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HERMITE-HADAMARD TYPE INEQUALITY FOR OPERATOR PREINVEX FUNCTIONS

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Abstract. In this paper we establish a Hermite- Hadamard type inequality for operator preinvex functions and an estimate of the right hand side of a Hermite- Hadamard type inequality in which some operator preinvex functions of selfadjoint operators in Hilbert spaces are involved.
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1. introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Dragomir and Agarwal in [3] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some convex functions are involved. The main results of [3] are given by the following theorems.

Theorem 1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.2}$$

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [8].

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

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We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},$$
$$g(x,y)(t):=f((1-t)x+ty),t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$. For any convex function defined on a segment $[x,y]\in X$, we have the Hermite- Hadamard integral inequality (see [4, p.2] and [5, p.2])

$$f\left(\frac{x+y}{2}\right)\le\int_0^1 f((1-t)x+ty)dt\le\frac{f(x)+f(y)}{2}, \tag{1.3}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0;1]\to\mathbb{R}$.

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator preinvex functions and operator convex functions.

In order to do that we need the following preliminary definitions and results. Let A be a bounded self adjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [7, p.3]). For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(f^*)=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the continuous functional calculus for a bounded selfadjoint operator A. If A is a bounded selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\ge 0$ for any $t\in Sp(A)$ implies that $f(A)\ge 0$, i.e., $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\ge g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\ge g(A) \tag{P}$$

in the operator order in $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A+\lambda B)\le(\ge)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order in $B(H)$, for all $\lambda\in[0,1]$ and for every bounded selfadjoint operators A and B in $B(H)$ whose spectra are contained in I.

Dragomir in [6] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

Theorem 2. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\left(f\left(\frac{A+B}{2}\right) \leq\right) \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \leq \int_0^1 f((1-t)A+tB))dt \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right]\left(\leq \frac{f(A)+f(B)}{2}\right).$$

In this paper we show that Theorem 3 holds for operator preinvex functions and establish an estimate of the right hand side of a Hermite- Hadamard type inequality in which some operator preinvex functions of selfadjoint operators in Hilbert spaces are involved.

2. OPERATOR PREINVEX FUNCTIONS

Definition 1. *Let X be a real vector space, a set $S \subseteq X$ is said to be invex with respect to the map $\eta : S \times S \to X$, if for every $x, y \in S$ and $t \in [0, 1]$,*

$$y + t\eta(x, y) \in S. \tag{2.1}$$

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x-y$, but there exist invex sets which are not convex (see [1]).

Let $S \subseteq X$ be an invex set with respect to $\eta : S \times S \to X$. For every $x, y \in S$ the $\eta-$path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y, x) : t \in [0, 1]\}.$$

The mapping η is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0, 1]$,

$$(C) \qquad \begin{aligned} \eta(y, y + t\eta(x, y)) &= -t\eta(x, y), \\ \eta(x, y + t\eta(x, y)) &= (1-t)\eta(x, y). \end{aligned}$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0, 1]$ from condition C we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y), \tag{2.2}$$

see [9, 10] for details.

Let $\mathcal{A}$ be a C^*-algebra, denote by $\mathcal{A}_{sa}$ the set of all self adjoint elements in $\mathcal{A}$.

Definition 2. *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$. Then, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to η on S, if for every $A, B \in S$ and $t \in [0, 1]$,*

$$f(A + t\eta(B, A)) \leq (1-t)f(A) + tf(B). \tag{2.3}$$

in the operator order in $B(H)$.

Every operator convex function is an operator preinvex with respect to the map $\eta(A, B) = A - B$ but the converse does not holds (see the following example).

Now, we give an example of some operator preinvex functions and invex sets with respect to the maps η which satisfy the conditions (C).

