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SOME NEW BOUNDS FOR TWO MAPPINGS RELATED TO THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. Some new results concerning two mappings associated to the celebrated Hermite-Hadamard integral inequality for convex function with applications for special means are given.

1. INTRODUCTION

The Hermite-Hadamard integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

In order to provide various refinements of this result, the first author introduced in 1991, see [2], the following associated mapping $H:[0,1]\to\mathbb{R}$ defined by

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

for a given convex function $f:[a,b]\to\mathbb{R}$.

Some of the main properties of H are as follows (see also [2], [3], [4] and [8]):

(1) H is convex on $[0,1]$;
(2) One has the bounds:

$$\inf_{t\in[0,1]}H(t)=H(0)=f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]}H(t)=H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx;$$

(3) H increases monotonically on $[0,1]$;

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(4) The following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \leq \int_0^1 H(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right]. \tag{1.1}$$

The corresponding double integral mapping in connection with the Hermite-Hadamard inequalities was considered first in [3] and is defined as

$$F:[0,1]\to\mathbb{R}, F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.$$

Some of the main results concerning this mapping [3] (see also [4]) are as follows:

(1) $F\left(\tau+\frac{1}{2}\right) = F\left(\frac{1}{2}-\tau\right)$ for all $\tau \in \left[0,\frac{1}{2}\right]$ and $F(t) = F(1-t)$ for all $t \in [0,1]$;
(2) F is convex on $[0,1]$;
(3) We have the bounds:

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy;$$

(4) The following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq F\left(\frac{1}{2}\right);$$

(5) F decreases monotonically on $\left[0,\frac{1}{2}\right]$ and increases monotonically on $\left[\frac{1}{2},1\right]$;
(6) We have the inequality:

$$H(t) \leq F(t) \text{ for all } t\in[0,1].$$

For other related results, see for instance the research papers [1], [10], [11], [12], [14], [13], [15], [16], [17], the monograph online [9] and the references therein.

In the recent paper [7] we proved the following result where upper and lower bounds for the associated functions

$$\frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)f\left(\frac{a+b}{2}\right) - H(t)$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx - F(t)$$

with $t\in[0,1]$, have been given.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$
\begin{aligned}
(1.2)\qquad & 0 \le 2\min\{t,1-t\} \\
& \times \left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\right] - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right] \\
& \le \frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)\,f\left(\frac{a+b}{2}\right) - H(t) \\
& \le 2\max\{t,1-t\} \\
& \times \left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\right] - \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(1.3)\qquad & 0 \le 2\min\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx - F\left(\frac{1}{2}\right)\right] \\
& \le \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \\
& \le 2\max\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx - F\left(\frac{1}{2}\right)\right],
\end{aligned}
$$

for any $t\in[0,1]$.

Motivated by the above results we establish in this paper some new bounds involving these two mappings. Applications for special means are also provided.

2. The Results

The following result holds:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$
\begin{aligned}
(2.1)\qquad & \frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)\,f\left(\frac{a+b}{2}\right) - H(t) \\
& \le t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(2.2)\qquad & \frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \\
& \le 2t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]
\end{aligned}
$$

for any $t\in[0,1]$.

Proof. Since the class of convex differentiable functions is dense in the uniform topology in the class of all convex functions defined on the interval $[a,b]$, we can assume that f is differentiable on (a,b).

Utilising the convexity of the function we can write the gradient inequality

$$f(u) - f(v) \geq f'(v)(u - v) \tag{2.3}$$

for any $u, v \in (a, b)$.

On making use of (2.3) we have

$$f(tx + (1-t)y) - f(x) \geq (1-t) f'(x)(y - x) \tag{2.4}$$

and

$$f(tx + (1-t)y) - f(y) \geq -t f'(y)(y - x) \tag{2.5}$$

for any $x, y \in (a, b)$ and $t \in (0, 1)$.

