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VECTORIAL ERROR ESTIMATES IN APPROXIMATING FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES BY A PERTURBED TAYLOR'S LIKE EXPANSION

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Abstract. On utilising the spectral representation of selfadjoint operators in Hilbert spaces, some vectorial error estimates in approximating functions of selfadjoint operators in Hilbert spaces by a perturbed Taylor's type expansion are given.

1. Introduction

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., . \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y \rangle, \tag{1.2}$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

By utilizing the spectral representation (1.1) the following trapezoid type approximation for a function of selfadjoint operators has been obtained in [4]:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral*

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family, then we have the inequalities

$$\left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.3}$$
$$\leq\frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.$$
$$\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]\bigvee_m^M(f)$$
$$\leq\frac{1}{2}\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

A generalized trapezoid type inequality was obtained in [7] and is as follows:

Theorem 2. *With the assumptions in Theorem 1 we have the inequalities.*

1. If $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$, then

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{1.4}$$
$$\leq\sup_{t\in[m,M]}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]\bigvee_m^M(f)$$
$$\leq\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\bigvee_m^M(f)\leq\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

2. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{1.5}$$
$$\leq L\int_{m-0}^{M}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]dt$$
$$\leq L(M-m)\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq L(M-m)\|x\|\|y\|$$

for any $x,y\in H$.

3. If $f:[m,M]\to\mathbb{R}$ is continuous and monotonic nondecreasing on $[m,M]$, then

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{1.6}$$
$$\leq\int_{m-0}^{M}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]df(t)$$
$$\leq\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)[f(M)-f(m)]\leq\|x\|\|y\|[f(M)-f(m)]$$

for any $x,y\in H$.

Utilising the spectral representation from (1.2) we also have established in [5] the following Ostrowski's type vector inequality:

Theorem 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned}
&|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \\
&\leq \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\
&+ \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\
&\leq \|x\| \|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right)
\end{aligned} \tag{1.7}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

A different result that compares the function of a selfadjoint operator with the integral mean is embodied in the following theorem [6]:

Theorem 4. *With the assumptions in Theorem 3 we have the inequalities*

$$\begin{aligned}
&\left| \langle x, y\rangle \cdot \frac{1}{M - m} \int_m^M f(s)\, ds - \langle f(A)x, y\rangle \right| \\
&\leq \frac{1}{M - m} \bigvee_m^M (f) \max_{t \in [m, M]} \Big[(M - t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\
&+ (t - m) \langle (1_H - E_t)x, x\rangle^{1/2} \langle (1_H - E_t)y, y\rangle^{1/2} \Big] \\
&\leq \|x\| \|y\| \bigvee_m^M (f)
\end{aligned} \tag{1.8}$$

for any $x, y \in H$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [8] and the references therein.

For other recent results see [1], [2], [3], [10], [11], [12] and [13].

In this paper, by utilizing the spectral representation (1.1) of selfadjoint operators in Hilbert spaces, some error estimates for the approximation of a function of selfadjoint operators by a perturbed Taylor's type formula are given. Applications for some elementary functions including the exponential and logarithmic functions are also provided.

2. Some Identities

The following result provides a perturbed Taylor's type representation for a function of selfadjoint operators in Hilbert spaces.

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family,*

I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then for any $c \in [m,M]$ we have the equalities

$$
\begin{aligned}
f(A) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\,(A - c1_H)^k \\
&\quad + \left[f(M) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\,(M-c)^k \right] 1_H \\
&\quad + V_n(f,c,m,M)
\end{aligned} \tag{2.1}
$$

where

$$
V_n(f,c,m,M) := \frac{(-1)^n}{(n-1)!} \int_{m-0}^{M} \left(\int_c^{\lambda} (t-\lambda)^{n-1}\, d\left(f^{(n)}(t)\right) \right) E_\lambda d\lambda. \tag{2.2}
$$

Proof. We utilize the Taylor's formula for functions $f : I \to \mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$
f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\,(\lambda - c)^k + \frac{1}{n!} \int_c^{\lambda} (\lambda - t)^n\, d\left(f^{(n)}(t)\right) \tag{2.3}
$$

for any $\lambda, c \in [m,M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m,M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$
\begin{aligned}
\int_{m-0}^{M} f(\lambda)\, dE_\lambda &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_{m-0}^{M} (\lambda - c)^k\, dE_\lambda \\
&\quad + \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^{\lambda} (\lambda - t)^n\, d\left(f^{(n)}(t)\right) \right) dE_\lambda
\end{aligned}
$$

which, by the spectral representation theorem, produces the equality

$$
f(A) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)\,(A - c1_H)^k + \frac{1}{n!} \int_{m-0}^{M} \left(\int_c^{\lambda} (\lambda - t)^n\, d\left(f^{(n)}(t)\right) \right) dE_\lambda \tag{2.4}
$$

that is of interest in itself as well.

