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INEQUALITIES IN ADMISSIBLE SPACES

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Abstract. Several forms of Bergstrom′s and Bohr′s inequalities will be given below, and other inequalities will be also investigated in this paper .

1. Introduction

We need to recall the used notions, see [7], [4].

Definition 1. *([4]) A locally convex space Z is called admissible in the Loynes sense if the following conditions are satisfied:*

(A.1): *Z is complete;*

(A.2): *there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);*

(A.3): *there is an involution in Z, $Z \ni z \to z^* \in Z$ $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in Z_+$ implies $z^* = z$;*

(A.4): *the topology of Z is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);*

(A.5): *any monotonously decreasing sequence in Z_+ is convergent.*

Remark 1. *([4]) A set $C \in Z$ is called solid if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.*

Definition 2. *([4]) Let Z be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes Z–space if satisfies the following properties:*

(L1): *$\mathcal{H}$ is endowed with an Z–valued inner product (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h,k) \to [h,k] \in Z$ having the properties:*

(G_1) $[h,h] \geq 0$; $[h,h] = 0$ *implies* $h = 0$;

(G_2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G_3) $[\lambda h, k] = \lambda [h,k]$;

(G_4) $[h,k]^* = [k,h]$;

for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

(L2): *The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h,h] \in Z$ is continuous.*

Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes Z–space.

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2. Inequalities in admissible spaces in the Loynes sense

We will consider from now $\mathcal{H}$ as being a Loynes Z-space (LVH-space), see [7]. A form of equality of O. T. Pop, [10] in admissible spaces in Loynes sense is presented below.

Proposition 1. *If $n \in \mathbb{N},\ n \geq 2,\ h_i \in \mathcal{H}$ and $a_i \in \mathbb{R} \setminus \{0\},\ i \in \{1, 2, ..., n\}$ with $\sum_{i=1}^n a_i \neq 0$ then*

$$\sum_{i=1}^n \frac{[h_i, h_i]}{a_i} - \frac{[\sum_{i=1}^n h_i, \sum_{j=1}^n h_j]}{\sum_{i=1}^n a_i} = \frac{1}{\sum_{i=1}^n a_i} \sum_{1 \leq i < j \leq n} \frac{[a_i h_j - a_j h_i, a_i h_j - a_j h_i]}{a_i a_j}.$$

Proof. By calculus we have:

$$\sum_{i=1}^n \frac{[h_i, h_i]}{a_i} - \frac{[\sum_{i=1}^n h_i, \sum_{j=1}^n h_j]}{\sum_{i=1}^n a_i} = \frac{1}{a_1 a_2 ... a_n \sum_{i=1}^n a_i} \{(a_2^2 a_3 a_4 ... a_n + a_2 a_3^2 a_4 ... a_n +$$

$$+ ... + a_2 a_3 ... a_{n-1} a_n^2)[h_1, h_1] + ... + (a_1^2 a_2 a_3 ... a_{n-1} + a_1 a_2^2 a_3 ... a_{n-1} + ... + a_1 a_2 ... a_{n-2} a_{n-1}^2) \cdot$$

$$\cdot [h_n, h_n] - a_1 a_2 ... a_n ([h_1, h_2] + [h_2, h_1] + ... + [h_1, h_n] + [h_n, h_1] + ... + [h_{n-1}, h_n] + [h_n, h_{n-1}])\} =$$

$$= \frac{1}{a_1 a_2 ... a_n \sum_{i=1}^n a_i} \{a_3 a_4 ... a_n (a_2^2 [h_1, h_1] + a_1^2 [h_2, h_2] - a_1 a_2 ([h_1, h_2] + [h_2, h_1])) + ... +$$

$$+ a_1 a_2 ... a_{n-2} (a_{n-1}^2 [h_n, h_n] + a_n^2 [h_{n-1}, h_{n-1}] - a_n a_{n-1} ([h_n, h_{n-1}] + [h_{n-1}, h_n]))\}$$

i.e the desired equality takes place.