Example 1. (a) Suppose that 1_H is the identity operator on a Hilbert space H, and

$$T := (-3 \times 1_H, -1 \times 1_H) = \{A \in B(H)_{sa} : -3 \times 1_H < A < -1 \times 1_H\}$$
$$U := (1_H, 4 \times 1_H) = \{A \in B(H)_{sa} : 1_H < A < 4 \times 1_H\}$$
$$S := T \cup U \subseteq B(H)_{sa}.$$

Suppose that the function $\eta_1 : S \times S \to B(H)_{sa}$ is defined by

$$\eta_1(A,B) = \begin{cases} A - B & A, B \in U, \\ A - B & A, B \in T, \\ 1_H - B & A \in T, B \in U, \\ -1_H - B & A \in U, B \in T. \end{cases}$$

Clearly η_1 satisfies condition C and S is an invex set with respect to η_1. The real function $f(t) = t^2$ is preinvex with respect to η_1 on S. Since f is an operator convex function, therefore for the cases which $\eta_1(A,B) = A-B$ the inequality (2.3) holds. Let $\eta_1(A,B) = 1_H - B$, since $1_H < A < 4 \times 1_H$ for every $x \in H$ by Mond-Pečarić inequality we have

$$\langle x, x\rangle^2 < \langle Ax, x\rangle^2 \leq \langle A^2 x, x\rangle$$

Therefore $1_H^2 < A^2$, and this implies that

$$\begin{aligned}(B + t\eta_1(A,B))^2 &= (B + t(1_H - B))^2 = ((1-t)B + t1_H)^2 \\ &\leq (1-t)B^2 + t1_H^2 \leq (1-t)B^2 + tA^2.\end{aligned}$$

Similarly for the case $\eta_1 = -1_H - B$, the inequality (2.3) holds.

But the real function $g(t) = a + bt, \quad a, b \in \mathbb{R}$ is not preinvex with respect to η_1 on S.

(b) Suppose that $V := (-2 \times 1_H, 0),\ W := (0, 2 \times 1_H),\ S := V \cup W \subseteq B(H)_{sa}$ and the function $\eta_2 : S \times S \to B(H)_{sa}$ is defined by

$$\eta_2(A,B) = \begin{cases} A - B & A, B \in V \text{ or } A, B \in W, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly η_2 satisfies condition C and S is an invex set with respect to η_2. The constant functions $f(t) = a,\ a \in \mathbb{R}$ is only preinvex functions with respect to η_2 on S. Because for $\eta_2 = 0$,

$$f(B + t\eta_2(A,B)) = f(B) \leq (1-t)f(B) + tf(A),$$

implies that $f(A) - f(B) \geq 0$. interchanging A ,B we get $f(B) - f(A) \geq 0$.

(c) The function $f(t) = -|t|$ is not a convex function, but it is a preinvex function with respect to η_3, where

$$\eta_3(A,B) = \begin{cases} A - B & A, B \geq 0 \text{ or } A, B \leq 0, \\ B - A & \text{otherwise}. \end{cases}$$

Proposition 1. *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that η satisfies condition C on S. Then for every $A, B \in S$ and $V = A + \eta(B,A)$ the function f is operator preinvex* *with respect to* η *on* $\eta-$*path* P_{AV} *if and only if the function* $\varphi_{x,A,B} : [0,1] \to \mathbb{R}$ *defined by*

$$\varphi_{x,A,B}(t) := \langle f(A + t\eta(B,A))x, x\rangle \tag{2.4}$$

is convex on $[0,1]$ *for every* $x \in H$ *with* $\|x\| = 1$.

Proof. Suppose that $x \in H$ with $\|x\| = 1$ and $\varphi_{x,A,B}$ is convex on $[0,1]$ and $C_1 := A + t_1\eta(B,A) \in P_{AV}, C_2 := A + t_2\eta(B,A) \in P_{AV}$. Fix $\lambda \in [0,1]$. By (2.4) we have

$$\begin{aligned}\langle f(C_1 + \lambda\eta(C_2, C_1))x, x\rangle &= \langle f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B,A))x, x\rangle \\ &= \varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) \\ &\leq (1-\lambda)\varphi_{x,A,B}(t_1) + \lambda\varphi_{x,A,B}(t_2) \\ &= (1-\lambda)\langle f(C_1)x, x\rangle + \lambda\langle f(C_2)x, x\rangle.\end{aligned} \tag{2.5}$$

Hence, f is operator preinvex with respect to η on $\eta-$path P_{AV}.

