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A GENERALIZATION OF WEIGHTED COMPANION OF OSTROWSKI INTEGRAL INEQUALITY FOR MAPPINGS OF BOUNDED VARIATION

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ABSTRACT. A weighted companion of Ostrowski type inequality is established. Some sharp inequalities are proved. Application to a quadrature rule is provided.

1. INTRODUCTION

In 1938, A. Ostrowski [1], proved the following inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \le M$, then the following inequality,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In [2], Dragomir proved the following Ostrowski's inequality for mappings of bounded variation

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \cdot \bigvee_a^b (f), \tag{1.2}$$

for any $x \in [a,b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $\frac{1}{2}$ is best possible.

A generalization of the above result is considered in [3]. In [4], Dragomir et al. have proved the following generalization of Ostrowski's inequality.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$, differentiable on (a,b) and whose derivative f' is bounded on (a,b). Denote $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.*

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Then,

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f(x)\right]-\int_a^b f(t)\,dt\right| \leq \left[\frac{(b-a)^2}{4}\left(\lambda^2+(1-\lambda)^2\right)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty . \tag{1.3}$$

for all $\lambda\in[0,1]$ *and* $a+\lambda\frac{b-a}{2}\leq x\leq b-\lambda\frac{b-a}{2}$.

A generalization of (1.3) was considered in [5]. In [6], Tseng et al. have proved the following weighted Ostrowski inequality for mappings of bounded variation:

Theorem 4. *Let* $0\leq\alpha\leq1$, $g:[a,b]\to[0,\infty)$ *continuous and positive on* (a,b) *and let* $h:[a,b]\to\mathbb{R}$ *be differentiable such that* $h'(t)=g(t)$ *on* $[a,b]$. *Let* $c=h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$ *and* $d=h^{-1}\left(\frac{\alpha}{2}h(a)+\left(1-\frac{\alpha}{2}\right)h(b)\right)$. *Suppose that* f *is of bounded variation on* $[a,b]$, *then for all* $x\in[c,d]$, *we have*

$$\left|\int_a^b f(t)g(t)\,dt-\left[(1-\alpha)f(x)+\alpha\frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt\right|\leq K\cdot\bigvee_a^b(f) \tag{1.4}$$

where,

$$K=\begin{cases}\frac{1-\alpha}{2}\int_a^b g(t)\,dt+\left|h(x)+\frac{h(a)+h(b)}{2}\right|, & 0\leq\alpha\leq\frac{1}{2}\\ \max\left\{\frac{1-\alpha}{2}\int_a^b g(t)\,dt+\left|h(x)+\frac{h(a)+h(b)}{2}\right|,\frac{\alpha}{2}\int_a^b g(t)\,dt\right\}, & \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{\alpha}{2}\int_a^b g(t)\,dt, & \frac{2}{3}\leq\alpha\leq1\end{cases}$$

and $\bigvee_a^b(f)$ *is the total variation of* f *over* $[a,b]$. *The constant* $\frac{1-\alpha}{2}$ *for* $0\leq\alpha\leq\frac{1}{2}$ *and the constant* $\frac{\alpha}{2}$ *for* $\frac{2}{3}\leq\alpha\leq1$ *are the best possible.*

for recent results concerning Ostrowski inequality for mappings of bounded variation see [7, 8].

Motivated by [9], S.S. Dragomir in [10] has proved the following companion of the Ostrowski inequality for mappings of bounded variation:

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$. *Then we have the inequalities:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b(f), \tag{1.5}$$

for any $x\in[a,\frac{a+b}{2}]$, *where* $\bigvee_a^b(f)$ *denotes the total variation of* f *on* $[a,b]$. *The constant* $\frac{1}{4}$ *is best possible.*

For other results see [11, 12]. In the recent work [13], M.W. Alomari has proved a companion of Ostrowski's inequality (1.3) for mappings of bounded variation:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$, we have the inequality*

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)\,dt\right| \tag{1.6}$$
$$\le \max\left\{\lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)a+\lambda b}{2}\right), \left(\frac{a+b}{2} - x\right)\right\} \cdot \bigvee_a^b (f)$$
$$= \max\left\{\lambda\frac{b-a}{2}, \frac{(1-\lambda)(b-a)}{4} + \left|x - \frac{(3-\lambda)a+(\lambda+1)b}{4}\right|\right\} \cdot \bigvee_a^b (f)$$

where, $\bigvee_a^b (f)$ denotes to the total variation of f over $[a,b]$.

