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Fekete-Szegö Problem for Concave Univalent Functions Defined by Sălăgean Operator

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ABSTRACT. Let $C_0(\alpha)$ denote the class of concave univalent functions defined in the open unit disk U. In this paper, we investigate the sharp upper bounds of Fekete-Szegö functional with real and complex parameter λ for the class of concave univalent functions defined by Sălăgean differential operator.

1. Introduction

Let S denote the class of all analytic and univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $S^*(\beta)$, $C(\beta)$ and $K(\alpha, \beta)$, the classes of starlike functions of order β, convex functions of order β and close-to-convex functions of order α type β respectively, which are analytically defined as follows:

(i) $S^*(\beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in U,\ 0 \le \beta < 1 \right\}$,

(ii) $C(\beta) = \left\{ f \in A : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in U,\ 0 \le \beta < 1 \right\}$,

(iii) $K(\alpha, \beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha,\ g(z) \in C(\beta),\ z \in U,\ 0 \le \alpha < 1,\ 0 \le \beta < 1 \right\}$.

In 1933, Fekete and Szegö [**9**] obtained the maximum value of $\left|a_3 - \lambda a_2^2\right|$ as a function of the real parameter λ, namely

$$\left|a_3 - \lambda a_2^2\right| \le 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right),$$

for the class S of analytic and univalent functions given by (1.1). This inequality is sharp for each $\lambda \in [0, 1)$. In the literature, there exists a large number of results of the Fekete-Szegö functional $\left|a_3 - \lambda a_2^2\right|$ for various subclasses of S, such as the class of $S^*(\beta)$, $C(\beta)$ and $K(\alpha, \beta)$. For instance, Keogh and Merkers [**13**], Kaplan [**12**],

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Koepf [**14**] solved the Fekete-Szegö problem for close-to-convex functions. Nasr and Gawad [**16**], Gawad and Thomas [**10**], Darus and Thomas [**8**], Ibrahim and Darus [**11**] and others generalized this result for the class of functions that are close-to-convex functions of order α and type β. Later, Avkhadiev et al. [**1**], [**2**] and Bhowmik et al. [**4**], [**5**], they gave another treatment of Fekete-Szegö problem by considering the class of concave univalent functions given by (1.1).

Also, there are several authors that proved this type of result for the Fekete-Szegö functional for the class of function defined by differential operator, see [**3**], [**6**], for example, by using the Sălăgean differential operator D^k [**15**], for $f \in S$ which is defined by

(i) $D^0 f(z) = f(z)$,
(ii) $D^1 f(z) = Df(z) = z + \sum_{n=2}^{\infty} n a_n z^n$,
(iii) $D^k f(z) = D\left(D^{k-1} f(z)\right) = z + \sum_{n=2}^{\infty} n^k a_n z^n$; $k = 1, 2, \ldots$.

Denote by S_k^*, the class of k-starlike functions which is analytically defined as follows:

$$S_k^* = \left\{ f(z) \in S : \operatorname{Re}\left(\frac{D^{k+1} f(z)}{D^k f(z)}\right) > 0,\ k = 0, 1, 2, \ldots,\ z \in U \right\}. \tag{1.2}$$

In this paper, we investigated the sharp upper bounds of Fekete-Szegö functional $\left|a_3 - \lambda a_2^2\right|$ for the class of concave univalent functions with real and complex parameter λ, where the function of f is defined by Sălăgean differential operator (1.2).

2. Preliminary results

A function $f : U \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

(a) f is analytic in U with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
(b) f maps conformally onto a set whose complement with respect to C is convex.
(c) The opening angle of $f(U)$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to concave univalent functions and for a detailed discussion about concave functions we refer to [**1**], [**2**], [**7**] and the references therein. Recently, the class $C_0(\alpha)$ of concave function was considered by Bhowmik et al. [**4**], [**5**].

We recall the analytic characterization for the functions in $C_0(\alpha)$, $\alpha \in (1, 2]$: $f \in C_0(\alpha)$ if and only if $\operatorname{Re} P_f(z) > 0$, $z \in U$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

In [**4**], [**5**] they used this characterization and proved the following theorem.