Now, integrating by parts in the Riemann-Stieltjes integral and using the Leibnitz formula for integrals with parameters, we have

$$
\begin{aligned}
(2.5)\qquad & \int_{m-0}^{M}\left(\int_{c}^{\lambda}(\lambda-t)^{n}\,d\left(f^{(n)}(t)\right)\right)dE_{\lambda} \\
&= E_{\lambda}\left(\int_{c}^{\lambda}(\lambda-t)^{n}\,d\left(f^{(n)}(t)\right)\right)\Bigg|_{m-0}^{M} \\
&- \int_{m-0}^{M}E_{\lambda}d\left(\int_{c}^{\lambda}(\lambda-t)^{n}\,d\left(f^{(n)}(t)\right)\right) \\
&= \left(\int_{c}^{M}(M-t)^{n}\,d\left(f^{(n)}(t)\right)\right)1_{H} \\
&- n\int_{m-0}^{M}\left(\int_{c}^{\lambda}(\lambda-t)^{n-1}\,d\left(f^{(n)}(t)\right)\right)E_{\lambda}d\lambda
\end{aligned}
$$

and, since by the Taylor's formula (2.3) we have

$$
(2.6)\qquad \frac{1}{n!}\int_{c}^{M}(M-t)^{n}\,d\left(f^{(n)}(t)\right)=f(M)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(M-c)^{k},
$$

then, by (2.4) and (2.6), we deduce the equality (2.1) with the integral representation for the remainder provided by (2.2). □

The following particular instances are of interest for applications:

Corollary 1. *With the assumptions of the above Theorem 5, we have the equalities*

$$
\begin{aligned}
(2.7)\qquad f(A) &= \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(m)(A-m1_{H})^{k} \\
&+ \left[f(M)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(m)(M-m)^{k}\right]1_{H}+T_{n}(f,c,m,M)
\end{aligned}
$$

where

$$
(2.8)\qquad T_{n}(f,m,M):=-\frac{1}{(n-1)!}\int_{m-0}^{M}\left(\int_{m}^{\lambda}(\lambda-t)^{n-1}\,d\left(f^{(n)}(t)\right)\right)E_{\lambda}d\lambda
$$

and

$$
\begin{aligned}
(2.9)\qquad f(A) &= \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left(A-\frac{m+M}{2}1_{H}\right)^{k} \\
&+ \left[f(M)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left(\frac{M-m}{2}\right)^{k}\right]1_{H} \\
&+ W_{n}(f,c,m,M)
\end{aligned}
$$

where

$$
(2.10)\qquad W_{n}(f,m,M):=\frac{(-1)^{n}}{(n-1)!}\int_{m-0}^{M}\left(\int_{\frac{m+M}{2}}^{\lambda}(t-\lambda)^{n-1}\,d\left(f^{(n)}(t)\right)\right)E_{\lambda}d\lambda
$$

and

$$f(A)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(M)(M1_H-A)^k+Y_n(f,c,m,M) \tag{2.11}$$

where

$$Y_n(f,m,M):=\frac{(-1)^{n+1}}{(n-1)!}\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)E_\lambda d\lambda, \tag{2.12}$$

respectively.

