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Coefficient Bounds for a Certain Subclass of Univalent Functions

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Abstract. Let $\widetilde{T}_n^{\alpha}(\beta)$ denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ such that $\left|\arg\left(\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right)\right| \leq \frac{\beta\pi}{2}$, $z \in U$, $n \in \{0\} \cup N$, $\alpha > 0$, $0 < \beta \leq 1$ and D^n is the Sălăgean differential operator. In this paper, we establish some coefficient bounds for functions of the class $\widetilde{T}_n^{\alpha}(\beta)$ introduced in [**4**] by Opoola.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let P denote the class consisting of analytic functions $p(z)$ of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{1.2}$$

such that $\operatorname{Re} p(z) > 0$, $z \in U$.

Furthermore, let $\widetilde{T}_n^{\alpha}(\beta)$ denote the class of functions in A which is satisfies the following condition:

$$\left|\arg\left(\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right)\right| \leq \frac{\beta\pi}{2}, \tag{1.3}$$

for $z \in U$, $n \in \{0, 1, 2, ...\}$, $\alpha > 0$, $0 < \beta \leq 1$ and D^n is the Sălăgean differential operator [**6**] defined as follows:

(i) $D^0 f(z) = f(z)$,
(ii) $D^1 f(z) = Df(z) = zf'(z)$,
(iii) $D^n f(z) = D\left[D^{n-1} f(z)\right] = z\left[D^{n-1} f(z)\right]'$.

Note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (z \in U).$$

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The class of functions which is defined in the half plane, namely

$$T_n^{\alpha}(\beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{D^n f(z)^{\alpha}}{z^{\alpha}} \right) > \beta, \right. \qquad (1.4)$$
$$\left. z \in U, n = 0, 1, 2, ..., \alpha > 0, 0 \leq \beta < 1 \right\},$$

was introduced by Opoola in 1994 [**4**]. This class of functions have been studied by several authors, such as Opoola et al. [**5**], Babalola [**3**] and others. Abdulhalim [**1**], [**2**] studied the case for $\beta = 0$. Some interesting properties of the class (1.4) were established. Among other things, it was shown that if $f(z)$ belongs to the class (1.4), then

$$|a_2| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+1)^n}, \text{ if } \alpha > 0,$$

$$|a_3| \leq \begin{cases} \dfrac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n(\alpha+1)^{2n}} \Big[(\alpha+1)^{2n} \\ \quad - (1-\beta)(\alpha-1)(\alpha+2)^n \alpha^{n-1} \Big]; & \text{if } 0 < \alpha < 1, \\ \dfrac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} & \text{if } \alpha \geq 1, \end{cases}$$

where $0 \leq \beta < 0$.

In this paper, we obtain some properties of functions of the class (1.3). In particular we obtain the upper bounds for $|a_n|$, $n = 2, 3, 4$.

In order to prove our results, we need the following lemma.

Lemma 1. *Let $p \in P$ and let it be of the form $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$. Then*

(i) $|c_i| \leq 2$ *for* $i \geq 1$,

(ii) $\left| c_2 - \mu c_1^2 \right| \leq 2 \max\{1, |1 - 2\mu|\}$ *for any* $\mu \in \mathbb{C}$.

2. Main results

In this section, we state and prove our main result.

Theorem 1. *Let $f(z)$ belongs to the class $\widetilde{T}_n^{\alpha}(\beta)$, $\alpha > 0$, $0 < \beta \leq 1$, $n \in \{0, 1, 2, ...\}$. Then the following inequalities hold.*

$$|a_2| \leq \frac{2\beta\alpha^{n-1}}{(\alpha+1)^n}; \quad \text{if } \alpha > 0, \qquad (2.1)$$

$$|a_3| \leq \begin{cases} \dfrac{2\beta\alpha^{n-1}}{(\alpha+2)^n} \\ \times \left(1 + \dfrac{(\alpha+2)^n}{(\alpha+1)^{2n}} (1-\alpha)\alpha^{n-1}\beta \right); & \text{if } 0 < \alpha < 1, \\ \dfrac{2\beta\alpha^{n-1}}{(\alpha+2)^n}; & \text{if } \alpha \geq 1, \end{cases} \qquad (2.2)$$

$$|a_4| \leq \begin{matrix} F_1 + F_3 + F_5; & \text{if } 0 < \alpha < 1, \\ F_1 + F_4 + F_5 + F_6; & \text{if } 1 \leq \alpha < 2, \\ F_1 + F_4 + F_5; & \text{if } \alpha \geq 2, \end{matrix} \qquad (2.3)$$

where

$$F_1 = \frac{2\beta\alpha^{n-1}}{3(\alpha+3)^n}\left[3+2(\beta-1)(\beta-2)\right], \tag{2.4}$$