In this paper, a weighted version of Alomari's inequality (1.6) is proved. Therefore, several weighted inequalities are deduced. Application to a quadrature rule is pointed out.

2. A weighted companion of Ostrowski type inequalities

Theorem 7. *Under the assumptions of Theorem 4, we have*

$$\left|\left[\alpha\frac{f(a)+f(b)}{2} + (1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt - \int_a^b f(t)g(t)\,dt\right|$$
$$\le K \cdot \bigvee_a^b (f) \tag{2.1}$$

where,

$$K = \begin{cases} \max\Big\{\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(x) - \left[\frac{3-\alpha}{4}h(a) + \frac{\alpha+1}{4}h(b)\right]\right|, \\ \frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(a+b-x) - \left[\frac{1+\alpha}{4}h(a) + \frac{3-\alpha}{4}h(b)\right]\right|\Big\}; & 0 \le \alpha \le \frac{1}{2} \\ \\ \max\Big\{\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(x) - \left[\frac{3-\alpha}{4}h(a) + \frac{\alpha+1}{4}h(b)\right]\right|, \frac{1-\alpha}{4}\int_a^b g(t)\,dt \\ + \left|h(a+b-x) - \left[\frac{1+\alpha}{4}h(a) + \frac{3-\alpha}{4}h(b)\right]\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\Big\}; & \frac{1}{2} < \alpha < \frac{2}{3} \\ \\ \frac{\alpha}{2}\int_a^b g(t)\,dt; & \frac{2}{3} \le \alpha \le 1 \end{cases}$$

for all $x \in [c, \frac{c+d}{2}]$, where, $\bigvee_a^b (f)$ denotes the total variation of f on $[a,b]$. Furthermore, the constant $\frac{1-\alpha}{4}$ for $0 \le \alpha \le \frac{1}{2}$ and the constant $\frac{\alpha}{2}$ for $\frac{2}{3} \le \alpha \le 1$ are the best possible.

Proof. Let $x \in [c, \frac{c+d}{2}]$. Define the mapping

$$s(t) = \begin{cases} h(t) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right], & t \in [a,x] \\ \\ h(t) - \frac{h(a)+h(b)}{2}, & t \in (x, a+b-x] \\ \\ h(t) - \left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right], & t \in (a+b-x, b] \end{cases}$$

for all $\alpha \in [0,1]$.

Using integration by parts, we have the following identity:

$$\begin{aligned}
\int_a^b s(t)\,df(t) &= \left[h(t) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right]\right]\cdot f(t)\Big|_a^x - \int_a^x f(t)\,g(t)\,dt \\
&\quad + \left[h(t) - \frac{h(a)+h(b)}{2}\right]\cdot f(t)\Big|_x^{a+b-x} - \int_x^{a+b-x} f(t)\,g(t)\,dt \\
&\quad + \left[h(t) - \left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right]\right]\cdot f(t)\Big|_{a+b-x}^b - \int_{a+b-x}^b f(t)\,g(t)\,dt \\
&= \left[\alpha\frac{f(a)+f(b)}{2} + (1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right][h(b)-h(a)] - \int_a^b f(t)\,g(t)\,dt \\
&= \left[\alpha\frac{f(a)+f(b)}{2} + (1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt.
\end{aligned}$$

Now, we use the fact that for a continuous function $p:[a,b]\to\mathbb{R}$ and a function $\nu:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu). \tag{2.2}$$