∎

The following result, (ii) contains a form of Bergstrom's inequality in admissible spaces in the Loynes sense.

Consequence 1. *(i) If $n \in \mathbb{N}$, $n \geq 2,\ h_1, h_2, ..., h_n \in \mathcal{H}$ and $a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\}$ with $\sum_{k=1}^n a_k \geq 0$, then*

(2.1)

$$\frac{[h_1, h_1]}{a_1} + \frac{[h_2, h_2]}{a_2} + ... + \frac{[h_n, h_n]}{a_n} \geq \frac{1}{\sum_{k=1}^n a_k} \sum_{1 \leq i < j \leq n} \frac{[a_i h_j - a_j h_i, a_i h_j - a_j h_i]}{a_i a_j}.$$

(ii) Under the above-stated conditions, if $a_1,\ a_2,\ ...., a_n \in (0, \infty)$ we also have,

$$\frac{[h_1, h_1]}{a_1} + \frac{[h_2, h_2]}{a_2} + ... + \frac{[h_n, h_n]}{a_n} \geq \frac{[h_1 + h_2 + ... + h_n, h_1 + h_2 + ... + h_n]}{a_1 + a_2 + ... + a_n},$$

with equality if and only if $a_i h_j = a_j h_i$, for any $i, j \in \{1, 2, ..., n\}$.

Consequence 2. *(i) If we take above instead of $[.,.]$ an inner product which takes values in $\mathbb{C}$ instead of Z and $\mathcal{H}$ is a Hilbert space instead of Loynes space then for $< Ah_i, Ah_i >= ||Ah_i||^2$ where $A \in \mathcal{B}(\mathcal{H})$ is a linear bounded operator on $\mathcal{H}$ we have*

$$\sum_{i=1}^n \frac{||Ah_i||^2}{a_i} - \frac{||A(\sum_{i=1}^n h_i)||^2}{\sum_{i=1}^n a_i} = \frac{1}{\sum_{i=1}^n a_i} \sum_{1 \leq i < j \leq n} \frac{||A(a_i h_j - a_j h_i)||^2}{a_i a_j}.$$

(ii) Under the above conditions if we take $A_i h$ instead of h_i, A_i being linear and bounded operators on $\mathcal{H}$, we have:

$$\sum_{i=1}^n \frac{||A_i||^2}{a_i} - \frac{||\sum_{i=1}^n A_i||^2}{\sum_{i=1}^n a_i} = \frac{1}{\sum_{i=1}^n a_i} \sum_{1 \leq i < j \leq n} \frac{||a_i A_j - a_j A_i||^2}{a_i a_j}.$$

Consequence 3. *Under the conditions of Proposition 1, if p is a monotonic seminorm in the admissible space Z and q_p is the corresponding seminorm in Loynes Z- space $\mathcal{H}$, given by $q_p(h) = (p([h,h])^{\frac{1}{2}}$, see [4] we will have*

$$|\sum_{i=1}^{n} a_i| p(\sum_{i=1}^{n} \frac{[h_i,h_i]}{|a_i|}) \leq p\left(\sum_{1\leq i<j\leq n} \frac{[a_i h_j - a_j h_i, a_i h_j - a_j h_i]}{a_i a_j}\right) + q_p^2(\sum_{i=1}^{n} h_i).$$

Proposition 2. *(i) Let $\alpha,\ \beta,\ \gamma \in \mathbb{R},\ \alpha > 0$ and $a,\ b \in \mathcal{H}$, $\mathcal{H}$ being a Loynes Z-space. Then we have:*

$$[\alpha a - b, \alpha a - b] + [\beta a - \gamma b, \beta a - \gamma b] = (\alpha + \beta\gamma)\left[\sqrt{\alpha} a - \frac{1}{\sqrt{\alpha}} b, \sqrt{\alpha} a - \frac{1}{\sqrt{\alpha}} b\right] +$$

$$+\beta(\beta - \alpha\gamma)[a,a] + \gamma(\gamma - \frac{\beta}{\alpha})[b,b]. \tag{2.2}$$