Conversely, let $A, B \in S$ and the function f be operator preinvex with respect to η on $\eta-$path P_{AV}. Suppose that $t_1, t_2 \in [0,1]$. Then, for every $\lambda \in [0,1]$ and $x \in H$ with $\|x\| = 1$ we have

$$\begin{aligned}\varphi_{x,A,B}((1-\lambda)t_1 + \lambda t_2) &= \langle f(A + ((1-\lambda)t_1 + \lambda t_2)\eta(B,A))x, x\rangle \\ &= \langle f\left(A + t_1\eta(B,A) + \lambda\eta(A + t_2\eta(B,A), A + t_1\eta(B,A)\right)x, x\rangle \\ &\leq \lambda\langle f(A + t_2\eta(B,A))x, x\rangle + (1-\lambda)\langle f(A + t_1\eta(B,A))x, x\rangle \\ &= \lambda\varphi_{x,A,B}(t_2) + (1-\lambda)\varphi_{x,A,B}(t_1).\end{aligned} \tag{2.6}$$

Therefore, $\varphi_{x,A,B}$ is convex on $[0,1]$. □

Theorem 3. *Let* $S \subseteq B(H)_{sa}$ *be an invex set with respect to* $\eta : S \times S \to B(H)_{sa}$ *and* η *satisfies condition* C. *If for every* $A, B \in S$ *and* $V = A + \eta(B,A)$ *the function* $f : I \to \mathbb{R}$ *is operator preinvex with respect to* η *on* $\eta-$*path* P_{AV} *with spectra of* A *and spectra of* V *in the interval* I. *Then we have the inequality*

$$\begin{aligned}f\left(\frac{A+V}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3A+V}{4}\right) + f\left(\frac{A+3V}{4}\right)\right] \\ &\leq \int_0^1 f(A + t\eta(B,A))dt \\ &\leq \frac{1}{2}\left[f\left(\frac{A+V}{2}\right) + \frac{f(A)+f(V)}{2}\right] \leq \frac{f(A)+f(B)}{2}.\end{aligned} \tag{2.7}$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$, we have

$$\langle (A + t\eta(B,A))x, x\rangle = \langle Ax, x\rangle + t\langle \eta(B,A)x, x\rangle \in I, \tag{2.8}$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Vx, x\rangle \in Sp(V) \subseteq I$.

Continuity of f and (2.8) imply that the operator valued integral $\int_0^1 f(A + t\eta(B,A))dt$ exists. Since η satisfied condition C, therefore for every $t \in [0,1]$ we have

$$A + \frac{1}{2}\eta(B,A) = A + t\eta(B,A) + \frac{1}{2}\eta(A + (1-t)\eta(B,A), A + t\eta(B,A)). \tag{2.9}$$

Preinvexity f with respect to η implies that

$$\begin{aligned} f(A+\frac{1}{2}\eta(B,A)) &\leq \frac{1}{2}f(A+t\eta(B,A))+\frac{1}{2}f(A+(1-t)\eta(B,A)) \\ &\leq \frac{1}{2}[(1-t)f(A)+tf(B)]+\frac{1}{2}[tf(A)+(1-t)f(B)] \\ &\leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{2.10}$$

Integrating the inequality (2.10) over $t\in[0,1]$ and taking into account that

$$\int_0^1 f(A+t\eta(B,A))dt=\int_0^1 f(A+(1-t)\eta(B,A))dt \tag{2.11}$$

then we deduce the Hermite-Hadamard inequality for operator preinvex functions

$$f\left(\frac{A+(A+\eta(B,A))}{2}\right)\leq\int_0^1 f(A+t\eta(B,A))dt\leq\frac{f(A)+f(B)}{2}.$$

that holds for any selfadjoint operators A and B with the spectra in I. Define the real-valued function $\varphi_{x,A,B}:[0,1]\to\mathbb{R}$ given by $\varphi_{x,A,B}(t)=\langle f(A+t\eta(B,A))x,x\rangle$. Since f is operator preinvex, by the previous proposition 1, $\varphi_{x,A,B}$ is a convex function on $[0,1]$. Utilizing the Hermite-Hadamard inequality for real-valued convex functions