Applying the inequality (2.2) for $p(t)=s(t)$, as above and $\nu(t)=f(t)$, $t\in[a,b]$, we get

$$\begin{aligned}
&\left|\left[\alpha\frac{f(a)+f(b)}{2} + (1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\right| \\
&\le \left|\int_a^b s(t)\,df(t)\right| \le \sup_{t\in[a,b]}|s(t)|\bigvee_a^b(f),
\end{aligned}$$

where,

$$\begin{aligned}
\sup_{t\in[a,b]}|s(t)| &= \max\left\{h(x) - \left[\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right], h(a+b-x) - \frac{h(a)+h(b)}{2},\right. \\
&\qquad \left.\frac{h(a)+h(b)}{2} - h(x), \left[\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right] - h(a+b-x), \frac{\alpha}{2}[h(b)-h(a)]\right\} \\
&= \max\left\{\frac{1-\alpha}{4}(h(b)-h(a)) + \left|h(x) - \left[\frac{3-\alpha}{4}h(a) + \frac{\alpha+1}{4}h(b)\right]\right|,\right. \\
&\qquad \left.\frac{1-\alpha}{4}(h(b)-h(a)) + \left|h(a+b-x) - \left[\frac{1+\alpha}{4}h(a) + \frac{3-\alpha}{4}h(b)\right]\right|, \frac{\alpha}{2}[h(b)-h(a)]\right\} \\
&= \max\left\{\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(x) - \left[\frac{3-\alpha}{4}h(a) + \frac{\alpha+1}{4}h(b)\right]\right|,\right. \\
&\qquad \left.\frac{1-\alpha}{4}\int_a^b g(t)\,dt + \left|h(a+b-x) - \left[\frac{1+\alpha}{4}h(a) + \frac{3-\alpha}{4}h(b)\right]\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\right\}
\end{aligned}$$

and thus we obtain the desired result in (2.1).

To prove the sharpness of the constant $\frac{1-\alpha}{4}$, for $0 \leq \alpha \leq \frac{1}{2}$ assume that (2.1) holds with a constant $C_1 > 0$, i.e.,

(2.3)
$$\left| \left[\alpha \frac{f(a)+f(b)}{2} + (1-\alpha) \frac{f(x)+f(a+b-x)}{2} \right] \int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \max \left\{ C_1 \int_a^b g(t)\,dt + \left| h(x) - \left[\frac{3-\alpha}{4} h(a) + \frac{\alpha+1}{4} h(b) \right] \right|, \right.$$
$$\left. C_1 \int_a^b g(t)\,dt + \left| h(a+b-x) - \left[\frac{1+\alpha}{4} h(a) + \frac{3-\alpha}{4} h(b) \right] \right| \right\} \cdot \bigvee_a^b (f).$$

Without loss of generality, assume that the maximum of the right hand side is the first term i.e.,

(2.4)
$$\left| \left[\alpha \frac{f(a)+f(b)}{2} + (1-\alpha) \frac{f(x)+f(a+b-x)}{2} \right] \int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \left[C_1 \int_a^b g(t)\,dt + \left| h(x) - \left[\frac{3-\alpha}{4} h(a) + \frac{\alpha+1}{4} h(b) \right] \right| \right] \cdot \bigvee_a^b (f)$$

Consider the mapping

$$f(t) = \begin{cases} 0, & t \in [a,b] \setminus \left\{ h^{-1}\left(\frac{(3-\alpha)h(a)+(1+\alpha)h(b)}{4} \right) \right\} \\ \frac{1}{2}, & t = h^{-1}\left(\frac{(3-\alpha)h(a)+(1+\alpha)h(b)}{4} \right) \end{cases}$$

Then f is with bounded variation on $[a,b]$, and $\int_a^b f(t)\,g(t)\,dt = 0$, $\bigvee_a^b (f) = 1$, and for $x = h^{-1}\left(\frac{(3-\alpha)h(a)+(1+\alpha)h(b)}{4} \right)$, making of use (2.4), we get

$$\frac{1-\alpha}{4} \leq C_1,$$

which implies that the constant $\frac{1-\alpha}{4}$ is the best possible.