Theorem 1. *Let $\alpha \in (1, 2]$. A function $f \in C_0(\alpha)$ if and only if there exist a starlike function $\phi \in S^*$ such that $f(z) = \Lambda_\phi(z)$ where*

$$\Lambda_\phi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}}\left(\frac{t}{\phi(t)}\right)^{(\alpha-1)/2} dt$$

and S^* denote the family of starlike functions g defined by $g \in S^*$ if and only if $\operatorname{Re}\left(\dfrac{zg'(z)}{g(z)}\right) > 0$.

The objective of the present paper is to give some generalizations of the result of Fekete-Szegö problem given by Bhowmik et al. [**4**] for the starlike function defined by Sălăgean differential operator $D^k f$, $k = 0, 1, 2, \ldots$, which is $f \in S_k^*$ is characterized by the condition (1.2).

In order to prove our main results, we need to recall the following lemma.

LEMMA 1. [**14**] *Let* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$. *Then* $\left|b_3 - \lambda b_2^2\right| \leq \max\{1, |3 - 4\lambda|\}$, *which is sharp for the Koebe function k if* $|\lambda - 3/4| \geq 1/4$ *and for* $(k(z))^{1/2} = \frac{z}{1-z^2}$ *if* $|\lambda - 3/4| \leq 1/4$.

3. Main result and its proof

We consider the Fekete-Szegö functional $\left|a_3 - \lambda a_2^2\right|$ for real and complex parameter λ. Our results are contained in the following theorems.

THEOREM 2. *Let $f \in C_0(\alpha)$ have the expansion given by (1.1), $\alpha \in (1, 2]$, $k = 0, 1, 2, \ldots$. If λ is real, then we have*

$$
12\left|a_3 - \lambda a_2^2\right| \leq \begin{cases}
\left(3+2^{2k}\right)(2-3\lambda)\alpha^2 + 3\left(1-2^{2k}\right)(1-2\alpha)\lambda + 6\left(1-3^k\right)\alpha + 2\left(3^{k+1}-2^{2k}\right), & \text{if } \lambda \leq \lambda_0; \\[2mm]
4\left[(2-3\lambda)\alpha^2 + 1\right], & \text{if } \lambda_0 \leq \lambda \leq \dfrac{2(\alpha-1)}{3\alpha}; \\[2mm]
\dfrac{4\left[(10-9\lambda)\alpha + (2-3\lambda)\right]}{3(2-\lambda) - (2-3\lambda)\alpha}, & \text{if } \dfrac{2(\alpha-1)}{3\alpha} \leq \lambda \leq \dfrac{2}{3}; \\[2mm]
12(1-\lambda)\alpha\sqrt{\frac{12(1-\lambda)}{(4-3\lambda)^2 - (3\lambda-2)^2\alpha^2}}, & \text{if } \frac{2}{3} \leq \lambda \leq \lambda_2; \\[2mm]
4\left[(3\lambda-2)\alpha^2 - 1\right], & \text{if } \lambda_2 \leq \lambda \leq \dfrac{2(\alpha+2)}{3(\alpha+1)}; \\[2mm]
\left(3+2^{2k}\right)(3\lambda-2)\alpha^2 + 3\left(2^{2k}-1\right)(1-2\alpha)\lambda + 6\left(3^k-1\right)\alpha + 2\left(2^{2k}-3^{k+1}\right), & \text{if } \lambda \geq \dfrac{2(\alpha+2)}{3(\alpha+1)};
\end{cases}
$$

where

$$
\lambda_0 = \frac{2^{2k-2}(\alpha+1) - 3^k}{3\left(2^{2k-3}\right)(\alpha-1)} \quad \text{and} \quad \lambda_2 = \frac{2}{3} + \frac{1}{6\alpha^2}\left(\sqrt{8\alpha^2+1} - 1\right).
$$

The inequalities are sharp.

