}(t)=F_{p}(1-t)$ *for all* $t\in[0,1]$;

(ii) F_{p} *is convex on* $[0,1]$;

(iii) We have the bounds:

$$\sup_{t\in[0,1]}F_{p}(t)=F_{p}(0)=F_{p}(1)=L_{p}^{p}(a,b)$$

and

$$\inf_{t\in[0,1]}F(t)=F_{p}\left(\frac{1}{2}\right)=\frac{1}{(b-a)^{2}}\int_{a}^{b}\int_{a}^{b}\left(\frac{x+y}{2}\right)^{p}dxdy;$$

(iv) The following inequality holds: $A^{p}(a,b)\leq F_{p}\left(\frac{1}{2}\right)$;

(v) F_{p} *decreases monotonically on* $\left[0,\frac{1}{2}\right]$ *and increases monotonically on* $\left[\frac{1}{2},1\right]$;

(va) We have the inequality:

$$H_{p}(t)\leq F_{p}(t)\ \text{ for all } t\in[0,1].$$

We can calculate the double integral

$$F_p\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b\left(\frac{x+y}{2}\right)^p dxdy$$

as follows.

Observe that for $p\neq -1$ we have

$$\int_a^b\left(\frac{x+y}{2}\right)^p dx=2\frac{\left(\frac{b+y}{2}\right)^{p+1}-\left(\frac{a+y}{2}\right)^{p+1}}{p+1}$$

and for $p\neq -2$ we have

$$\int_a^b\left(\frac{b+y}{2}\right)^{p+1} dy=2\frac{b^{p+2}-\left(\frac{b+a}{2}\right)^{p+2}}{p+2}$$

and

$$\int_a^b\left(\frac{a+y}{2}\right)^{p+1} dy=2\frac{\left(\frac{a+b}{2}\right)^{p+2}-a^{p+2}}{p+2}.$$

Then we get

$$\int_a^b\int_a^b\left(\frac{x+y}{2}\right)^p dxdy=\frac{4}{(p+1)(p+2)}\left[b^{p+2}-2\left(\frac{b+a}{2}\right)^{p+2}+a^{p+2}\right],$$

which gives that

$$F_p\left(\frac{1}{2}\right)=\frac{4}{(b-a)^2(p+1)(p+2)}\left[b^{p+2}-2\left(\frac{b+a}{2}\right)^{p+2}+a^{p+2}\right] \tag{3.5}$$

for $p\neq -2,-1$.

The case $p=-2$ gives that

$$\int_a^b\left(\frac{x+y}{2}\right)^{-2} dx=4\left(\frac{1}{a+y}-\frac{1}{b+y}\right),$$

and

$$\int_a^b\left(\int_a^b\left(\frac{x+y}{2}\right)^{-2} dx\right)dy=4\int_a^b\left(\frac{1}{a+y}-\frac{1}{b+y}\right)dy$$
$$=4\ln\left(\left[\frac{A(a,b)}{G(a,b)}\right]^2\right)=8\ln\left(\frac{A(a,b)}{G(a,b)}\right),$$

where $G(a,b)=\sqrt{ab}$ is the geometric mean of the positive numbers a and b.

Therefore

$$F_{-2}\left(\frac{1}{2}\right)=\frac{8}{(b-a)^2}\ln\left(\frac{A(a,b)}{G(a,b)}\right). \tag{3.6}$$

Finally, on making use of the inequality (2.2) we can state that:

Proposition 4. *We have the inequalities:*

$$0 \le 2 \min \{t, 1-t\} \left[L_p^p (a,b) - F_p \left(\frac{1}{2} \right) \right] \le L_p^p (a,b) - F_p (t) \le 2 \max \{t, 1-t\} \left[L_p^p (a,b) - F_p \left(\frac{1}{2} \right) \right], \tag{3.7}$$

for any $t \in [0,1]$.

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[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.