$$\varphi\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\varphi(s)ds\leq\frac{\varphi(a)+\varphi(b)}{2}$$

with $a=0,b=\frac{1}{2}$ we have

$$\left\langle f\left(\frac{3A+V}{4}\right)x,x\right\rangle\leq 2\int_0^{\frac{1}{2}}\varphi_{x,A,B}(t)dt\leq\left\langle\frac{f(A)+f\left(\frac{A+V}{2}\right)}{2}x,x\right\rangle$$

and with $a=\frac{1}{2},b=1$ we have

$$\left\langle f\left(\frac{A+3V}{4}\right)x,x\right\rangle\leq 2\int_{\frac{1}{2}}^1\varphi_{x,A,B}(t)dt\leq\left\langle\frac{f(V)+f\left(\frac{A+V}{2}\right)}{2}x,x\right\rangle$$

which by summation and division by two produces

$$\begin{aligned} \left\langle\frac{1}{2}\left[f\left(\frac{3A+V}{4}\right)+f\left(\frac{A+3V}{4}\right)\right]x,x\right\rangle &\leq\int_0^1\langle f(A+t\eta(B,A))x,x\rangle dt \\ &\leq\left\langle\frac{1}{2}\left[f\left(\frac{A+V}{2}\right)+\frac{f(A)+f(V)}{2}\right]x,x\right\rangle \end{aligned}$$

Finally, from the continuity of the function f we have

$$\int_0^1\langle f(A+t\eta(B,A))x,x\rangle dt=\left\langle\int_0^1 f(A+t\eta(B,A))dtx,x\right\rangle,$$

and the inequality (2.9) implies that

$$f\left(\frac{A+V}{2}\right)\leq\frac{1}{2}\left[f\left(\frac{3A+V}{4}\right)+f\left(\frac{A+3V}{4}\right)\right]\leq\frac{f(A)+f(B)}{2}.$$

Hence we deduce the desired result (2.7).

□

A simple consequence of the above theorem is that the integral is closer to the left bound than to the right, namely we can state:

Corollary 1. *With the assumptions in Theorem 3 we have the inequality*

$$0 \le \int_0^1 f(A + t\eta(B,A))dt - f\left(\frac{A+V}{2}\right) \le \frac{f(A)+f(B)}{2} - \int_0^1 f(A+t\eta(B,A))dt.$$

Example 2. *Let S, f, η_1 be as in Example 1, then we have*

$$\begin{aligned}\left(\frac{A+V}{2}\right)^2 &\le \frac{1}{2}\left[\left(\frac{3A+V}{4}\right)^2 + \left(\frac{A+3V}{4}\right)^2\right] \\ &\le \int_0^1 (A + t\eta_1(B,A))^2 dt \\ &\le \frac{1}{2}\left[\left(\frac{A+V}{2}\right)^2 + \frac{A^2+V^2}{2}\right] \le \frac{A^2+B^2}{2},\end{aligned}$$

for every $A, B \in S$ and $V = A + \eta_1(B,A)$.

The following Theorem is a generalization of Theorem 3.1 in [2].

Theorem 4. *Let the function $f : I \to \mathbb{R}^+$ is continuous, $S \subseteq B(H)_{sa}$ be an open invex set with respect to $\eta : S \times S \to B(H)_{sa}$ and η satisfies condition C. If for every $A, B \in S$ and $V = A + \eta(B,A)$ the function f is operator preinvex with respect to η on $\eta-$path P_{AV} with spectra of A and spectra of V in I. Then, for every $a, b \in (0,1)$ with $a < b$ and every $x \in H$ with $\|x\| = 1$ the following inequality holds,*

$$\begin{aligned}(2.12)\quad &\left|\frac{1}{2}\left\langle \int_0^a f(A+s\eta(B,A))ds\ x, x\right\rangle + \frac{1}{2}\left\langle \int_0^b f(A+s\eta(B,A))ds\ x, x\right\rangle \right. \\ &\left. -\frac{1}{b-a}\int_a^b \left\langle \int_0^t f(A+s\eta(B,A))ds\ x, x\right\rangle dt\right| \\ &\le \frac{b-a}{8}\{\langle f(A+a\eta(B,A))x, x\rangle + \langle f(A+b\eta(B,A))x, x\rangle\}.\end{aligned}$$