THEOREM 3. *Let $f \in C_0(\alpha)$ have the expansion given by (1.1), $\alpha \in (1, 2]$, $k = 0, 1, 2, \ldots$. If λ are complex numbers, then we have*

$$
\left|a_3 - \lambda a_2^2\right| \leq \max\left\{1, \ \frac{1}{12}(\alpha+1)\,\nu(\alpha, \lambda)\right\},
$$

where

$$\nu(\alpha,\lambda) = |(2-3\lambda)(\alpha+1)+2| + 2(\alpha-1)|3\lambda-2| + \left(\frac{\alpha-1}{\alpha+1}\right)|6+[2-3(\alpha-1)\lambda]|.$$

PROOF. We recall that from Theorem 1 and making use the Sălăgean differential operator, that is $f \in C_0(\alpha)$ if and only if there exist a function $\phi(z) = z + \sum_{n=2}^{\infty}\phi_n z^n \in S_k^*$, $k = 0,1,2,...$ such that

$$f'(z) = \frac{1}{(1-z)^{\alpha+1}}\left(\frac{z}{D^n\phi(z)}\right)^{(\alpha-1)/2}, \tag{3.1}$$

where f has the form given by (1.1) Comparing the coefficients of z and z^2 on the both sides of the series expansion (3.1), we obtain that

$$a_2 = \frac{(\alpha+1)}{2} - 2^{k-2}(\alpha-1)\phi_2$$

and

$$a_3 = \frac{1}{6}(\alpha+1)(\alpha+2) - \frac{2^{k-1}}{3}\left(\alpha^2-1\right)\phi_2 - \frac{3^{k-1}}{2}(\alpha-1)\phi_3 + \frac{2^{2k-3}}{3}(\alpha-1)\phi_2^2$$

respectively.

A computation yields that

$$a_3 - \lambda a_2^2 = \frac{(\alpha+1)^2}{4}\left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right] + 2^{k-2}\left(\alpha^2-1\right)\left(\lambda - \frac{2}{3}\right)\phi_2 - \frac{3^{k-1}}{2}(\alpha-1) \times \left[\phi_3 - \left(\frac{2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)}{3^k}\right)\phi_2^2\right]. \tag{3.2}$$

Now, we need to investigate the maximum values of the function $\left|a_3 - \lambda a_2^2\right|$ by considering several cases of λ.

Case 1: Consider the first case for all $\lambda \leq \dfrac{2^{2k-2}(\alpha+1) - 3^k}{3\left(2^{2k-3}\right)(\alpha-1)}$.

We observe that the assumption on λ is seen to be equivalent to

$$\frac{1}{3^k}\left[2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)\right] \geq 1$$

and the first term in equation (3.2) is nonnegative. Hence, using the Lemma 1 for the last term in (3.2), we have

$$\left|\phi_3 - \left(\frac{2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)}{3^k}\right)\phi_2^2\right| \leq \frac{2^{2k}(\alpha+1) - 3\lambda\left(2^{2k-1}\right)(\alpha-1)}{3^k} - 3$$

and noticing that for $\phi \in S_k^*$, $|\phi_n| \leq n^{1-k}$, $k = 2, 3, \ldots$, we have from the equality (3.2) that

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| &\leq \frac{(\alpha+1)^2}{4}\left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right] + 2^{k-2}\left(\alpha^2-1\right)\left(\frac{2}{3}-\lambda\right)|\phi_2| \\
&\quad + \frac{3^{k-1}}{2}(\alpha-1)\left|\phi_3 - \left(\frac{2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)}{3^k}\right)\phi_2^2\right| \\
&= \frac{(\alpha+1)(\alpha+2)}{6} - \frac{\lambda}{4}(\alpha+1)^2 + \frac{\left(\alpha^2-1\right)}{2}\left(\frac{2}{3}-\lambda\right) \\
&\quad + \frac{3^{k-1}}{2}(\alpha-1)\left(\frac{2^{2k}(\alpha+1) - 3\lambda\left(2^{2k-1}\right)(\alpha-1)}{3^k} - 3\right).
\end{aligned}$$

It can be simplified to

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{12}&\left[\left(3+2^{2k}\right)(2-3\lambda)\alpha^2 + 3\left(1-2^{2k}\right)(1-2\alpha)\lambda\right. \\
&\left.+6\left(1-3^k\right)\alpha + 2\left(3^{k+1}-2^{2k}\right)\right],
\end{aligned}$$

for $\lambda \in \left(\infty, \dfrac{2^{2k-2}(\alpha+1) - 3^k}{3\left(2^{2k-3}\right)(\alpha-1)}\right)$.

