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# SOME REVERSES OF A CAUCHY–SCHWARZ INEQUALITY FOR COMPLEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some reverses of a Cauchy–Schwarz inequality for complex functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given. Several examples for particular functions of interest are provided as well.

## 1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called *positive* if $\langle Ax, x\rangle \geq 0$ holds for every $x \in \mathscr{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. Let $A \in \mathbb{B}(\mathscr{H})$ be selfadjoint. The *continuous functional calculus* $f \mapsto f(A)$ establishes a $*$-isometrically isomorphism $\Phi$ between the $C^*$-algebra $C(\mathrm{sp}(A))$ of all continuous complex-valued functions defined on the spectrum $\mathrm{sp}(A)$ of $A$ and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_{\mathscr{H}}$ on $\mathscr{H}$ (see [13]). If $f$ and $g$ are real valued functions on $\mathrm{sp}(A)$ then the following property holds:

$$f(A) \leq g(A) \Leftrightarrow f(t) \leq g(t) \;\; (t \in \mathrm{sp}(A)).$$

For recent results on various inequalities for functions of selfadjoint operators, see [11, 4, 3, 5] and the references therein.

Let $A \in \mathbb{B}(\mathscr{H})$ be selfadjoint with the spectrum included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of a Riemann–Stieltjes integral*:

$$f(A) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(A)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{1.2}$$

for any $x, y \in \mathscr{H}$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of bounded variation on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

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for any $x, y \in \mathscr{H}$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is monotonic nondecreasing and right continuous on $[m, M]$.

The classical Cauchy–Schwarz inequality asserts that if $x$ and $y$ are elements of a semi-inner product space, then $|\langle x, y\rangle| \le \|x\|\,\|y\|$. There are mainly two types of reverse Cauchy–Schwarz inequality. In the additive approach (initiated by Ozeki [14]) we look for an inequality of the form $\kappa + |\langle x, y\rangle| \ge \|x\|\,\|y\|$ for some suitable positive constant $\kappa$. In the multiplicative approach (initiated by Polya and Szegö [15]) we seek for an appropriate positive constant $\kappa$ such that $|\langle x, y\rangle| \ge \kappa\|x\|\,\|y\|$. There are many generalizations and applications of the Cauchy–Schwarz inequality and its reverse for integrals, weighted sums and isotone functionals; see the monograph [2]. Moreover, some reverse Cauchy–Schwarz inequalities for Hilbert space operators were presented in [9, 10, 8, 12, 1].

In order to provide upper bounds for the nonnegative quantity $\|f(A)x\|^2 - \langle f(A)x, x\rangle^2$, where $x \in \mathscr{H}$ with $\|x\| = 1$ and $f : [m, M] \to \mathbb{R}$ is a continuous real-valued function defined on $[m, M]$ which contains the spectrum of the selfadjoint operator $A$ the first author obtained in [7] the following result:

**Theorem 1** (Dragomir, 2008)**.** *Let $A \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator with $\operatorname{sp}(A) \subseteq [m, M]$ for some scalars $m < M$. If $f : [m, M] \to \mathbb{R}$ is continuous on $[m, M]$ and $\delta := \min_{t\in[m,M]} f(t)$, $\Delta := \max_{t\in[m,M]} f(t)$, then*

$$\begin{aligned} 0 &\le \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \qquad (1.3)\\ &\le \frac{1}{4}\cdot(\Delta-\delta)^2 - \begin{cases} [\langle \Delta x - f(A)x, f(A)x - \delta x\rangle], \\ \left|\langle f(A)x, x\rangle - \frac{\Delta+\delta}{2}\right|^2 \end{cases}\\ &\le \frac{1}{4}\cdot(\Delta-\delta)^2 \end{aligned}$$

*for any $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Moreover if $\delta$ is positive, then we also have*

$$0 \le \|f(A)x\|^2 - \langle f(A)x, x\rangle^2 \le \begin{cases} \frac{1}{4}\cdot\frac{(\Delta-\delta)^2}{\Delta\delta}\langle f(A)x, x\rangle^2, \\ \left(\sqrt{\Delta}-\sqrt{\delta}\right)^2\langle f(A)x, x\rangle. \end{cases} \qquad (1.4)$$