Now, assume that the maximum of the right hand side is the second term i.e.,

(2.5)
$$\left| \left[\alpha \frac{f(a)+f(b)}{2} + (1-\alpha) \frac{f(x)+f(a+b-x)}{2} \right] \int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \left[C_1 \int_a^b g(t)\,dt + \left| h(a+b-x) - \left[\frac{1+\alpha}{4} h(a) + \frac{3-\alpha}{4} h(b) \right] \right| \right] \cdot \bigvee_a^b (f)$$

Consider the mapping

$$f(t) = \begin{cases} 0, & t \in [a,b] \setminus \left\{ a+b-h^{-1}\left(\frac{(1+\alpha)h(a)+(3-\alpha)h(b)}{4} \right) \right\} \\ \frac{1}{2}, & t = a+b-h^{-1}\left(\frac{(1+\alpha)h(a)+(3-\alpha)h(b)}{4} \right) \end{cases}$$

Then f is with bounded variation on $[a,b]$, and $\int_a^b f(t)\,g(t)\,dt = 0$, $\bigvee_a^b(f) = 1$, and for $x = a+b-h^{-1}\left(\frac{(1+\alpha)h(a)+(3-\alpha)h(b)}{4}\right)$, making of use (2.5), we get

$$\frac{1-\alpha}{4} \le C_1,$$

which implies that the constant $\frac{1-\alpha}{4}$ is the best possible. Therefore, $\frac{1-\alpha}{4}$ is the best possible for (2.3).

Now, to prove the sharpness of the constant $\frac{\alpha}{2}$, for $\frac{2}{3} \le \alpha \le 1$ assume that (2.1) holds with a constant $C_2 > 0$, i.e.,

(2.6)

$$\left|\left[\alpha\frac{f(a)+f(b)}{2} + (1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\right| \le C_2\int_a^b g(t)\,dt \cdot \bigvee_a^b(f)$$

Consider the mapping

$$f(t) = \begin{cases} 0, & t \in [a,b]\setminus\{a\} \\ 1, & t = a \end{cases}$$

Then f is with bounded variation on $[a,b]$, and $\int_a^b f(t)\,g(t)\,dt = 0$, $\bigvee_a^b(f) = 1$, and for $x = h^{-1}\left(\frac{h(a)+h(b)}{2}\right)$, making of use (2.6), we get

$$\frac{\alpha}{2} \le C_2,$$

which implies that the constant $\frac{\alpha}{2}$ is the best possible. Thus, the proof of (2.1) is completely established. $\square$

Remark 1. *If we choose $h(t) = t$ and $g(t) = 1$, then the inequality (2.1) reduces to (1.6).*

Corollary 1. *In (2.1), choose $\alpha = 0$, then we get*

$$\left|\frac{f(x)+f(a+b-x)}{2}\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\right| \le K_1 \cdot \bigvee_a^b(f), \tag{2.7}$$

where,

$$K_1 = \max\left\{\frac{1}{4}\int_a^b g(t)\,dt + \left|h(x) - \frac{3h(a)+h(b)}{4}\right|, \right.$$
$$\left. \frac{1}{4}\int_a^b g(t)\,dt + \left|h(a+b-x) - \frac{h(a)+3h(b)}{4}\right|\right\},$$

for all $x \in [a, \frac{a+b}{2}]$, which is the "weighted companion of Ostrowski" inequality. Furthermore, if we choose $h(t) = t$ and $g(t) = 1$, then the inequality (2.7) reduces to (1.5).

Remark 2. *In Corollary 1, choose*

(1) $x = a$, *then we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right|\le\frac{1}{2}\int_a^b g(t)\,dt\cdot\bigvee_a^b(f), \tag{2.8}$$

which is the "weighted trapezoid" inequality.