Remark 1. *In order to give some examples we use the simplest representation, namely (2.11) for the exponential and the logarithmic functions.*

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then we have the representation

$$\begin{aligned} e^A=&\,e^M\sum_{k=0}^{n}\frac{(-1)^k}{k!}(M1_H-A)^k \\ &+\frac{(-1)^{n+1}}{(n-1)!}\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^{n-1}e^t dt\right)E_\lambda d\lambda. \end{aligned} \tag{2.13}$$

In the case when A is positive definite, i.e., $m>0$, then we have the representation

$$\begin{aligned} \ln A=&\,(\ln M)1_H-\sum_{k=1}^{n}\frac{(M1_H-A)^k}{kM^k} \\ &-n\int_{m-0}^{M}\left(\int_{\lambda}^{M}\frac{(t-\lambda)^{n-1}}{t^{n+1}}dt\right)E_\lambda d\lambda. \end{aligned} \tag{2.14}$$

3. Error Bounds for $f^{(n)}$ of Bounded Variation

We start with the following result that provides an approximation for an n-time differentiable function of selfadjoint operators in Hilbert spaces:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ (the interior of I) and let n be an integer with $n\geq1$. If $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded*

variation on the interval $[m,M]$*, then for any* $c\in[m,M]$ *we have the inequalities*

$$
\begin{aligned}
(3.1)\qquad & \left|\left\langle V_n\left(f,c,m,M\right)x,y\right\rangle\right| \\
& \le \frac{1}{(n-1)!}\int_{m-0}^{c}(c-\lambda)^{n-1}\bigvee_{\lambda}^{c}\left(f^{(n)}\right)\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& + \frac{1}{(n-1)!}\int_{c}^{M}(\lambda-c)^{n-1}\bigvee_{c}^{\lambda}\left(f^{(n)}\right)\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& \le \frac{1}{(n-1)!}\bigvee_{m}^{c}\left(f^{(n)}\right)\int_{m-0}^{c}(c-\lambda)^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& + \frac{1}{(n-1)!}\bigvee_{c}^{M}\left(f^{(n)}\right)\int_{c}^{M}(\lambda-c)^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& \le \frac{1}{(n-1)!}\max\left\{\bigvee_{m}^{c}\left(f^{(n)}\right),\bigvee_{c}^{M}\left(f^{(n)}\right)\right\}\int_{m-0}^{M}\left|\lambda-c\right|^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& \le \frac{1}{n!}\max\left\{\bigvee_{m}^{c}\left(f^{(n)}\right),\bigvee_{c}^{M}\left(f^{(n)}\right)\right\}B_n(c,m,M,x,y),
\end{aligned}
$$

for any $x,y\in H$*, where*

$$
(3.2)\qquad B_n(c,m,M,x,y):=\begin{cases}\left[(M-c)^n+(c-m)^n\right]\|x\|\,\|y\|\,; \\[2mm] C_n(c,m,M,x,y); \\[2mm] n\left[\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right]^{n-1} \\ \times\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(M1_H-A\right)y,y\right\rangle\right]^{1/2}\end{cases}
$$

and

$$
\begin{aligned}
(3.3)\qquad & C_n(c,m,M,x,y) \\
& :=\left[\left\langle\left[(M-c)^n1_H-sgn\left(A-c1_H\right)\left|A-c1_H\right|^n\right]x,x\right\rangle\right]^{1/2} \\
& \times\left[\left\langle\left[(M-c)^n1_H-sgn\left(A-c1_H\right)\left|A-c1_H\right|^n\right]y,y\right\rangle\right]^{1/2}.
\end{aligned}
$$

Here the operator function $sgn\left(A-c1_H\right)\left|A-c1_H\right|^n$ *is generated by the continuous function* $sgn\left(\cdot-c\right)\left|\cdot-c\right|^n$ *defined on the interval* $[m,M]$*.*

Proof. From the identities (2.1) and (2.2) we have

$$\begin{aligned}(3.4)\qquad & \left|\left\langle V_n(f,c,m,M)x,y\right\rangle\right| \\ &= \left|\frac{1}{(n-1)!}\int_{m-0}^{M}\left(\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)\left\langle E_\lambda x,y\right\rangle d\lambda\right| \\ &\leq \frac{1}{(n-1)!}\left|\int_{m-0}^{c}\left(\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)\left\langle E_\lambda x,y\right\rangle d\lambda\right| \\ &+ \frac{1}{(n-1)!}\left|\int_{c}^{M}\left(\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right)\left\langle E_\lambda x,y\right\rangle d\lambda\right| \\ &\leq \frac{1}{(n-1)!}\int_{m-0}^{c}\left|\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right|\left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda \\ &+ \frac{1}{(n-1)!}\int_{c}^{M}\left|\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right|\left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda\end{aligned}$$

for any $x,y\in H$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$(3.5)\qquad \left|\int_a^b p(t)\,dv(t)\right| \leq \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

By the same property (3.5) we have

$$(3.6)\qquad \left|\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right| \leq (c-\lambda)^{n-1}\bigvee_\lambda^c\left(f^{(n)}\right)$$

for $\lambda\in[m,c]$ and

$$(3.7)\qquad \left|\int_c^{\lambda}(t-\lambda)^{n-1}d\left(f^{(n)}(t)\right)\right| \leq (\lambda-c)^{n-1}\bigvee_c^\lambda\left(f^{(n)}\right)$$

for $\lambda\in[c,M]$.