Now, if we multiply (2.4) by t and (2.5) by $1 - t$, with $t \in (0, 1)$, and add the obtained inequalities, we get

$$\begin{aligned} & f(tx + (1-t)y) - tf(x) - (1-t)f(y) \\ & \geq t(1-t)[f'(x) - f'(y)](y - x) \end{aligned}$$

which is equivalent with

$$\begin{aligned} & tf(x) + (1-t)f(y) - f(tx + (1-t)y) \\ & \leq t(1-t)[f'(y) - f'(x)](y - x) \end{aligned} \tag{2.6}$$

for any $x, y \in (a, b)$ and $t \in (0, 1)$.

If we choose $y = \frac{a+b}{2}$ in (2.6) and integrate over x on $[a, b]$ then we get

$$\begin{aligned} & t\int_a^b f(x)\,dx + (1-t) f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx \\ & \leq t(1-t)\int_a^b \left[f'(x) - f'\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right)dx \end{aligned} \tag{2.7}$$

which holds for any $t \in [0, 1]$ (we notice that for either $t = 0$ or $t = 1$, the inequality reduces to the equality $0 = 0$).

Now, observe that

$$\begin{aligned} & \int_a^b \left[f'(x) - f'\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right)dx \\ & = \int_a^b f'(x)\left(x - \frac{a+b}{2}\right)dx \\ & = f(x)\left(x - \frac{a+b}{2}\right)\Big|_a^b - \int_a^b f(x)\,dx \\ & = \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(x)\,dx, \end{aligned}$$

which by (2.7) produces the desired result (2.1).

Further, if we integrate the inequality (2.6) over x and y on $[a,b]$ we have

$$\begin{aligned} &t\,(b-a)\int_a^b f(x)\,dx + (1-t)\,(b-a)\int_a^b f(y)\,dy \\ &-\int_a^b\int_a^b f(tx+(1-t)\,y)\,dxdy \\ &\le t\,(1-t)\int_a^b\int_a^b \left[f'(y)-f'(x)\right](y-x)\,dxdy \end{aligned} \tag{2.8}$$

for any $t\in[0,1]$.

Observe that

$$\begin{aligned} &\int_a^b\int_a^b \left[f'(y)-f'(x)\right](y-x)\,dxdy \\ &=(b-a)\int_a^b f'(x)\,xdx+(b-a)\int_a^b f'(y)\,ydy \\ &-\int_a^b f'(x)\,dx\int_a^b ydy-\int_a^b xdx\int_a^b f'(y)\,dy \\ &=2\left[(b-a)\int_a^b f'(x)\,xdx-\frac{b^2-a^2}{2}\left(f(b)-f(a)\right)\right] \\ &=2\,(b-a)\int_a^b f'(x)\left(x-\frac{a+b}{2}\right)dx \\ &=2\,(b-a)\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right] \end{aligned}$$

and by (2.8) we get the desired result (2.2). □

Remark 1. *By replacing t with $1-t$ in (2.1), adding the obtained results and dividing by 2 we get the symmetric inequality*

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{H(t)+H(1-t)}{2} \\ &\le t\,(1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned} \tag{2.9}$$

for any $t\in[0,1]$.

Since it is known that for convex functions whose lateral derivatives $f'_+(a)$ and $f'_-(b)$ are finite we have the inequality

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a)$$

with the sharp constant $\frac{1}{8}$ (see [5]), then we can obtain the simpler, however coarser upper bounds as follows:

Corollary 1. *With the above assumptions and if the lateral derivatives* $f'_{+}(a)$ *and* $f'_{-}(b)$ *are finite, then we have*

$$\frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)\,f\left(\frac{a+b}{2}\right)-H(t) \leq \frac{1}{8}t(1-t)\left[f'_{-}(b)-f'_{+}(a)\right](b-a) \tag{2.10}$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx-F(t) \leq \frac{1}{8}t(1-t)\left[f'_{-}(b)-f'_{+}(a)\right](b-a) \tag{2.11}$$

for any $t\in[0,1]$.