(ii) If $\alpha + \beta\gamma < 0$ and $\alpha > 0$, where $a,\ b \in \mathcal{H}$ then

$$[\alpha a - b, \alpha a - b] + [\beta a - \gamma b, \beta a - \gamma b] \leq \beta(\beta - \alpha\gamma)[a,a] + \gamma(\gamma - \frac{\beta}{\alpha})[b,b]. \tag{2.3}$$

(iii) If $a,\ b,\ c,\ d \in \mathbb{R}\setminus\{0\},\ a,b > 0$ and $a_1,\ b_1 \in \mathcal{H}$ then

$$[aa_1 - bb_1, aa_1 - bb_1] + [ca_1 - db_1, ca_1 - db_1] =$$

$$= (ab+cd)\left[\sqrt{\frac{a}{b}} a_1 - \sqrt{\frac{b}{a}} b_1, \sqrt{\frac{a}{b}} a_1 - \sqrt{\frac{b}{a}} b_1\right] + c^2(1 - \frac{ad}{cb})[a_1,a_1] + d^2(1 - \frac{cb}{ad})[b_1,b_1].$$

(iv) If $ab + cd < 0$ and $\frac{a}{b} > 0$, where $a_1,\ b_1 \in \mathcal{H}$ then

$$[aa_1 - bb_1, aa_1 - bb_1] + [ca_1 - db_1, ca_1 - db_1] \leq c^2(1 - \frac{ad}{cb})[a_1,a_1] + d^2(1 - \frac{cb}{ad})[b_1,b_1].$$

Proof. Taking into account that

$$[\alpha a - b, \alpha a - b] + [\beta a - \gamma b, \beta a - \gamma b] = (\alpha^2 + \beta^2)[a,a] + (\gamma^2 + 1)[b,b] - (\alpha + \beta\gamma)([a,b] + [b,a])$$

and using the same method as in [5] equality (2.2) takes place.

∎

The following result is a generalization on admissible spaces of Theorem 4 from [2] and of Theorem 7 from [1].

Proposition 3. *Let $n > 0$ be any integer, $h_i \in \mathcal{H}$ $(i = 1,2,...,n)$ and $p_{ij},\ q_{ij} \in \mathbb{R}$ such that $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$ for $1 \leq i < j \leq n$.*

(i) If $p_{ij} > 1$ for all $1 \leq i < j \leq n$ then

$$\left[\sum_{i=1}^{n} h_i, \sum_{j=1}^{n} h_j\right] \leq \left(\sum_{j=2}^{n} p_{1j} + 2 - n\right)[h_1,h_1] + \tag{2.1}$$

$$+\sum_{k=2}^{n-1}\left(\sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n\right)[h_k,h_k] + \left(\sum_{j=1}^{n-1} q_{jn} + 2 - n\right)[h_n,h_n].$$

(ii) If $p_{ij} < 1$ for all $1 \leq i < j \leq n$ then the reverse inequality is obtained.

Proof. By calculus we obtain,

$$\left[\sum_{i=1}^{n} h_i, \sum_{j=1}^{n} h_j\right] - \sum_{i=1}^{n}[h_i, h_i] = \sum_{1\le i<j\le n}([h_i, h_j] + [h_j, h_i]) =$$

$$= \sum_{1\le i<j\le n}([h_i+h_j, h_i+h_j]-([h_i, h_i]+[h_j, h_j])) \le \sum_{1\le i<j\le n}((p_{ij}-1)[h_i, h_i]+(q_{ij}-1)[h_j, h_j]).$$