(2) $x = \frac{3a+b}{4}$, *then we get*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right|\le K_2\cdot\bigvee_a^b(f), \tag{2.9}$$

where,

$$K_2=\max\left\{\frac{1}{4}\int_a^b g(t)\,dt+\left|h\left(\frac{3a+b}{4}\right)-\frac{3h(a)+h(b)}{4}\right|,\right.$$
$$\left.\frac{1}{4}\int_a^b g(t)\,dt+\left|h\left(\frac{a+3b}{4}\right)-\frac{h(a)+3h(b)}{4}\right|\right\},$$

(3) $x = \frac{a+b}{2}$, *then we get*

$$\left|f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right|\le K_3\cdot\bigvee_a^b(f), \tag{2.10}$$

where,

$$K_3=\max\left\{\frac{1}{4}\int_a^b g(t)\,dt+\left|h\left(\frac{a+b}{2}\right)-\frac{3h(a)+h(b)}{4}\right|,\right.$$
$$\left.\frac{1}{4}\int_a^b g(t)\,dt+\left|h\left(\frac{a+b}{2}\right)-\frac{h(a)+3h(b)}{4}\right|\right\}$$

which is the "weighted midpoint" inequality.

Corollary 2. *If we choose $h(t) = t$, $g(t) = 1$ and $x = \frac{a+b}{2}$ in (2.1), then we have the following inequality*

$$\left|(b-a)\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)\,f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt\right| \le K'(b-a)\cdot\bigvee_a^b(f) \tag{2.11}$$

where,

$$K'=\begin{cases}\frac{(1-\alpha)}{2}, & 0\le\alpha\le\frac{1}{2}\\ \left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right], & \frac{1}{2}<\alpha<\frac{2}{3}\\ \frac{\alpha}{2}, & \frac{2}{3}\le\alpha\le 1\end{cases}.$$

which is the "generalized Bullen's inequality", for details (see [7] *and* [14]*, p. 141).*

Corollary 3. *Let $0 \le \alpha \le 1$. Let $f \in C^{(1)}[a,b]$. Then we have the inequality*

(2.12)
$$\left|\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right| \le K\cdot\left\|f'\right\|_1,$$

for all $x \in [c, \frac{c+d}{2}]$, where $\left\|\cdot\right\|_1$ is the L_1 norm, namely $\left\|f'\right\|_1 := \int_a^b \left|f'(t)\right| dt$.

Corollary 4. *Let $0 \le \alpha \le 1$. Let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality*

(2.13)
$$\left|\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right| \le KL(b-a),$$

for all $x \in [c, \frac{c+d}{2}]$.

Corollary 5. *Let $0 \le \alpha \le 1$. Let $f : [a,b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality*

(2.14)
$$\left|\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)\frac{f(x)+f(a+b-x)}{2}\right]\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right| \le K\left|f(b)-f(a)\right|,$$

for all $x \in [c, \frac{c+d}{2}]$.

3. Application to a quadrature rule

Let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$ and $c_i = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(x_i)+\frac{\alpha}{2}h(x_{i+1})\right)$, $d_i = h^{-1}\left(\frac{\alpha}{2}h(x_i)+\left(1-\frac{\alpha}{2}\right)h(x_{i+1})\right)$ $\xi_i \in [c_i, \frac{c_i+d_i}{2}]$ $(i = 0, 1, \cdots, n-1)$. Put $L_i = h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t)dt$, and define the sum

(3.1)
$$A_\alpha(f,g,h,I_n,\xi)=\sum_{i=0}^{n-1}\left[\alpha\cdot\frac{f(x_i)+f(x_{i+1})}{2}+(1-\alpha)\cdot\frac{f(\xi_i)+f(x_i+x_{i+1}-\xi_i)}{2}\right]L_i$$

for all $\alpha \in [0,1]$. In the following we propose an approximation for the integral $\int_a^b f(t)\,g(t)\,dt$.

Theorem 8. *Let f, g, h be defined as in Theorem 7, then we have*

$$\int_a^b f(t)\,g(t)\,dt = A_\alpha(f,g,h,I_n,\xi)+R_\alpha(f,g,h,I_n,\xi). \tag{3.2}$$

where, $A_\alpha(f,g,h,I_n,\xi)$ is given in (3.1) and the remainder $R_\alpha(f,g,h,I_n,\xi)$ satisfies the bounds

$$R_\alpha(f,g,h,I_n,\xi) \le \sum_{i=0}^{n-1} K_{i,\alpha}\cdot\bigvee_{x_i}^{x_{i+1}}(f) \le M_{1,\alpha}\cdot\bigvee_a^b(f) \le M_{2,\alpha}\cdot\bigvee_a^b(f),$$