*for any $x \in \mathscr{H}$ with $\|x\| = 1$.*

**Remark 1.** *We notice that the first inequality in (1.4) is equivalent with*

$$\|f(A)x\| \le \frac{\Delta+\delta}{2\sqrt{\Delta\delta}}\langle f(A)x, x\rangle \qquad (1.5)$$

*while the second inequality is equivalent with*

$$0 \le \frac{\|f(A)x\|^2}{\langle f(A)x, x\rangle} - \langle f(A)x, x\rangle \le \left(\sqrt{\Delta}-\sqrt{\delta}\right)^2 \qquad (1.6)$$

*for any $x \in \mathscr{H}$ with $\|x\| = 1$.*

Motivated by the above results we investigate in the current paper the problem of finding revrse of $|\langle f(A)x,x\rangle| \leq \|f(A)x\|$, where $x$ is a unit vector of a complex Hilbert space $(\mathscr{H};\langle\cdot,\cdot\rangle)$ for different classes of continuous complex valued functions $f:[m,M]\to\mathbb{C}$ and selfadjoint operators $A\in\mathbb{B}(\mathscr{H})$ with $\operatorname{sp}(A)\subseteq[m,M]$ for some scalars $m<M$. Some applications are also presented.

## 2. The results

The following result holds:

**Theorem 2.** *If $f:[m,M]\to\mathbb{C}$ is continuous on $[m,M]$ and $a\in\mathbb{C}$, $r>0$ with $|a|>r$ and such that*

$$|f(t)-a|\leq r \text{ for any } t\in[m,M], \tag{2.1}$$

*then for any selfadjoint operator $A\in\mathbb{B}(\mathscr{H})$ with $\operatorname{sp}(A)\subseteq[m,M]$,*

$$\begin{aligned}\langle|f(A)|x,x\rangle &\leq \|f(A)x\| \\ &\leq \frac{\operatorname{Re}(a)\operatorname{Re}\langle f(A)x,x\rangle+\operatorname{Im}(a)\operatorname{Im}\langle f(A)x,x\rangle}{\sqrt{|a|^2-r^2}} \\ &\leq \frac{|\langle f(A)x,x\rangle|}{\sqrt{1-\frac{r^2}{|a|^2}}}\end{aligned} \tag{2.2}$$

*for any $x\in\mathscr{H}$ with $\|x\|=1$.*

*Proof.* We have from (2.7) that

$$|f(t)|^2-2\operatorname{Re}[f(t)\bar{a}]+|a|^2\leq r^2 \tag{2.3}$$

for any $t\in[m,M]$.

Let $\{E_t\}_t$ be the spectral family for the operator $A$. For a given $x\in\mathscr{H}$ with $\|x\|=1$, the function $g_x(t):=\langle E_tx,x\rangle$ is monotonic nondecreasing and integrating inequality (2.3) over $g_x$ on the interval $[m-\varepsilon,M]$ with $\varepsilon>0$ and then taking the limit for $\varepsilon\to0^+$, we get

$$\begin{aligned}\int_{m-0}^{M}|f(t)|^2\,d\langle E_tx,x\rangle-2\operatorname{Re}\left[\bar{a}\int_{m-0}^{M}f(t)\,d\langle E_tx,x\rangle\right]& \\ +|a|^2\int_{m-0}^{M}d\langle E_tx,x\rangle\leq r^2\int_{m-0}^{M}d\langle E_tx,x\rangle&\end{aligned} \tag{2.4}$$

for any $x\in\mathscr{H}$ with $\|x\|=1$.