Now, on making use of (3.4) and (3.6)-(3.7) we deduce

$$\begin{aligned}& \left|\left\langle V_n(f,c,m,M)x,y\right\rangle\right| \\ &\leq \frac{1}{(n-1)!}\int_{m-0}^{c}(c-\lambda)^{n-1}\bigvee_\lambda^c\left(f^{(n)}\right)\left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda \\ &+ \frac{1}{(n-1)!}\int_{c}^{M}(\lambda-c)^{n-1}\bigvee_c^\lambda\left(f^{(n)}\right)\left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda\end{aligned}$$

for any $x,y\in H$ which proves the first part of (3.1).

The second and the third inequalities follow by the properties of the integral.

For the last part we observe that

$$\int_{m-0}^{M} |\lambda - c|^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \leq \max_{\lambda\in[m,M]} |\langle E_\lambda x, y\rangle| \int_{m}^{M} |\lambda - c|^{n-1} \, d\lambda$$
$$\leq \frac{1}{n} \|x\| \|y\| [(M-c)^n + (c-m)^n]$$

for any $x, y \in H$,and the proof for the first branch of $B(c, m, M, x, y)$ is complete.

Now, to prove the inequality for the second branch of $B(c, m, M, x, y)$ we use the fact that if P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality that provides a generalization of the Schwarz inequality in H can be stated

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \tag{3.8}$$

for any $x, y \in H$.

If we use (3.8) and the Cauchy-Buniakowski-Schwarz weighted integral inequality we can write that

$$\begin{aligned} &\int_{m-0}^{M} |\lambda - c|^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\ &\leq \int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \, d\lambda \\ &\leq \left(\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle \, d\lambda\right)^{1/2} \left(\int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda y, y\rangle \, d\lambda\right)^{1/2} \end{aligned} \tag{3.9}$$

for any $x, y \in H$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$
\begin{aligned}
&\int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_\lambda x,x\rangle\,d\lambda \qquad (3.10)\\
&=\int_{m-0}^{c}(c-\lambda)^{n-1}\langle E_\lambda x,x\rangle\,d\lambda+\int_{c}^{M}(\lambda-c)^{n-1}\langle E_\lambda x,x\rangle\,d\lambda\\
&=\frac{1}{n}\left[-\int_{m-0}^{c}\langle E_\lambda x,x\rangle\,d(c-\lambda)^n+\int_{c}^{M}\langle E_\lambda x,x\rangle\,d(\lambda-c)^n\right]\\
&=\frac{1}{n}\left[-(c-\lambda)^n\langle E_\lambda x,x\rangle\big|_{m-0}^{c}+\int_{m-0}^{c}(c-\lambda)^n\,d\langle E_\lambda x,x\rangle\right]\\
&+\frac{1}{n}\left[\langle E_\lambda x,x\rangle(\lambda-c)^n\big|_{c}^{M}-\int_{c}^{M}(\lambda-c)^n\,d\langle E_\lambda x,x\rangle\right]\\
&=\frac{1}{n}\int_{m-0}^{c}(c-\lambda)^n\,d\langle E_\lambda x,x\rangle\\
&+\frac{1}{n}\left[\|x\|^2(M-c)^n-\int_{c}^{M}(\lambda-c)^n\,d\langle E_\lambda x,x\rangle\right]\\
&=\frac{1}{n}\|x\|^2(M-c)^n\\
&+\frac{1}{n}\left[\int_{m-0}^{c}(c-\lambda)^n\,d\langle E_\lambda x,x\rangle-\int_{c}^{M}(\lambda-c)^n\,d\langle E_\lambda x,x\rangle\right]\\
&=\frac{1}{n}\left[\|x\|^2(M-c)^n-\int_{m-0}^{M}sgn(\lambda-c)|\lambda-c|^n\,d\langle E_\lambda x,x\rangle\right]\\
&=\frac{1}{n}\left[\left\langle\left[(M-c)^n1_H-sgn(A-c1_H)|A-c1_H|^n\right]x,x\right\rangle\right]
\end{aligned}
$$