That means

$$\left[\sum_{i=1}^{n} h_i, \sum_{j=1}^{n} h_j\right] \le \sum_{i=1}^{n}[h_i, h_i] + \sum_{j=2}^{n}(p_{1j} - 1)[h_1, h_1] + \sum_{j=1}^{n-1}(q_{jn} - 1)[h_n, h_n]+$$

$$+[h_n, h_n] + \sum_{k=2}^{n-1}(1 + \sum_{j=k+1}^{n}(p_{kj} - 1) + \sum_{j=1}^{k-1}(q_{jk} - 1))[h_k, h_k] =$$

$$= (\sum_{j=2}^{n} p_{1j}+2-n)[h_1, h_1]+(\sum_{j=1}^{n-1} q_{jn}+2-n)[h_n, h_n]+\sum_{k=2}^{n-1}(\sum_{j=k+1}^{n} p_{kj}+\sum_{j=1}^{k-1} q_{jk}+2-n)[h_k, h_k].$$

∎

Corollary 1. *Let $m, n, p, q \in \mathbb{C}$ and $a, b \in \mathcal{H}$ such that $\frac{\overline{m}}{\overline{n}} \ge 0$ and $m\overline{n} + p\overline{q} \in \mathbb{R}$. Then we have:*

$$[ma-nb, ma-nb]+[pa-qb, pa-qb] = (m\overline{n}+p\overline{q})\{[\sqrt{\frac{\overline{m}}{\overline{n}}}a-\sqrt{\frac{\overline{n}}{\overline{m}}}b, \sqrt{\frac{\overline{m}}{\overline{n}}}a-\sqrt{\frac{\overline{n}}{\overline{m}}}b]\}+$$

$$+p\overline{q}(\frac{\overline{p}}{\overline{q}} - \frac{\overline{m}}{\overline{n}})[a, a] + q\overline{p}(\frac{\overline{q}}{\overline{p}} - \frac{\overline{n}}{\overline{m}})[b, b].$$

If $m\overline{n} + p\overline{q} \ge 0$ then

$$[ma - nb, ma - nb] + [pa - qb, pa - qb] \ge p\overline{q}(\frac{\overline{p}}{\overline{q}} - \frac{\overline{m}}{\overline{n}})[a, a] + q\overline{p}(\frac{\overline{q}}{\overline{p}} - \frac{\overline{n}}{\overline{m}})[b, b].$$

If $m\overline{n} + p\overline{q} \le 0$ then the converse inequality takes place.

Proof. We will use the same idea as in Proposition 2. Thus it is necessary to compute $[ma - nb, ma - nb] + [pa - qb, pa - qb]$ and using the fact that $m\overline{n} + p\overline{q} = \overline{m}n + p\overline{q}$ we can write

$$m\overline{m} + p\overline{p} = (m\overline{n} + p\overline{q})(\frac{\overline{m}}{\overline{n}} + \frac{\overline{p}}{\overline{q}}) - p\overline{q}\frac{\overline{m}}{\overline{n}} - m\overline{n}\frac{\overline{p}}{\overline{q}}$$

and

$$n\overline{n} + q\overline{q} = (\overline{m}n + \overline{p}q)(\frac{\overline{n}}{\overline{m}} + \frac{\overline{q}}{\overline{p}}) - q\overline{p}\frac{\overline{n}}{\overline{m}} - n\overline{m}\frac{\overline{q}}{\overline{p}}$$

That means

$$[ma - nb, ma - nb] + [pa - qb, pa - qb] =$$

$$= (m\overline{n} + p\overline{q})\{(\frac{\overline{m}}{\overline{n}} + \frac{\overline{p}}{\overline{q}})[a, a] - [a, b] - [b, a] + (\frac{\overline{n}}{\overline{m}} + \frac{\overline{q}}{\overline{p}})[b, b]\}-$$

$$-(p\overline{q}\frac{\overline{m}}{\overline{n}} + m\overline{n}\frac{\overline{p}}{\overline{q}})[a, a] - (q\overline{p}\frac{\overline{n}}{\overline{m}} + n\overline{m}\frac{\overline{q}}{\overline{p}})[b, b]$$

which leads to our equality.