Case 2: Let $\lambda \geq \dfrac{2(\alpha+2)}{3(\alpha+1)}$.

For this case, the first term in (3.2) is nonnegative. The condition on λ in particular gives $\lambda \geq \frac{2}{3}$ and therefore our assumption on λ implies that

$$\frac{2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)}{3^k} \leq \frac{2^{2k}}{3^k}\left(\frac{1}{2}\right).$$

Again, it follows from Lemma 1, that

$$\begin{aligned}
&\left|\phi_3 - \frac{2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)}{3^k}\phi_2^2\right| \\
&\leq 3 - \frac{2^{2k}(\alpha+1) - 3\lambda\left(2^{2k-1}\right)(\alpha-1)}{3^k}.
\end{aligned}$$

In view of these observation and an use of the inequality that $|\phi_2| \leq 2^{1-k}$, the equality (3.2) gives

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| &\leq \frac{(\alpha+1)^2}{4}\left[\lambda - \frac{2(\alpha+2)}{3(\alpha+1)}\right] + 2^{k-2}\left(\alpha^2-1\right)\left(\lambda-\frac{2}{3}\right)\left(2^{1-k}\right) \\
&\quad + \frac{3^{k-1}}{2}(\alpha-1)\left(3 - \frac{2^{2k}(\alpha+1) - 3\lambda\left(2^{2k-1}\right)(\alpha-1)}{3^k}\right).
\end{aligned} \tag{3.3}$$

Thus, simplifying the right hand side expression (3.3), we obtain that

$$\begin{aligned}
\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{12}&\left[\left(3+2^{2k}\right)(3\lambda-2)\alpha^2 + 3\left(2^{2k}-1\right)(1-2\alpha)\lambda\right. \\
&\left.+6\left(3^k-1\right)\alpha - 2\left(3^{k+1}-2^{2k}\right)\right],
\end{aligned}$$

for $\lambda \in \left[\frac{2(\alpha+2)}{3(\alpha+1)}, \infty\right)$.

Case 3: Consider λ, where

$$\lambda \in \left(\frac{2^{2k-2}(\alpha+1) - 3^k}{3(2^{2k-3})(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)} \right).$$

Now we deal with the case by using the formulas (3.1) and (3.2) together with the representation formula for $\phi(z) \in S_k^*$. Let us define $w(z)$ by

$$\frac{D^{k+1}\phi(z)}{D^k\phi(z)} = \frac{1+zw(z)}{1-zw(z)}; \quad (w(z) \neq 1) \tag{3.4}$$

where $w : U \to \overline{U}$ is a function analytic in U with the Taylor series

$$w(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Comparing the coefficients of z and z^2 in (3.4), we get that

$$\phi_2 = 2^{1-k} c_0 \quad \text{and} \quad \phi_3 = \frac{1}{3^k}\left(c_1 + 3c_0^2\right). \tag{3.5}$$

Inserting these resulting formulas (3.5) into (3.2) yields

$$\begin{aligned} a_3 - \lambda a_2^2 &\le \frac{(\alpha+1)^2}{4}\left[\frac{2(\alpha+2)}{3(\alpha+1)} - \lambda\right] \\ &+ 2^{k-2}\left(\alpha^2-1\right)\left(\lambda - \frac{2}{3}\right)\left(2^{1-k}c_0\right) \\ &+ \frac{3^{k-1}}{2}(\alpha-1)\left[\frac{1}{3^k}\left(c_1+3c_0^2\right)\right. \\ &- \left.\left(\frac{2^{2k-2}(\alpha+1) - 3\lambda\left(2^{2k-3}\right)(\alpha-1)}{3^k}\right)\left(2^{2-2k}\right)c_0^2\right] \\ &= A + Bc_0 + Cc_0^2 + Dc_1, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} A &= \frac{1}{6}(\alpha+2)(\alpha+1) - \frac{\lambda}{4}(\alpha+1)^2, \\ B &= \frac{1}{6}\left(\alpha^2-1\right)(3\lambda-2), \\ C &= -\frac{1}{12}(\alpha-1)\left[4-2\alpha+3\lambda(\alpha-1)\right], \\ D &= -\frac{1}{6}(\alpha-1). \end{aligned}$$