for any $x\in H$, and a similar relation for y, namely

$$
\begin{aligned}
&\int_{m-0}^{M}|\lambda-c|^{n-1}\langle E_\lambda y,y\rangle\,d\lambda \qquad (3.11)\\
&=\frac{1}{n}\left[\left\langle\left[(M-c)^n1_H-sgn(A-c1_H)|A-c1_H|^n\right]y,y\right\rangle\right]
\end{aligned}
$$

for any $y\in H$.

The inequality (3.9) and the equalities (3.10) and (3.11) produce the second bound in (3.2).

Finally, observe also that

$$
\begin{aligned}
(3.12)\qquad & \int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle \, d\lambda \\
&= \int_{m-0}^{c} (c-\lambda)^{n-1} \langle E_\lambda x, x\rangle \, d\lambda + \int_{c}^{M} (\lambda - c)^{n-1} \langle E_\lambda x, x\rangle \, d\lambda \\
&\le (c-m)^{n-1} \int_{m-0}^{c} \langle E_\lambda x, x\rangle \, d\lambda + (M-c)^{n-1} \int_{c}^{M} \langle E_\lambda x, x\rangle \, d\lambda \\
&\le \max\left\{ (c-m)^{n-1}, (M-c)^{n-1} \right\} \int_{m-0}^{M} \langle E_\lambda x, x\rangle \, d\lambda \\
&= \left[\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right]^{n-1} \\
&\times \left[\langle E_\lambda x, x\rangle \lambda \Big|_{m-0}^{M} - \int_{m-0}^{M} \lambda d \langle E_\lambda x, x\rangle \right] \\
&= \left[\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right]^{n-1} \langle (M1_H - A) x, x\rangle
\end{aligned}
$$

for any $x \in H$ and similarly,

$$
\begin{aligned}
(3.13)\qquad & \int_{m-0}^{M} |\lambda - c|^{n-1} \langle E_\lambda x, x\rangle \, d\lambda \\
&\le \left[\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right]^{n-1} \langle (M1_H - A) y, y\rangle
\end{aligned}
$$

for any $y \in H$.

On making use of (3.9), (3.12) and (3.13) we deduce the last bound provided in (3.2). □

The following particular cases are of interest for applications

Corollary 2. *With the assumption of Theorem 6 we have the inequalities*

$$
\begin{aligned}
(3.14)\qquad & |\langle T_n(f, m, M) x, y\rangle| \\
&\le \frac{1}{(n-1)!} \int_{m-0}^{M} (\lambda - m)^{n-1} \bigvee_{m}^{\lambda} \left(f^{(n)} \right) |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\le \frac{1}{(n-1)!} \bigvee_{m}^{M} \left(f^{(n)} \right) \int_{m-0}^{M} (\lambda - m)^{n-1} |\langle E_\lambda x, y\rangle| \, d\lambda \\
&\le \frac{1}{n!} \bigvee_{m}^{M} \left(f^{(n)} \right) B_n(m, M, x, y),
\end{aligned}
$$

for any $x, y \in H$, where

$$B_n(m, M, x, y) \tag{3.15}$$

$$:= \begin{cases} (M-m)^n \|x\| \|y\|; \\ C_n(m, M, x, y); \\ n(M-m)^{n-1} \left[\langle (M1_H - A)\, x, x \rangle \langle (M1_H - A)\, y, y \rangle \right]^{1/2} \end{cases}$$

and

$$C_n(m, M, x, y) := \left[\langle [(M-m)^n 1_H - (A - m1_H)^n]\, x, x \rangle \right]^{1/2} \times \left[\langle [(M-m)^n 1_H - (A - m1_H)^n]\, y, y \rangle \right]^{1/2}. \tag{3.16}$$

The proof follows from Theorem 6 by choosing $c = m$ and performing the corresponding calculations.