∎

If we consider the coefficients arbitrary elements in $\mathbb{C}$ then the equality from Proposition 2 can take the following form:

Proposition 4. *Let $m,\ n,\ p,\ q \in \mathbb{C}$ with $m = m_1 + im_2,\ n = n_1 + in_2,\ p = p_1 + ip_2,\ q = q_1 + iq_2$ and $a, b \in \mathcal{H}$, where $\mathcal{H}$ is a Loynes Z-space. If the products $m_1n_1,\ m_1n_2,\ p_1q_1,\ p_2q_1$ are positive then the following equality takes place:*

$$[ma - nb, ma - nb] + [pa - qb, pa - qb] =$$

$$= (m_1n_1 + m_2n_2)\left[\sqrt{\frac{m_1}{n_1}}a - \sqrt{\frac{n_1}{m_1}}b, \sqrt{\frac{m_1}{n_1}}a - \sqrt{\frac{n_1}{m_1}}b\right] +$$

$$+(m_1n_2 - m_2n_1)\left[i\sqrt{\frac{m_1}{n_2}}a + \sqrt{\frac{n_2}{m_1}}b, i\sqrt{\frac{m_1}{n_2}}a + \sqrt{\frac{n_2}{m_1}}b\right] +$$

$$+(p_1q_1 + p_2q_2)\left[\sqrt{\frac{p_1}{q_1}}a - \sqrt{\frac{q_1}{p_1}}b, \sqrt{\frac{p_1}{q_1}}a - \sqrt{\frac{q_1}{p_1}}b\right] +$$

$$+(p_2q_1 - p_1q_2)\left[\sqrt{\frac{p_2}{q_1}}a + i\sqrt{\frac{q_1}{p_2}}b, \sqrt{\frac{p_2}{q_1}}a + i\sqrt{\frac{q_1}{p_2}}b\right] -$$

$$-\frac{(m_1n_2 - m_2n_1)(m_1n_1 + m_2n_2)}{n_1n_2}[a, a] - \frac{(p_2q_1 - p_1q_2)(p_2q_2 + p_1q_1)}{p_1p_2}[b, b].$$

Proof. The equality will results by calculus. ∎

Applying now the same kind of equality for the first and the third term and for the second and the fourth term we will obtain :

Consequence 4. *Under the above conditions we obtain:*

$$[ma - nb, ma - nb] + [pa - qb, pa - qb] =$$

$$= \{(m_1n_1 + m_2n_2)^2 + (p_1q_1 + p_2q_2)^2\}\left[\sqrt{\frac{m_1}{n_1}}a - \sqrt{\frac{n_1}{m_1}}b, \sqrt{\frac{m_1}{n_1}}a - \sqrt{\frac{n_1}{m_1}}b\right] =$$

$$+\{(m_1n_2 - m_2n_1)^2 + (p_2q_1 - q_1p_2)^2\}\left[\sqrt{\frac{m_1}{n_2}}a - i\sqrt{\frac{n_2}{m_1}}b, \sqrt{\frac{m_1}{n_2}}a - i\sqrt{\frac{n_2}{m_1}}b\right] +$$

$$+\{(p_1q_1+p_2q_2)^2(\frac{p_1}{q_1}-\frac{m_1}{n_1})+(p_2q_1-q_1p_2)^2(\frac{p_2}{q_1}+\frac{m_1}{n_2})-(\frac{m_1}{n_1}-\frac{m_2}{n_2})(m_1n_1+m_2n_2)\}[a, a]+$$

$$+\{(p_1q_1+p_2q_2)^2(\frac{q_1}{p_1}-\frac{n_1}{m_1})+(p_1q_2-p_2q_1)^2(\frac{q_1}{p_2}+\frac{n_2}{m_1})-\frac{(p_1q_1 + p_2q_2)(p_2q_1 - p_1q_2)}{p_1p_2}\}[b, b].$$