where,

$$K_{i,\alpha} := \begin{cases} \max\left\{\frac{1-\alpha}{4}L_i + \left|h(\xi_i) - \left[\frac{3-\alpha}{4}h(x_i) + \frac{\alpha+1}{4}h(x_{i+1})\right]\right|,\right. \\ \left.\frac{1-\alpha}{4}L_i + \left|h(x_i + x_{i+1} - \xi_i) - \left[\frac{1+\alpha}{4}h(x_i) + \frac{3-\alpha}{4}h(x_{i+1})\right]\right|\right\}, & 0 \le \alpha \le \frac{1}{2} \\ \max\left\{\frac{1-\alpha}{4}L_i + \left|h(\xi_i) - \left[\frac{3-\alpha}{4}h(x_i) + \frac{\alpha+1}{4}h(x_{i+1})\right]\right|, \frac{1-\alpha}{4}L_i\right. \\ \left. + \left|h(x_i + x_{i+1} - \xi_i) - \left[\frac{1+\alpha}{4}h(x_i) + \frac{3-\alpha}{4}h(x_{i+1})\right]\right|, \frac{\alpha}{2}L_i\right\}, & \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}L_i, & \frac{2}{3} \le \alpha \le 1 \end{cases}$$

$(i = 0, 1, 2, \cdots, n-1)$.

$$M_{1,\alpha} := \begin{cases} \max\limits_{i=0,1,\ldots,n-1}\left\{\max\left\{\frac{1-\alpha}{4}L_i + \left|h(\xi_i) - \left[\frac{3-\alpha}{4}h(x_i) + \frac{\alpha+1}{4}h(x_{i+1})\right]\right|,\right.\right. \\ \left.\left.\frac{1-\alpha}{4}L_i + \left|h(x_i + x_{i+1} - \xi_i) - \left[\frac{1+\alpha}{4}h(x_i) + \frac{3-\alpha}{4}h(x_{i+1})\right]\right|\right\}\right\}, & 0 \le \alpha \le \frac{1}{2} \\ \max\limits_{i=0,1,\ldots,n-1}\left\{\max\left\{\frac{1-\alpha}{4}L_i + \left|h(\xi_i) - \left[\frac{3-\alpha}{4}h(x_i) + \frac{\alpha+1}{4}h(x_{i+1})\right]\right|, \frac{1-\alpha}{4}L_i\right.\right. \\ \left.\left. + \left|h(x_i + x_{i+1} - \xi_i) - \left[\frac{1+\alpha}{4}h(x_i) + \frac{3-\alpha}{4}h(x_{i+1})\right]\right|, \frac{\alpha}{2}L_i\right\}\right\}, & \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\nu(L), & \frac{2}{3} \le \alpha \le 1 \end{cases}$$

$$M_{2,\alpha} := \begin{cases} \max\left\{\frac{1-\alpha}{4}\nu(L) + \max\limits_{i=0,1,\ldots,n-1}\left|h(\xi_i) - \left[\frac{3-\alpha}{4}h(x_i) + \frac{\alpha+1}{4}h(x_{i+1})\right]\right|,\right. & 0 \le \alpha \le \frac{1}{2} \\ \left.\frac{1-\alpha}{4}\nu(L) + \max\limits_{i=0,1,\ldots,n-1}\left|h(x_i + x_{i+1} - \xi_i) - \left[\frac{1+\alpha}{4}h(x_i) + \frac{3-\alpha}{4}h(x_{i+1})\right]\right|\right\}, & \\ \max\left\{\frac{1-\alpha}{4}\nu(L) + \max\limits_{i=0,1,\ldots,n-1}\left|h(\xi_i) - \left[\frac{3-\alpha}{4}h(x_i) + \frac{\alpha+1}{4}h(x_{i+1})\right]\right|,\right. & \frac{1}{2} < \alpha < \frac{2}{3} \\ \left.\frac{1-\alpha}{4}\nu(L) + \max\limits_{i=0,1,\ldots,n-1}\left|h(x_i + x_{i+1} - \xi_i) - \left[\frac{1+\alpha}{4}h(x_i) + \frac{3-\alpha}{4}h(x_{i+1})\right]\right|, \frac{\alpha}{2}\nu(L)\right\}, & \\ \frac{\alpha}{2}\nu(L), & \frac{2}{3} \le \alpha \le 1 \end{cases}$$