Corollary 3. *With the assumption of Theorem 6 we have the inequalities*

$$\begin{aligned} &|\langle Y_n(f, m, M)\, x, y \rangle| \\ &\le \frac{1}{(n-1)!} \int_{m-0}^{M} (M-\lambda)^{n-1} \bigvee_{\lambda}^{M} \left(f^{(n)} \right) |\langle E_\lambda x, y \rangle|\, d\lambda \\ &\le \frac{1}{(n-1)!} \bigvee_{m}^{M} \left(f^{(n)} \right) \int_{m-0}^{M} (M-\lambda)^{n-1} |\langle E_\lambda x, y \rangle|\, d\lambda \\ &\le \frac{1}{n!} \bigvee_{m}^{M} \left(f^{(n)} \right) \tilde{B}_n(m, M, x, y), \end{aligned} \tag{3.17}$$

for any $x, y \in H$, where

$$\tilde{B}_n(m, M, x, y) \tag{3.18}$$

$$:= \begin{cases} (M-m)^n \|x\| \|y\|; \\ \tilde{C}_n(m, M, x, y); \\ n(M-m)^{n-1} \left[\langle (M1_H - A)\, x, x \rangle \langle (M1_H - A)\, y, y \rangle \right]^{1/2} \end{cases}$$

and

$$\tilde{C}_n(m, M, x, y) := \left[\langle (M1_H - A)^n\, x, x \rangle \right]^{1/2} \left[\langle (M1_H - A)^n\, y, y \rangle \right]^{1/2}. \tag{3.19}$$

The proof follows from Theorem 6 by choosing $c = M$ and performing the corresponding calculations.

The best bound we can get is incorporated in

Corollary 4. *With the assumption of Theorem 6 we have the inequalities*

$$(3.20)\quad \begin{aligned} &|\langle W_n(f,m,M)x,y\rangle| \\ &\leq \frac{1}{(n-1)!}\int_{m-0}^{\frac{m+M}{2}}\left(\frac{m+M}{2}-\lambda\right)^{n-1}\bigvee_{\lambda}^{\frac{m+M}{2}}\left(f^{(n)}\right)|\langle E_\lambda x,y\rangle|\,d\lambda \\ &+\frac{1}{(n-1)!}\int_{\frac{m+M}{2}}^{M}\left(\lambda-\frac{m+M}{2}\right)^{n-1}\bigvee_{\frac{m+M}{2}}^{\lambda}\left(f^{(n)}\right)|\langle E_\lambda x,y\rangle|\,d\lambda \\ &\leq \frac{1}{(n-1)!}\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right)\int_{m-0}^{\frac{m+M}{2}}\left(\frac{m+M}{2}-\lambda\right)^{n-1}|\langle E_\lambda x,y\rangle|\,d\lambda \\ &+\frac{1}{(n-1)!}\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right)\int_{\frac{m+M}{2}}^{M}\left(\lambda-\frac{m+M}{2}\right)^{n-1}|\langle E_\lambda x,y\rangle|\,d\lambda \\ &\leq \frac{1}{(n-1)!}\max\left\{\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right),\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right)\right\} \\ &\times\int_{m-0}^{M}\left|\lambda-\frac{m+M}{2}\right|^{n-1}|\langle E_\lambda x,y\rangle|\,d\lambda \\ &\leq \frac{1}{n!}\max\left\{\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right),\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right)\right\}\breve{B}_n(m,M,x,y), \end{aligned}$$

for any $x,y\in H$, *where*

$$(3.21)\quad \begin{aligned} &\breve{B}_n(m,M,x,y) \\ &:=\begin{cases}\frac{(M-m)^n}{2^{n-1}}\|x\|\|y\|; \\ \\ \breve{C}(m,M,x,y) \\ \\ \frac{n}{2^{n-1}}(M-m)^{n-1}\left[\langle(M1_H-A)x,x\rangle\langle(M1_H-A)y,y\rangle\right]^{1/2}\end{cases}\end{aligned}$$

and

$$(3.22)\quad \begin{aligned} &\breve{C}_n(m,M,x,y) \\ &:=\left[\left\langle\left[\frac{(M-m)^n}{2^n}1_H-sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^n\right]x,x\right\rangle\right]^{1/2} \\ &\times\left[\left\langle\left[\frac{(M-m)^n}{2^n}1_H-sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^n\right]y,y\right\rangle\right]^{1/2}.\end{aligned}$$