Since by the spectral representation theorem (1.2) we have

$$\int_{m-0}^{M}|f(t)|^2\,d\langle E_tx,x\rangle=\|f(A)x\|^2,$$

$$\int_{m-0}^{M}f(t)\,d\langle E_tx,x\rangle=\langle f(A)x,x\rangle$$

and $\int_{m-0}^{M} d\langle E_t x, x\rangle = 1$ for any $x \in \mathscr{H}$ with $\|x\| = 1$, we have, by (2.4), that

$$\|f(A)x\|^2 + \left(\sqrt{|a|^2 - r^2}\right)^2 \leq 2\operatorname{Re}\left(\langle f(A)x, x\rangle \bar{a}\right) = 2\left[\operatorname{Re}(a)\operatorname{Re}\langle f(A)x, x\rangle + \operatorname{Im}(a)\operatorname{Im}\langle f(A)x, x\rangle\right] \quad (2.5)$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Utilizing the elementary inequality $\alpha^2 + \beta^2 \geq 2\alpha\beta$ if $\alpha, \beta \in \mathbb{R}$ we obtain

$$2\|f(A)x\|\sqrt{|a|^2 - r^2} \leq \|f(A)x\|^2 + \left(\sqrt{|a|^2 - r^2}\right)^2 \quad (2.6)$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

On making use of (2.5) and (2.6) we get the second inequality in (2.2).

Since $\operatorname{Re}\left(\langle f(A)x, x\rangle \bar{a}\right) \leq |\langle f(A)x, x\rangle \bar{a}| = |\langle f(A)x, x\rangle|\,|a|$, it follows from (2.5) that the third inequality in (2.2) also holds.

The first inequality is a consequence of the Hölder–McCarthy inequality $\langle Ax, x\rangle^2 \leq \langle A^2x, x\rangle$ $(\|x\| = 1)$. We, however, present a direct proof. By the Cauchy–Bunyakovsky–Schwarz integral inequality for the Riemann–Stieltjes integral with monotonic nondecreasing integrators we also have that

$$\left[\int_{m-0}^{M} |f(t)|\, d\langle E_t x, x\rangle\right]^2 \leq \int_{m-0}^{M} |f(t)|^2\, d\langle E_t x, x\rangle \int_{m-0}^{M} d\langle E_t x, x\rangle$$

which means that

$$\langle |f(A)|\, x, x\rangle^2 \leq \|f(A)x\|^2$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$, and thus we get the first inequality in (2.2). ∎

**Remark 2.** *If $\delta := \min_{t\in[m,M]} f(t)$ and $\Delta := \max_{t\in[m,M]} f(t)$, then the condition (2.7) holds with $a = \frac{\delta+\Delta}{2}$ and $r = \frac{\Delta-\delta}{2}$.*

**Corollary 1.** *Let $f : [m, M] \to \mathbb{C}$ be continuous on $[m, M]$, $a \in \mathbb{C}$, $r > 0$ with $|a| > r$ and*

$$|f(t) - a| \leq r \text{ for any } t \in [m, M], \quad (2.7)$$

*Then for any selfadjoint operators $A_j \in \mathbb{B}(\mathscr{H})$ with $\operatorname{sp}(A_j) \subseteq [m, M]$; $j \in \{1, \cdots, n\}$,*

$$\begin{aligned}\sum_{j=1}^{n} \langle |f(A_j)|\, x_j, x_j\rangle &\leq \|(f(A_1)x_1, \ldots, f(A_n)x_n)\| \\ &\leq \sum_{j=1}^{n} \frac{\operatorname{Re}(a)\operatorname{Re}\langle f(A_j)x_j, x_j\rangle + \operatorname{Im}(a)\operatorname{Im}\langle f(A_j)x_j, x_j\rangle}{\sqrt{|a|^2 - r^2}} \\ &\leq \frac{\left|\left\langle \sum_{j=1}^{n} f(A_j)x_j, x_j\right\rangle\right|}{\sqrt{1 - \frac{r^2}{|a|^2}}}\end{aligned}$$

*for each $x_j \in \mathscr{H}$; $j \in \{1, \cdots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.*

*Proof.* On considering

$$\widetilde{x}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathscr{H}^n, \quad \widetilde{A}=\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix} \in \mathbb{B}(\mathscr{H}^n).$$

and applying Theorem 2 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired results. ∎

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f:[a,b]\to\mathbb{C} \mid \operatorname{Re}\left[ (\Gamma - f(t)) \left( \overline{f(t)} - \bar{\gamma} \right) \right] \geq 0 \text{ for each } t\in[a,b] \right\} \tag{2.8}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f:[a,b]\to\mathbb{C} \mid \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma-\gamma| \text{ for each } t\in[a,b] \right\}. \tag{2.9}$$

The following representation result may be stated.