Hence, by using the well known inequalities that $|c_0| \le 1$ and $|c_1| \le 1 - |c_0|^2$, from (3.6) we obtain that

$$\left|a_3 - \lambda a_2^2\right| \le \left|A + Bc_0 + Cc_0^2\right| + \frac{1}{6}(\alpha-1)\left(1 - |c_0|^2\right). \tag{3.7}$$

Now, in order to determine the maximum value of (3.7), let $c_0 = re^{i\theta}$, then we consider the quadratic expression

$$f(r,\theta) = \left|A + Bc_0 + Cc_0^2\right|^2 = \left(A - Cr^2\right)^2 + B^2r^2 + 2Br\left(A + Cr^2\right)\cos\theta + 4ACr^2\cos^2\theta \tag{3.8}$$

where $\cos\theta \in [-1,1]$, $r \in (0,1]$. For getting the upper bounds of $\left|a_3 - \lambda a_2^2\right|$, we have to find the biggest value of (3.8) for r in the interval $(0,1]$. So, let $x = \cos\theta$, then from (3.8) we have

$$h(x) = \left(A - Cr^2\right)^2 + B^2r^2 + 2Br\left(A + Cr^2\right)x + 4ACr^2x^2. \tag{3.9}$$

We have to determine the maximum value of (3.9) for $x \in [-1,1]$. So, for this, we need to consider the several subclasses of λ, where

$$\lambda \in \left(\frac{2^{2k-2}(\alpha+1) - 3^k}{3\left(2^{2k-3}\right)(\alpha-1)}, \frac{2(\alpha+2)}{3(\alpha+1)}\right).$$

Case 3A: First, consider

$$\lambda \in \left(\frac{2^{2k-2}(\alpha+1) - 3^k}{3\left(2^{2k-3}\right)(\alpha-1)}, \frac{2(\alpha-2)}{3(\alpha-1)}\right).$$

We observe that for λ in this interval, we have $A > 0$, $B < 0$, $C > 0$ and $A + Cr^2 > 0$ for $r \in (0,1]$, and (3.9) attains its maximum value at $x = -1$. Therefore, it gives that

$$\left|a_3 - \lambda a_2^2\right| \le g(r) = A - Br + Cr^2 + \frac{1}{3}(\alpha-1)\left(1 - r^2\right). \tag{3.10}$$

By a simple calculation, we show that the maximum value of (3.10) attains at the boundary of r, i.e. $r = 1$. Therefore

$$g(r) \le g(1) = A - B + C = \frac{1}{3}\left[(2 - 3\lambda)\alpha^2 + 1\right].$$

Case 3B: Let $\lambda = \dfrac{2(\alpha-2)}{3(\alpha-1)}$.

In this case, we get $C = 0$, therefore $h(x)$ becomes a linear function,

$$h(x) = A^2 + B^2r^2 + 2BrAx. \tag{3.11}$$

It is easy to show that the maximum value of (3.11) occurs at $x = -1$ and $r = 1$. Again we get the maximum value of $\left|a_3 - \lambda a_2^2\right|$ as the previous case.

Case 3C: Let $\lambda \in \left(\dfrac{2(\alpha-2)}{3(\alpha-1)}, \dfrac{2(\alpha-1)}{3\alpha}\right)$.