$$F2 = \frac{4\beta(\beta-1)\alpha^{n-1}}{(\alpha+3)^n},$$

$$F_3 = \frac{4\beta^2(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n},$$

$$F_4 = \frac{4\beta^2(\beta-1)(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n},$$

$$F_5 = \frac{4\beta^3(1-\alpha)^2\alpha^{3n-3}}{(\alpha+1)^{3n}},$$

$$F_6 = \frac{4\beta^3(1-\alpha)(\alpha-2)\alpha^{3n-3}}{3(\alpha+1)^{3n}}.$$

PROOF. From definition (1.3) for $f \in \widetilde{T}_n^{\alpha}(\beta)$, it suggests that there exists $p \in P$ such that for $z \in U$

$$D^n f(z)^{\alpha} = \alpha^n z^n \left[p(z)\right]^{\beta}. \tag{2.5}$$

Using the Sălăgean differential operator $D^n f(z)^{\alpha}$ as $z\left[D^{n-1}f(z)^{\alpha}\right]'$ and

$$p(z) = 1 + \sum_{i=1}^{\infty} p_i z^i$$

in (2.5), we have that

$$\begin{aligned} \left[D^{n-1}f(z)^{\alpha}\right]' &= \alpha^n z^{\alpha-1}\left[1+\sum_{i=1}^{\infty} c_i z^i\right]^{\beta} \\ &= \alpha^n\left[z^{\alpha-1}+\mu_1 z^{\alpha}+\mu_2 z^{\alpha+1}+\mu_3 z^{\alpha+2}+\mu_4 z^{\alpha+3}+...\right], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \mu_1 &= \beta_1 p_1, \\ \mu_2 &= \beta_1 p_2 + \beta_2 p_1^2, \\ \mu_3 &= \beta_1 p_3 + 2\beta_2 p_1 p_2 + \beta_3 p_1^3, \\ \mu_4 &= \beta_1 p_4 + 2\beta_2 p_1 p_3 + \beta_2 p_2^2 + 3\beta_3 p_1^2 p_2 + \beta_4 p_1^4, \end{aligned}$$

and

$$\beta_j = \binom{\beta}{j} = \frac{\beta!}{j!(\beta-j)!}, j = 1,2,3,... .$$

Integrating both sides (2.6) along the line segment 0 to z, we obtain that

$$\begin{aligned} \frac{D^{n-1}f(z)^{\alpha}}{z^{\alpha}} &= \alpha^{n-1}\left[1+\frac{\mu_1\alpha}{(\alpha+1)}z+\frac{\mu_2\alpha}{(\alpha+2)}z^2\right. \\ &+ \left.\frac{\mu_3\alpha}{(\alpha+3)}z^3+\frac{\mu_4\alpha}{(\alpha+4)}z^4+...\right], \end{aligned}$$

where $f(0) = 0$.

Now, repeating the process, we are able to establish the following relation which holds in general for any k $(k = 0, 1, 2, ...)$,

$$\frac{D^{n-k}f(z)^{\alpha}}{z^{\alpha}} = \alpha^{n-k}\left[1 + \frac{\mu_1\alpha^k}{(\alpha+1)^k}z + \frac{\mu_2\alpha^k}{(\alpha+2)^k}z^2 + \frac{\mu_3\alpha^k}{(\alpha+3)^k}z^3 + \frac{\mu_4\alpha^k}{(\alpha+4)^k}z^4 + ...\right]. \tag{2.7}$$

In particular, for $n = k$ we have

$$\left(\frac{f(z)}{z}\right)^{\alpha} = 1 + \frac{\mu_1\alpha^n}{(\alpha+1)^n}z + \frac{\mu_2\alpha^n}{(\alpha+2)^n}z^2 + \frac{\mu_3\alpha^n}{(\alpha+3)^n}z^3 + \frac{\mu_4\alpha^n}{(\alpha+4)^n}z^4 + ... \tag{2.8}$$

On comparing coefficients in (2.8) with $f(z)$ given by (1.1), we obtain that

$$\alpha_1 a_2 = \frac{\mu_1\alpha^n}{(\alpha+1)^n}, \tag{2.9}$$

$$\alpha_1 a_3 = \frac{\mu_2\alpha^n}{(\alpha+2)^n} - \alpha_2 a_2^2, \tag{2.10}$$

$$\alpha_1 a_4 = \frac{\mu_3\alpha^n}{(\alpha+3)^n} - 2\alpha_2 a_2 a_3 - \alpha_3 a_2^3, \tag{2.11}$$

where $\alpha_j = \binom{\alpha}{j} = \frac{\alpha!}{j!\,(\alpha-j)!}$, $j = 1, 2, 3, ...$.