**Proposition 1.** *Let $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$. Then $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{2.10}$$

*Proof.* We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[ (\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2 - \left| z - \frac{\gamma+\Gamma}{2} \right|^2 = \operatorname{Re}\left[ (\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z\in\mathbb{C}$.

The equality (2.10) is thus a simple consequence of this fact. ∎

On making an application of the properties of the complex numbers we can also state that:

**Corollary 2.** *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, it holds that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma,\Gamma) = \{ f:[a,b]\to\mathbb{C} \mid\ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for each } t\in[a,b] \}. \end{aligned} \tag{2.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{ f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b] \}. \tag{2.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma).$$

Making use of the classes of functions defined above we can provide a generalization of inequality (1.5) as follows:

**Corollary 3.** *Let $A \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator with $\operatorname{sp}(A) \subseteq [m,M]$ for some scalars $m < M$. If $f : [m,M] \to \mathbb{C}$ is continuous on $[m,M]$ and there exist two numbers $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, with $\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma) > 0$ and such that $f \in \bar{U}_{[m,M]}(\gamma,\Gamma)\left(= \bar{\Delta}_{[m,M]}(\gamma,\Gamma)\right)$, then*

$$\begin{aligned} &\langle |f(A)|\, x, x\rangle \\ &\leq \|f(A)\, x\| \\ &\leq \frac{[\operatorname{Re}(\gamma) + \operatorname{Re}(\Gamma)]\operatorname{Re}\langle f(A)\, x, x\rangle + [\operatorname{Im}(\gamma) + \operatorname{Im}(\Gamma)]\operatorname{Im}\langle f(A)\, x, x\rangle}{4\sqrt{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)}} \\ &\leq \frac{|\gamma+\Gamma|\, |\langle f(A)\, x, x\rangle|}{4\sqrt{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)}} \end{aligned} \tag{2.13}$$

*for any $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Proof.* We apply Theorem 2 for $a = \frac{\gamma+\Gamma}{2}$ and $r = \frac{1}{2}|\Gamma - \gamma|$.

We observe that

$$\begin{aligned} |a|^2 - r^2 &= \left|\frac{\gamma+\Gamma}{2}\right|^2 - \left|\frac{\Gamma-\gamma}{2}\right|^2 = 4\operatorname{Re}(\Gamma\bar{\gamma}) \\ &= 4[\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)] > 0 \end{aligned}$$

and by (2.2) we get the desired result (2.13). ∎

**Remark 3.** *If $f : [m,M] \to (0,\infty)$ is continuous on $[m,M]$ and $0 < \delta := \min_{t\in[m,M]} f(t)$, $\Delta := \max_{t\in[m,M]} f(t)$, then $f \in \bar{S}_{[a,b]}(\delta,\Delta) \subseteq \bar{U}_{[a,b]}(\delta,\Delta)$ and by (2.13) we get inequality (1.5).*

The following result, where the condition $|a| > r$ from Theorem 2 is dropped may be stated as well:

**Theorem 3.** *If $f : [m,M] \to \mathbb{C}$ is continuous on $[m,M]$ and $a \in \mathbb{C}\setminus\{0\}$, $r > 0$ such that*

$$|f(t) - a| \leq r \text{ for any } t \in [m,M],$$

*then for any selfadjoint operator* $A \in \mathbb{B}(\mathscr{H})$ *with* $\operatorname{sp}(A) \subseteq [m, M]$,

$$\begin{aligned}\langle |f(A)| x, x\rangle &\le \|f(A) x\| \\ &\le \frac{\operatorname{Re}(a)\operatorname{Re}\langle f(A)x,x\rangle + \operatorname{Im}(a)\operatorname{Im}\langle f(A)x,x\rangle}{|a|} + \frac{r^2}{2|a|} \\ &\le |\langle f(A)x,x\rangle| + \frac{r^2}{2|a|}\end{aligned} \tag{2.14}$$

*for any* $x \in \mathscr{H}$ *with* $\|x\| = 1$.