and $\nu(L) := \max\{L_i : i = 0, 1, \ldots, n-1\}$. *In the last inequality the constant* $\frac{1-\alpha}{4}$ *for* $0 \le \alpha \le \frac{1}{2}$ *and the constant* $\frac{\alpha}{2}$ *for* $\frac{2}{3} \le \alpha \le 1$ *are the best possible.*

Proof. Applying Theorem 7 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\left|\left[\alpha\frac{f(x_i) + f(x_{i+1})}{2} + (1-\alpha)\frac{f(\xi_i) + f(x_i + x_{i+1} - \xi_i)}{2}\right]\int_{x_i}^{x_{i+1}} g(t)\,dt - \int_{x_i}^{x_{i+1}} f(t)\,g(t)\,dt\right| \le K_i \cdot \bigvee_{x_i}^{x_{i+1}}(f)$$

for all $i = 0, 1, \cdots, n-1$.

Using this and the generalized triangle inequality, we have

$$R_{\alpha}\left(f,g,h,I_n,\xi\right)$$
$$\leq \sum_{i=0}^{k-1}\left|\left[\alpha\frac{f\left(x_i\right)+f\left(x_{i+1}\right)}{2}+\left(1-\alpha\right)\frac{f\left(\xi_i\right)+f\left(x_i+x_{i+1}-\xi_i\right)}{2}\right]\int_{x_i}^{x_{i+1}}g\left(t\right)dt-\int_{x_i}^{x_{i+1}}f\left(t\right)g\left(t\right)dt\right|$$
$$\leq \sum_{i=0}^{k-1}K_i\cdot\bigvee_{x_i}^{x_{i+1}}\left(f\right)\leq \max_{i=0,1,\dots,n-1}\left\{K_i\right\}\cdot\sum_{i=0}^{k-1}\bigvee_{x_i}^{x_{i+1}}\left(f\right)=M_1\cdot\bigvee_a^b\left(f\right)\leq M_2\cdot\bigvee_a^b\left(f\right)$$

□

Corollary 6. *In Theorem 8, choose*

(1) $\alpha = 0$, *then we get*

$$\int_a^b f\left(t\right)g\left(t\right)dt = A_0\left(f,g,h,I_n,\xi\right)+R_0\left(f,g,h,I_n,\xi\right). \tag{3.3}$$

where, $A_0\left(f,g,h,I_n,\xi\right)$ *is given in (3.1) and the remainder* $R_0\left(f,g,h,I_n,\xi\right)$ *satisfies the bounds*

$$R_0\left(f,g,h,I_n,\xi\right)\leq\sum_{i=0}^{n-1}K_{i,0}\cdot\bigvee_{x_i}^{x_{i+1}}\left(f\right)\leq M_{1,0}\cdot\bigvee_a^b\left(f\right)\leq M_{2,0}\cdot\bigvee_a^b\left(f\right),$$

(2) $\alpha = 1$, *then we get*

$$\int_a^b f\left(t\right)g\left(t\right)dt = A_1\left(f,g,h,I_n,\xi\right)+R_1\left(f,g,h,I_n,\xi\right). \tag{3.4}$$

where, $A_1\left(f,g,h,I_n,\xi\right)$ *is given in (3.1) and the remainder* $R_1\left(f,g,h,I_n,\xi\right)$ *satisfies the bounds*

$$R_1\left(f,g,h,I_n,\xi\right)\leq\sum_{i=0}^{n-1}K_{i,1}\cdot\bigvee_{x_i}^{x_{i+1}}\left(f\right)\leq M_{1,1}\cdot\bigvee_a^b\left(f\right)\leq M_{2,1}\cdot\bigvee_a^b\left(f\right),$$

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