For all $\alpha > 0$, $\beta > 0$ and since $|c_1| \leq 2$, it follows easily from (2.9) that

$$|a_2| \leq \frac{2\beta\alpha^{n-1}}{(1+\alpha)^n}.$$

Eliminating a_2 in (2.10), we get

$$a_3 = \frac{\beta\alpha^{n-1}}{(\alpha+2)^n}p_2 + \frac{\beta(\beta-1)\alpha^{n-1}}{2(\alpha+2)^n}p_1^2 + \frac{(1-\alpha)\beta^2\alpha^{2n-2}}{(\alpha+1)^{2n}}p_1^2. \tag{2.12}$$

Using Lemma 1 with $|p_i| \leq 2$, $i = 1, 2, ...$ and then, consider the cases for all $0 < \beta \leq 1$ with $0 < \alpha < 1$ and $\alpha \geq 1$, both inequalities in (2.2) are easily obtained.

Now, we prove inequalities (2.3). Substituting the equalities (2.9) and (2.10) in (2.11) gives us that

$$\begin{aligned} a_4 = {} & \frac{\beta\alpha^{n-1}}{(\alpha+3)^n}p_3 + \frac{\beta(\beta-1)\alpha^{n-1}}{(\alpha+3)^n}p_1p_2 \\ & + \frac{\beta(\beta-1)(\beta-2)\alpha^{n-1}}{6(\alpha+3)^n}p_1^3 \\ & + \frac{\beta^2(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n}p_1p_2 + \frac{\beta^2(\beta-1)(1-\alpha)\alpha^{2n-2}}{2(\alpha+1)^n(\alpha+2)^n}p_1^3 \\ & + \frac{\beta^3(1-\alpha)^2\alpha^{3n-3}}{2(\alpha+1)^{3n}}p_1^3 + \frac{\beta^3(1-\alpha)(\alpha-2)\alpha^{3n-3}}{6(\alpha+1)^{3n}}p_1^3. \end{aligned} \tag{2.13}$$

Hence, for $\alpha > 0$, $0 < \beta \leq 1$ and applying the well known inequality that $|p_i| \leq 2, i = 1, 2, ...$, we get

$$|a_4| \leq \frac{2\beta\alpha^{n-1}}{3(\alpha+3)^3}[3+2(\beta-1)(\beta-2)] - \frac{4\beta(\beta-1)\alpha^{n-1}}{(\alpha+3)^n} + \frac{4\beta^2|(1-\alpha)|\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n} - \frac{4\beta^2|(1-\alpha)|(\beta-1)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n} + \frac{8\beta^3(1-\alpha)^2\alpha^{3n-3}}{(\alpha+1)^{3n}} + \frac{4\beta^3|(1-\alpha)(\alpha-2)|\alpha^{3n-3}}{3(\alpha+1)^{3n}}, \tag{2.14}$$

Thus, the coefficient bounds $|a_4|$ depends on α in the intervals $(0,1)$, $[1,2)$ and $[2,\infty)$, hence from (2.14) we get that

$$|a_4| \leq \begin{cases} F_1 + F_3 + F_5\ ; & \text{for } 0 < \alpha < 1, \\ F_1 + F_4 + F_5 + F_6; & \text{for } 1 \leq \alpha < 2, \\ F_1 + F_4 + F_5; & \text{for } \alpha \geq 2, \end{cases}$$

where F_1, F_2, F_3, F_4, F_5 and F_6 is shown in (2.4). □

References

[1] S. Abdulhalim, On a class of analytic functions involving the Sălăgean differential operator, *Tamkang J. Maths.* **23**(1992), pp. 51-58.

[2] S. Abdul Halim, Coefficients estimates for functions in $B_n(\alpha)$, *Inter. J. of Maths and Math. Sci.* **59**(2003), pp. 3761-3767.

[3] K. O. Babalola, Bounds on the coefficients of certain analytic and univalent functions, *Mathematica, Tome* **50**(73)(3008), pp. 139-148.

[4] T. O. Opoola, On a new subclass of univalent functions, *Mathematica Tome*, **36**, 59(2)(1994), pp. 195-200.

[5] T. O. Opoola, K. O. Babalola, O. A. Fadipe-Joseph and K. Rauf, On coefficient bounds of a subclass of univalent functions, *Jour. of the Nigerian Math. Soc.* **23**(2004), pp. 87-92.

[6] G. S. Sălăgean, *Subclasses of Univalent Functions, Lecture* Note in Math, Springer-Verlag, Berlin, Heidelberg and New York(1983), pp. 362-372.

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