*Proof.* As in the proof of Theorem 2 we can state that

$$\begin{aligned}\|f(A)x\|^2 + |a|^2 &\le 2\operatorname{Re}(\langle f(A)x,x\rangle \bar{a}) + r^2 \\ &= 2[\operatorname{Re}(a)\operatorname{Re}\langle f(A)x,x\rangle + \operatorname{Im}(a)\operatorname{Im}\langle f(A)x,x\rangle] + r^2\end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Since

$$2\|f(A)x\|\,|a| \le \|f(A)x\|^2 + |a|^2$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$, then we get

$$2\|f(A)x\|\,|a| \le 2[\operatorname{Re}(a)\operatorname{Re}\langle f(A)x,x\rangle + \operatorname{Im}(a)\operatorname{Im}\langle f(A)x,x\rangle] + r^2$$

which proves the second inequality in (2.14).

The rest is obvious and the theorem is proved. ∎

**Corollary 4.** *Assume that* $A \in \mathbb{B}(\mathscr{H})$ *is a selfadjoint operator with* $\operatorname{sp}(A) \subseteq [m, M]$ *for some scalars* $m < M$. *If* $f : [m, M] \to \mathbb{C}$ *is continuous on* $[m, M]$ *and there exists two numbers* $\gamma, \Gamma \in \mathbb{C}$ *with* $\gamma \neq \pm\Gamma$ *such that* $f \in \bar{U}_{[m,M]}(\gamma, \Gamma)\ \left(= \bar{\Delta}_{[m,M]}(\gamma, \Gamma)\right)$, *then*

$$\begin{aligned}&\langle |f(A)| x, x\rangle \\ &\le \|f(A)x\| \\ &\le \frac{[\operatorname{Re}(\gamma) + \operatorname{Re}(\Gamma)]\operatorname{Re}\langle f(A)x,x\rangle + [\operatorname{Im}(\gamma) + \operatorname{Im}(\Gamma)]\operatorname{Im}\langle f(A)x,x\rangle}{|\gamma + \Gamma|} \\ &+ \frac{|\Gamma - \gamma|^2}{4|\gamma + \Gamma|} \\ &\le |\langle f(A)x,x\rangle| + \frac{|\Gamma - \gamma|^2}{4|\gamma + \Gamma|}\end{aligned} \tag{2.15}$$

*for any* $x \in \mathscr{H}$ *with* $\|x\| = 1$.

**Remark 4.** *If* $f : [m, M] \to \mathbb{R}$ *is continuous on* $[m, M]$ *and* $\delta := \min_{t \in [m,M]} f(t)$, $\Delta := \max_{t \in [m,M]} f(t)$ *with* $\Delta \neq -\delta$ *then* $f \in \bar{S}_{[a,b]}(\delta, \Delta) \subseteq \bar{U}_{[a,b]}(\delta, \Delta)$ *and by (2.15) we get*

$$\|f(A)x\| \le |\langle f(A)x,x\rangle| + \frac{(\Delta - \delta)^2}{4|\Delta + \delta|} \tag{2.16}$$

*for any* $x \in \mathscr{H}$ *with* $\|x\| = 1$.

## 3. Applications

The above results can be applied to various continuous complex valued functions. However, here we will present only a simple example as follows:

**Proposition 2.** *Let $A \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator with the property that there exists the real numbers $m, M$ with $[m, M] \subseteq \left[0, \frac{\pi}{2}\right]$ and*

$$m1_{\mathscr{H}} \leq A \leq M1_{\mathscr{H}}. \tag{3.1}$$

*Then*

$$\begin{aligned} 1 &\leq \left\|e^{iA}x\right\| \\ &\leq \frac{\cos\left(\frac{M-m}{2}\right)}{2\sqrt{\cos\left(M-m\right)}} \\ &\quad\times\left[\cos\left(\frac{M+m}{2}\right)\langle\cos Ax, x\rangle+\sin\left(\frac{M+m}{2}\right)\langle\sin Ax, x\rangle\right] \\ &\leq \frac{\cos\left(\frac{M-m}{2}\right)}{2\sqrt{\cos\left(M-m\right)}}\left[\langle\cos Ax, x\rangle^2+\langle\sin Ax, x\rangle^2\right]^{1/2}, \end{aligned} \tag{3.2}$$