Moreover the following inequality takes place:

$$[ma - nb, ma - nb] + [pa - qb, pa - qb] \geq$$

$$\geq \{(p_1q_1+p_2q_2)^2(\frac{p_1}{q_1}-\frac{m_1}{n_1})+(p_2q_1-q_1p_2)^2(\frac{p_2}{q_1}+\frac{m_1}{n_2})-(\frac{m_1}{n_1}-\frac{m_2}{n_2})(m_1n_1+m_2n_2)\}[a, a]+$$

$$+\{(p_1q_1+p_2q_2)^2(\frac{q_1}{p_1}-\frac{n_1}{m_1})+(p_1q_2-p_2q_1)^2(\frac{q_1}{p_2}+\frac{n_2}{m_1})-\frac{(p_1q_1 + p_2q_2)(p_2q_1 - p_1q_2)}{p_1p_2}\}[b, b].$$

Let $\mathcal{H}$ and $\mathcal{K}$ be two Loynes Z- spaces. We can obtain also for Loynes Z-spaces a similar result of [8].

Remark 2. *Let* $\alpha,\ \beta,\ \gamma \in \mathbb{R}$ *and* $T,\ S \in \mathcal{L}^*(\mathcal{H},\mathcal{K})$ *two linear operators which admit gramian adjoint from* $\mathcal{H}$ *to* $\mathcal{K}$ *such that* $T^*S = S^*T$ *and* $\alpha[Th,Th] + \beta[Sh,Sh] = \gamma[h,h]$, $(\forall)\ h \in \mathcal{H}$. *Then we have:*

$$\alpha\beta[Th+Sk,Th+Sk] + [\beta Sh - \alpha Tk, \beta Sh - \alpha Tk] = \beta\gamma[h,h] + \alpha\gamma[k,k],$$

for all $h,\ k \in \mathcal{H}$.

As an application of Proposition 2, we can find another upper bound for the expression $||\delta^2_{A,B}(X)||_2^2 + ||\delta^2_{(p-1)A,-B}(X)||_2^2$ from Theorem 4, see [6].

Proposition 5. *Let* $A,\ B \in \mathcal{B}(\mathcal{H})$ *be two normal operators in Hilbert spaces and* $m,\ n,\ p,\ q \in \mathbb{R}\setminus\{0\}$ *with* $m,\ n > 0$ *see Proposition 2, (iii). If* $mn + pq < 0$ *then*

$$||\delta^2_{mA,nB}(X)||_2^2 + ||\delta^2_{pA,qB}(X)||_2^2 \le ||\delta^2_{\frac{m}{n}A,\frac{n}{m}B}(X) + (1-\frac{mq}{np})|pA|^2X + (1-\frac{np}{mq}X|qB|^2)||_2^2,$$

for every $X \in \mathcal{B}(\mathcal{H})$.

Also if $m,\ n,\ p,\ q \in \mathbb{R}\setminus\{0\}$ *with* $m,\ n > 0$ *the following inequality takes place:*

$$\frac{1}{2}||\delta^2_{\frac{m}{n}A,\frac{n}{m}B}(X) + (1-\frac{mq}{np})|pA|^2X + (1-\frac{np}{mq}X|qB|^2)||_2^2 \le ||\delta^2_{mA,nB}(X)||_2^2 + ||\delta^2_{pA,qB}(X)||_2^2$$

for every $X \in \mathcal{B}(\mathcal{H})$.