Corollary 2. *With the above assumptions, we have*

$$\begin{aligned}&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-H\left(\frac{1}{2}\right)\\ &\leq \frac{1}{4}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]\\ &\leq \frac{1}{32}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]\\ &\leq \frac{1}{16}\left[f'_{-}(b)-f'_{+}(a)\right](b-a).\end{aligned} \tag{2.13}$$

Remark 2. *We observe that the first inequality in (2.12) is equivalent with*

$$\frac{3}{4}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{2}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\leq H\left(\frac{1}{2}\right)$$

while the first inequality in (2.13) is equivalent with

$$\frac{3}{2}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{4}\leq F\left(\frac{1}{2}\right).$$

3. Applications for L_p-means

Let us consider the convex mapping $f:(0,\infty)\to\mathbb{R},\ f(x)=x^p,\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the mapping

$$H_p(t):=\frac{1}{b-a}\int_a^b (tx+(1-t)\,A(a,b))^p\,dx,\ t\in[0,1].$$

It is obvious that $H_p(0) = A^p(a,b)$, $H_p(1) = L_p^p(a,b)$ where, we recall that $A(a,b) = \frac{a+b}{2}$,

$$L_p^p(a,b) := \frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a},\ p \in (-\infty,0) \cup [1,\infty) \setminus \{-1\}$$

and for $t \in (0,1)$ we have

$$\begin{aligned} H_p(t) &= \frac{1}{[tb+(1-t)A(a,b)]-[ta+(1-t)A(a,b)]}\int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} y^p dy \\ &= L_p^p(ta+(1-t)A(a,b), tb+(1-t)A(a,b)). \end{aligned} \tag{3.1}$$

Now, consider the function

$$F_p(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b (tx+(1-t)y)^p\, dxdy.$$

We observe that $F_p(1) = F_p(0) = L_p^p(a,b)$ and for $t \in (0,1)$ we have

$$\begin{aligned} F_p(t) &= \frac{1}{b-a}\int_a^b \left(\frac{1}{b-a}\int_a^b (tx+(1-t)y)^p\, dx\right) dy \\ &= \frac{1}{b-a}\int_a^b \left(\frac{1}{[tb+(1-t)y]-[ta+(1-t)y]}\int_{ta+(1-t)y}^{tb+(1-t)y} s^p ds\right) dy \\ &= \frac{1}{b-a}\int_a^b L_p^p(ta+(1-t)y, tb+(1-t)y)\, dy. \end{aligned} \tag{3.2}$$

We can calculate the double integral

$$\begin{aligned} F_p\left(\frac{1}{2}\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{x+y}{2}\right)^p dxdy \\ &= \begin{cases} \frac{4}{(b-a)^2(p+1)(p+2)}\left[b^{p+2} - 2\left(\frac{b+a}{2}\right)^{p+2} + a^{p+2}\right] & p \neq -2, \\ \frac{8}{(b-a)^2}\ln\left(\frac{A(a,b)}{G(a,b)}\right) & p = -2 \end{cases} \end{aligned}$$

for $p \neq -1$, where $G(a,b)$ denotes the geometric mean of a, b (see [7]).

We can state the following result:

Proposition 1. *We have the following inequalities:*

$$\begin{aligned} &tL_p^p(a,b) + (1-t)A^p(a,b) - H_p(t) \\ &\le t(1-t)\left[A(a^p,b^p) - L_p^p(a,b)\right] \end{aligned} \tag{3.3}$$

and

$$L_p^p(a,b) - F_p(t) \le 2t(1-t)\left[A(a^p,b^p) - L_p^p(a,b)\right] \tag{3.4}$$

for any $t \in [0,1]$.

In particular, for $t = \frac{1}{2}$ we get

$$\begin{aligned} &A\left(L_p^p(a,b), A^p(a,b)\right) - L_p^p(A(a,A(a,b)), A(A(a,b),b)) \\ &\le \frac{1}{4}\left[A(a^p,b^p) - L_p^p(a,b)\right] \end{aligned} \tag{3.5}$$

$$L_p^p(a,b) - F_p\left(\frac{1}{2}\right) \leq \frac{1}{2}\left[A(a^p,b^p) - L_p^p(a,b)\right]. \tag{3.6}$$

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