In this interval, the quadratic function (3.9) has maximum value at

$$x(r) = -\frac{B}{4}\left(\frac{1}{Cr} + \frac{r}{A}\right),$$

where $x(r)$ is monotonic increasing in $r \in (0,1]$ and $x(1) < -1$. Hence we get the upper bound as in Cases 3A and 3B. As conclusion, from the Cases 3A, 3B and 3C give us that

$$\left|a_3 - \lambda a_2^2\right| \le \frac{1}{3}\left[(2 - 3\lambda)\alpha^2 + 1\right]$$

for

$$\lambda \in \left(\frac{2^{2k-2}(\alpha+1) - 3^k}{3\left(2^{2k-3}\right)(\alpha-1)}, \frac{2(\alpha-1)}{3\alpha}\right).$$

Case 3D: Let $\lambda \in \left[\dfrac{2(\alpha-1)}{3\alpha}, \dfrac{2}{3}\right)$.

From the Case 3C, the inequality $x(1) < -1$ gives that

$$\frac{2\left(3\lambda + 4\alpha^2 - 12\alpha^2\lambda + 9\alpha^2\lambda^2 - 4\right)}{\left[(3\lambda-4)+\alpha(3\lambda-2)\right]\left[\alpha(3\lambda-2)-(3\lambda-4)\right]} < 0,$$

hence it shows that

$$p(\lambda) = 9\alpha^2\lambda^2 + \left(3 - 12\alpha^2\right)\lambda + 4(\alpha^2 - 1) < 0$$

where $\lambda < \frac{2}{3}$. Factorizing $p(\lambda)$, we have

$$\lambda_1 = \frac{2}{3} - \frac{1}{6\alpha^2}\left(1 + \sqrt{8\alpha^2+1}\right) \tag{3.12}$$

and

$$\lambda_2 = \frac{2}{3} - \frac{1}{6\alpha^2}\left(1 - \sqrt{8\alpha^2+1}\right). \tag{3.13}$$

It is clear that $\lambda_1 < \lambda_2$. Therefore, for $\lambda \in \left[\dfrac{2(\alpha-1)}{3\alpha}, \lambda_1\right)$, the functions (3.9) and (3.10) has its maximum value at

$$x = -1 \text{ and } r_m = \frac{-3B}{-6C + \alpha - 1} \in (0, 1]$$

respectively. Hence the upper bound of Fekete-Szegö functional is given by

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &\le g(r_m) = A - Br_m + Cr_m^2 + \frac{1}{3}(\alpha-1)\left(1-r_m^2\right) \\ &= \frac{4\left[(10-9\lambda)\alpha + (2-3\lambda)\right]}{3(2-\lambda) - (2-3\lambda)\alpha}. \end{aligned} \tag{3.14}$$

Next, we consider for $\lambda \in \left[\lambda_1, \dfrac{2}{3}\right)$. In this interval, the quadratic equation (3.9) attains its maximum value at

$$x(r) = \frac{-B\left(A + Cr^2\right)}{4ACr}$$

with

$$h(x(r)) = -\frac{1}{4AC}\left(B^2 - 4AC\right)(A - Cr)^2.$$

Hence, the Fekete-Szegö functional satisfies the following inequality

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &\le \sqrt{h(x(r))} + \frac{(\alpha-1)}{6}\left(1-r^2\right) \\ &= (A - Cr)\sqrt{1 - \frac{B^2}{4AC}} + \frac{(\alpha-1)}{6}\left(1-r^2\right) = k(r). \end{aligned} \tag{3.15}$$

The maximum value of $g(r)$,

$$g(r) = A - Br + Cr^2 + \frac{(\alpha-1)}{6}\left(1-r^2\right)$$

and (3.15) occurs at

$$r_m = \frac{-B}{-2C + \frac{(\alpha-1)}{3}} \text{ and } r_0 = \frac{B}{2C + \sqrt{1 - \frac{B^2}{4AC}}}$$

respectively. It is easy to show that (3.15) is monotonic decreasing for $r \geq r_0$. Hence, the maximum value of $\left|a_3 - \lambda a_2^2\right|$ is given by (3.14).