Proof. We will use Proposition 2 and the proof will be as in [6], using the inequalities $x^2 + y^2 \le (x+y)^2$ and $2(x^2+y^2) \ge (x+y)^2$ for $x,\ y \ge 0$. Thus

$$||\delta^2_{mD_1,nD_2}(X)||_2^2 + ||\delta^2_{pD_1,qD_2}(X)||_2^2 = \sum_{i,j=1}^{\infty} | < \delta^2_{mD_1,nD_1}(X)f_j, e_i > |^2 +$$

$$+ \sum_{i,j=1}^{\infty} | < \delta^2_{pD_1,qD_2}(X)f_j, e_i > |^2 = \sum_{i,j=1}^{\infty} |m\lambda_i - n\mu_j|^4 | < Xf_j, e_i > |^2 +$$

$$+ \sum_{i,j=1}^{\infty} |p\lambda_i - q\mu_j|^4 | < Xf_j, e_i > |^2 \le \sum_{i,j=1}^{\infty} (|m\lambda_i - n\mu_j|^2 + |p\lambda_i - q\mu_j|^2)^2 | < Xf_j, e_i > |^2 =$$

$$= \sum_{i,j=1}^{\infty} \{(mn+pq)|\sqrt{\frac{m}{n}}\lambda_i - \sqrt{\frac{n}{m}}\mu_j|^2 + p^2(1-\frac{mq}{pn})|\lambda_i|^2 + q^2(1-\frac{np}{mq}|\mu_j|^2)\}^2 | < Xf_j, e_i > |^2 =$$

$$= ||\delta^2_{\sqrt{\frac{m}{n}}D_1,\sqrt{\frac{n}{m}}D_2}(X) + (1-\frac{mq}{np})|pD_1|^2X + (1-\frac{np}{mq})X|qD_2|^2||_2^2.$$

∎

3. Inequalities for Z- valued sesquilinear forms

Let now $\mathcal{H}$ be a complex vector space, Z an admissible space in the Loynes sense and $\mathcal{F}(\mathcal{H},Z)$ the set of sesquilinear Z- valued functions on $\mathcal{H}$, i.e.

$$\varphi : \mathcal{H}\times\mathcal{H} \to Z,$$

which satisfy

$$\varphi(\alpha_1 h_1 + \alpha_2 h_2, k) = \alpha_1\varphi(h_1,k) + \alpha_2\varphi(h_2,k),$$

and

$$\varphi(h, \beta_1 k_1 + \beta_2 k_2) = \overline{\beta_1}\varphi(h,k_1) + \overline{\beta_2}\varphi(h,k_2),$$

for any $h,\ h_j,\ k,\ k_j \in \mathcal{H}$; $\alpha_j,\ \beta_j \in \mathbb{C}$ $(j = 1,2)$.

The elements of $\mathcal{F}(\mathcal{H}, Z)$ are called sesquilinear forms (or functionals).

We shall say that the sesquilinear Z-form φ is positive if satisfies

$$\varphi(h,h) \geq 0, \ h \in \mathcal{H}.$$

Remark 3. *If $\varphi : \mathcal{H} \times \mathcal{H} \to Z$ is a positive Z- sesquilinear form (i.e. $\varphi(h,h) \geq 0$, $(\forall) h \in \mathcal{H}$) and $m,\ n,\ p,\ q \in \mathbb{R}$, $m,\ n > 0$, $a,\ b \in \mathcal{H}$ then*

$$\varphi(ma-nb, ma-nb)+\varphi(pa-qb, pa-qb) = (mn+pq)\varphi(\sqrt{\frac{m}{n}}a-\sqrt{\frac{n}{m}}b, \sqrt{\frac{m}{n}}a-\sqrt{\frac{n}{m}}b)+$$

$$+p^2(1-\frac{mq}{pn})\varphi(a,a)+q^2(1-\frac{pn}{mq})\varphi(b,b).$$

Let $\mathcal{H}$ be now a linear space and Λ an arbitrary set, $\mathcal{K}$ a Loynes Z- space and a family of linear operators defined by: $D : \Lambda \to \mathcal{L}(\mathcal{H}, \mathcal{K})$. The kernel $C : \Lambda \times \Lambda \to \mathcal{F}(\mathcal{H}, Z)$ defined by

$$C(s,t)(h,k) = [D(t)h, D(s)k]_{\mathcal{K}}$$

is $\mathcal{F}(\mathcal{H}, Z)$-valued and positively defined, i.e.

$$\sum_{j,l=1}^{n} C(s_j, s_l)(h_l, h_j) \geq 0, \ (\forall) \ n \in \mathbb{N}, \ s_1, ..., s_n \in \Lambda$$

and $h_1, ..., h_n \in \mathcal{H}$.