Moreover we have

$$\begin{aligned}(2.13)\quad &\left\|\frac{1}{2}\int_0^a f(A+s\eta(B,A))ds + \frac{1}{2}\int_0^b f(A+s\eta(B,A))ds \right. \\ &\left. -\frac{1}{b-a}\int_a^b\int_0^t f(A+s\eta(B,A))dsdt\right\| \\ &\le \frac{b-a}{8}\|f(A+a\eta(B,A)) + f(A+b\eta(B,A))\| \\ &\le \frac{b-a}{8}[\ \|f(A+a\eta(B,A))\| + \|f(A+b\eta(B,A))\|\ ].\end{aligned}$$

Proof. Let $A, B \in S$ and $a, b \in (0,1)$ with $a < b$. For $x \in H$ with $\|x\| = 1$ we define the function $\varphi : [0,1] \to \mathbb{R}^+$ by

$$\varphi(t) := \left\langle \int_0^t f(A+s\eta(B,A))ds\ x, x\right\rangle.$$

Utilizing the continuity of the function f, the continuity property of the inner product and the properties of the integral of operator-valued functions we have

$$\left\langle \int_0^t f(A + s\eta(B,A))ds\ x, x \right\rangle = \int_0^t \langle f(A + s\eta(B,A))\ x, x\rangle\, ds.$$

Since $f(A + s\eta(B,A)) \geq 0$, therefore $\varphi(t) \geq 0$ for all $t \in I$. Obviously for every $t \in (0,1)$ we have

$$\varphi'(t) = \langle f(A + t\eta(B,A))x, x\rangle \geq 0,$$

hence, $|\varphi'(t)| = \varphi'(t)$. Since f is operator preinvex with respect to η on $\eta-$path P_{AV}, by Proposition 1 the function φ' is convex. Applying Theorem 1 to the function φ implies that

$$\left| \frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(s)ds \right| \leq \frac{(b-a)\left(\varphi'(a) + \varphi'(b)\right)}{8},$$

and we deduce that (2.12) holds. Taking supremum over both side of inequality (2.12) for all x with $\|x\| = 1$, we deduce that the inequality (2.13) holds. □

3. Application for operator convex functions

If we consider $\eta(B,A) = B - A$ in Theorem 3 then $f : I \to \mathbb{R}$ will be an operator convex function and $V = B$. Hence we can conclude Theorem 2 as a result of Theorem 3.

As an application of Theorem 4 we state the following Theorem, which is a generalization of Theorem 2.1 in [2].

Theorem 5. *Let $f : I \to \mathbb{R}^+$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $a, b \in (0,1)$ with $a < b$ the following inequality holds,*

$$\begin{aligned}(3.1)\quad &\left| \frac{1}{2}\left\langle \int_0^a f((1-s)A + sB)ds\ x, x \right\rangle + \frac{1}{2}\left\langle \int_0^b f((1-s)A + sB)ds\ x, x \right\rangle \right. \\ &\qquad \left. - \frac{1}{b-a}\int_a^b \left\langle \int_0^t f((1-s)A + sB)ds\ x, x \right\rangle dt \right| \\ &\qquad \leq \frac{b-a}{8}\left[\langle f((1-a)A + aB)x, x\rangle + \langle f((1-b)A + bB)x, x\rangle\right].\end{aligned}$$

Moreover we have

$$\begin{aligned}(3.2)\quad &\left\| \frac{1}{2}\int_0^a f((1-s)A + sB))ds + \frac{1}{2}\int_0^b f((1-s)A + sB))ds \right. \\ &\qquad \left. - \frac{1}{b-a}\int_a^b \int_0^t f((1-s)A + sB))dsdt \right\| \\ &\leq \frac{b-a}{8}\| f((1-a)A + aB)) + f((1-b)A + bB))\| \\ &\qquad \leq \frac{b-a}{8}[\ \| f((1-a)A + aB))\| + \| f((1-b)A + bB))\|\ ].\end{aligned}$$

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