For $\lambda = \frac{2}{3}$, we get $B = 0$ and $C = \frac{1}{6}(1-\alpha)$. Thus, the maximum value

$$\left|a_3 - \lambda a_2^2\right| = \frac{\alpha}{3}, \tag{3.16}$$

occurs at $x = \cos\theta = 0$ and $r \in (0,1]$. From (3.14), (3.15) and (3.16) we concluded that

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{4\left[(10-9\lambda)\alpha + (2-3\lambda)\right]}{3(2-\lambda) - (2-3\lambda)\alpha}$$

for $\lambda \in \left[\frac{2(\alpha-1)}{3\alpha}, \frac{2}{3}\right]$.

Case 3E: Let $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$, where λ_2 is given by (3.13).

In this interval, we have $B > 0$. So that (3.9) attains its maximum value at $x = 1$. Then, we consider the function

$$l(r) = h(1) = A + Br + Cr^2 + \frac{(\alpha-1)}{6}\left(1-r^2\right).$$

Again, by a simple calculation shows that the maximum value of $l(r)$ to be occured at

$$r_n = \frac{B}{-2C + \frac{(\alpha-1)}{3}},$$

hence the maximum of the function (3.15) to be attained at

$$r_1 = \frac{B}{-2C\left(1+\sqrt{1-\frac{B^2}{4AC}}\right)} \in (0,1].$$

It is easily to prove that $r_1 < r_n \leq 1$. Since $k(r)$ is monotonic increasing function, then

$$k(r) \leq k(1) = (A-C)\sqrt{1-\frac{B^2}{4AC}},$$

which gives that

$$\left|a_3 - \lambda a_2^2\right| \leq k(1) = (1-\lambda)\alpha\sqrt{\frac{12(1-\lambda)}{(4-3\lambda)^2 - (3\lambda-2)^2\alpha^2}}$$

for $\lambda \in \left(\frac{2}{3}, \lambda_2\right]$.

Case 3F: Finally, we consider the case for $\lambda \in \left(\lambda_2, \frac{2(\alpha+2)}{3(\alpha+1)}\right)$.

For these λ, we see that $A < 0$, $B > 0$, $C < 0$, $A + Cr^2 < 0$ and the maximum value of function (3.7) is attained for $x = -1$, i.e.

$$\eta(x) = -A + Br - Cr^2 + \frac{(\alpha-1)}{6}\left(1-r^2\right).$$

We get $\eta(r) \leq \eta(1)$ for all λ in these interval and hence

$$\left|a_3 - \lambda a_2^2\right| \leq -A + B - C = \frac{1}{3}\left[(3\lambda-2)\alpha^2 - 1\right].$$

Thus, the proof of Theorem 2 is complete.

Further, substitute (3.5) into (3.2) yields

$$12\left(a_3-\lambda a_2^2\right)=(\alpha+1)\left[(2-3\lambda)(\alpha+1)+2\right]+2\left(\alpha^2-1\right)(3\lambda-2)c_0 \\ +(\alpha-1)\left(6+\left[2-3(\alpha-1)\lambda\right]\right)c_0^2+2(1-\alpha)c_1.$$

Hence for λ complex numbers, we have

$$12\left|a_3-\lambda a_2^2\right|\leq(\alpha+1)\left|(2-3\lambda)(\alpha+1)+2\right| \\ +2(1-\alpha)|c_1|+2\left(\alpha^2-1\right)|3\lambda-2||c_0| \\ +(\alpha-1)\left|6+\left[2-3(\alpha-1)\lambda\right]\right||c_0|^2. \tag{3.17}$$

Using the well known inequality that $|c_0|\leq 1$ and $|c_1|\leq 1-|c_0|^2$, then from (3.17) we get

$$12\left|a_3-\lambda a_2^2\right|\leq\frac{1}{12}(\alpha+1)\nu(\alpha,\lambda)$$

for $\mathrm{Re}\{\nu(\alpha,\lambda)\}>0$, where

$$\nu(\alpha,\lambda)=\left|(2-3\lambda)(\alpha+1)+2\right|+2(1-\alpha)|3\lambda-2| \\ +\frac{(\alpha-1)}{\alpha+1}\left|6+\left[2-3(\alpha-1)\lambda\right]\right|.$$

Thus, the proof of Theorem 3 is complete. □

Remark 1. *Taking $k=0$ and λ real numbers, we deduce a result of Bhowmik et al.* [**4**].

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