*for any $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Proof.* Consider the function $f : [m, M] \to \mathbb{C}$ given by $f(t) = e^{it} = \cos t + i \sin t$. Then $\operatorname{Re} f(t) = \cos t, \operatorname{Im} f(t) = \sin t$ and for $t \in [m, M]$ we have $\operatorname{Re} f(t) \in [\cos M, \cos m]$ and $\operatorname{Im} f(t) \in [\sin m, \sin M]$ and if we define $\Gamma = \cos m + i \sin M$ and $\gamma = \cos M + i \sin m$ then we have

$$\cos M = \operatorname{Re}\gamma \leq \operatorname{Re} f(t) \leq \operatorname{Re}\Gamma = \cos m$$

and

$$\sin m = \operatorname{Im}\gamma \leq \operatorname{Im} f(t) \leq \operatorname{Im}\Gamma = \sin M.$$

Now, if we write inequality (2.13) for $f(t) = e^{it}$ and $\Gamma, \gamma$ as above, we have

$$\begin{aligned} 1 &\leq \left\|e^{iA}x\right\| \\ &\leq \frac{(\cos M+\cos m)\langle\cos Ax, x\rangle+(\sin m+\sin M)\langle\sin Ax, x\rangle}{4\sqrt{\cos m\cos M+\sin M\sin m}} \\ &\leq \frac{|\cos M+i\sin m+\cos m+i\sin M|\,|\langle f(A)x, x\rangle|}{4\sqrt{\cos m\cos M+\sin M\sin m}} \end{aligned} \tag{3.3}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

We have

$$\begin{aligned} & |\cos M+i\sin m+\cos m+i\sin M|^2 \\ =\ & (\cos M+\cos m)^2+(\sin m+\sin M)^2 \\ =\ & 2+2(\cos m\cos M+\sin m\sin M) \\ =\ & 2+2\cos(M-m)=4\cos^2\left(\frac{M-m}{2}\right) \end{aligned}$$

and by (3.3) we get (3.2). ∎

The following result also holds:

**Proposition 3.** *With the assumptions as in Proposition 2,*

$$\begin{aligned}
1 &\leq \left\|e^{iA}x\right\| \\
&\leq \cos\left(\frac{M+m}{2}\right)\langle \cos Ax, x\rangle + \sin\left(\frac{M+m}{2}\right)\langle \sin Ax, x\rangle \\
&+ \frac{1}{2}\tan\left(\frac{M-m}{2}\right)\sin\left(\frac{M-m}{2}\right) \\
&\leq \left[\langle \cos Ax, x\rangle^2 + \langle \sin Ax, x\rangle^2\right]^{1/2} \\
&+ \frac{1}{2}\tan\left(\frac{M-m}{2}\right)\sin\left(\frac{M-m}{2}\right),
\end{aligned} \tag{3.4}$$

*for any $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Proof.* We write inequality (2.15) for for $f(t) = e^{it}$ and $\Gamma = \cos m + i\sin M$ and $\gamma = \cos M + i\sin m$ to get

$$\begin{aligned}
1 &\leq \left\|e^{iA}x\right\| \\
&\leq \frac{(\cos M + \cos m)\langle \cos Ax, x\rangle + (\sin m + \sin M)\langle \sin Ax, x\rangle}{2\cos\left(\frac{M-m}{2}\right)} \\
&+ \frac{|\cos M + i\sin m - (\cos m + i\sin M)|^2}{8\cos\left(\frac{M-m}{2}\right)} \\
&\leq \left[\langle \cos Ax, x\rangle^2 + \langle \sin Ax, x\rangle^2\right]^{1/2} \\
&+ \frac{|\cos M + i\sin m - (\cos m + i\sin M)|^2}{8\cos\left(\frac{M-m}{2}\right)}
\end{aligned} \tag{3.5}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

It follows from

$$\begin{aligned}
& |\cos M + i\sin m - (\cos m + i\sin M)|^2 \\
= \;& (\cos M - \cos m)^2 + (\sin M - \sin m)^2 \\
= \;& 2 - (\cos m\cos M + \sin m\sin M) \\
= \;& 2 - 2\cos(M-m) = 4\sin^2\left(\frac{M-m}{2}\right),
\end{aligned}$$

and (3.5) that the desired result (3.4) holds. ∎

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