Consequence 5. *Under the above conditions, and Remark 3 we obtain an analogue of Proposition 2 for kernels:*

$$[D(t)(ma-nb), D(t)(ma-nb)]_{\mathcal{K}} + [D(t)(pa-qb), D(t)(pa-qb)]_{\mathcal{K}} \leq$$

$$\leq m^2(1-\frac{np}{mq})[D(t)a, D(t)a]_{\mathcal{K}} + q^2(1-\frac{mq}{np})[D(t)b, D(t)b]_{\mathcal{K}}$$

or

$$C(t,t)(ma-nb, ma-nb) + C(t,t)(pa-qb, pa-qb) \leq$$

$$\leq m^2(1-\frac{np}{mq})C(t,t)(a,a) + q^2(1-\frac{mq}{np})C(t,t)(b,b).$$

or

$$[mD(t)a-nD(s)b, mD(t)a-nD(s)b]_{\mathcal{K}} + [pD(t)a-qD(s)b, pD(t)a-qD(s)b]_{\mathcal{K}} \leq$$

$$\leq m^2(1-\frac{np}{mq})[D(t)a, D(t)a]_{\mathcal{K}} + q^2(1-\frac{mq}{np})[D(s)b, D(s)b]_{\mathcal{K}}$$

As in Proposition 1 we can state the following result:

Remark 4. *If $n \in \mathbb{N}$, $n \geq 2$, $h_i \in \mathcal{H}$ and $a_i \in \mathbb{R} \setminus \{0\}$, $i \in \{1, 2, ..., n\}$ with $\sum_{i=1}^{n} a_i \neq 0$, $\varphi : \mathcal{H} \times \mathcal{H} \to Z$ is a positive Z- sesquilinear form then*

$$\sum_{i=1}^{n} \frac{\varphi(h_i, h_i)}{a_i} - \frac{\varphi(\sum_{i=1}^{n} h_i, \sum_{j=1}^{n} h_j)}{\sum_{i=1}^{n} a_i} = \frac{1}{\sum_{i=1}^{n} a_i} \sum_{1 \leq i < j \leq n} \frac{\varphi(a_i h_j - a_j h_i, a_i h_j - a_j h_i)}{a_i a_j}.$$

Proposition 3 can we rewritten also for Z- valued sesquilinear forms.

Remark 5. *Let $n > 0$ be any integer, $h_i \in \mathcal{H}$ $(i = 1, 2, ..., n)$ and $p_{ij},\ q_{ij} \in \mathbb{R}$ such that $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$ for $1 \leq i < j \leq n$ and $\varphi : \mathcal{H} \times \mathcal{H} \to Z$ is a positive Z- sesquilinear form.*

(i) If $p_{ij} > 1$ for all $1 \leq i < j \leq n$ then

$$\varphi(\sum_{i=1}^{n} h_i, \sum_{j=1}^{n} h_j) \leq \left(\sum_{j=2}^{n} p_{1j} + 2 - n \right) \varphi(h_1, h_1) + \tag{2.1}$$

$$+ \sum_{k=2}^{n-1} \left(\sum_{j=k+1}^{n} p_{kj} + \sum_{j=1}^{k-1} q_{jk} + 2 - n \right) \varphi(h_k, h_k) + \left(\sum_{j=1}^{n-1} q_{jn} + 2 - n \right) \varphi(h_n, h_n)$$

(ii) If $p_{ij} < 1$ for all $1 \leq i < j \leq n$ then the reverse inequality is obtained.

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