4. Error Bounds for $f^{(n)}$ Lipschitzian

The case when the n-th derivative is Lipschitzian is incorporated in the following result:

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family,*

I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset \mathring{I}$ (the interior of I) and let n be an integer with $n\geq 1$. If $f:I\rightarrow \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n>0$ on the interval $[m,M]$, then for any $c\in [m,M]$ we have the inequalities

$$
\begin{aligned}
(4.1)\qquad & \left|\left\langle V_n\left(f,c,m,M\right)x,y\right\rangle\right| \\
& \leq \frac{1}{n!}L_n\int_{m-0}^{M}\left|\lambda-c\right|^{n}\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& \leq \frac{1}{(n+1)!}L_n \\
& \times\begin{cases}
\left[(M-c)^{n+1}+(c-m)^{n+1}\right]\left\|x\right\|\left\|y\right\|; \\[2ex]
\left[\left\langle\left[(M-c)^{n+1}1_H-sgn\left(A-c1_H\right)\left|A-c1_H\right|^{n+1}\right]x,x\right\rangle\right]^{1/2} \\
\times\left[\left\langle\left[(M-c)^{n+1}1_H-sgn\left(A-c1_H\right)\left|A-c1_H\right|^{n+1}\right]y,y\right\rangle\right]^{1/2}; \\[2ex]
(n+1)\left[\frac{1}{2}\left(M-m\right)+\left|c-\frac{m+M}{2}\right|\right]^{n} \\
\times\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(M1_H-A\right)y,y\right\rangle\right]^{1/2};
\end{cases}
\end{aligned}
$$

for any $x,y\in H$.

Proof. From the inequality (3.4) in the proof of Theorem 6 we have

$$
\begin{aligned}
(4.2)\qquad & \left|\left\langle V_n\left(f,c,m,M\right)x,y\right\rangle\right| \\
& \leq \frac{1}{(n-1)!}\int_{m-0}^{c}\left|\int_{\lambda}^{c}\left(t-\lambda\right)^{n-1}d\left(f^{(n)}\left(t\right)\right)\right|\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda \\
& +\frac{1}{(n-1)!}\int_{c}^{M}\left|\int_{c}^{\lambda}\left(t-\lambda\right)^{n-1}d\left(f^{(n)}\left(t\right)\right)\right|\left|\left\langle E_\lambda x,y\right\rangle\right|d\lambda
\end{aligned}
$$

for any $x,y\in H$.

Further, we utilize the fact that for an $L-$Lipschitzian function, $p:[\alpha,\beta]\rightarrow\mathbb{C}$ and a Riemann integrable function $v:[\alpha,\beta]\rightarrow\mathbb{C}$, the Riemann-Stieltjes integral $\int_{\alpha}^{\beta}p\left(s\right)dv\left(s\right)$ exists and

$$
\left|\int_{\alpha}^{\beta}p\left(s\right)dv\left(s\right)\right|\leq L\int_{\alpha}^{\beta}\left|p\left(s\right)\right|ds.
$$

On making use of this property we have for $\lambda\in[m,c]$ that

$$
\left|\int_{\lambda}^{c}\left(t-\lambda\right)^{n-1}d\left(f^{(n)}\left(t\right)\right)\right|\leq L_n\int_{\lambda}^{c}\left(t-\lambda\right)^{n-1}dt=\frac{1}{n}L_n\left(c-\lambda\right)^{n}
$$

and for $\lambda\in[c,M]$ that

$$
\left|\int_{c}^{\lambda}\left(t-\lambda\right)^{n-1}d\left(f^{(n)}\left(t\right)\right)\right|\leq L_n\int_{c}^{\lambda}\left(\lambda-t\right)^{n-1}dt=\frac{1}{n}L_n\left(\lambda-c\right)^{n}
$$

which, by (4.2) produces the inequality

$$
\begin{aligned}
(4.3)\qquad & \left|\left\langle V_n\left(f, c, m, M\right) x, y\right\rangle\right| \\
& \leq \frac{1}{n!} L_n \int_{m-0}^{c}(c-\lambda)^n\left|\left\langle E_\lambda x, y\right\rangle\right| d \lambda+\frac{1}{n!} L_n \int_c^M(\lambda-c)^n\left|\left\langle E_\lambda x, y\right\rangle\right| d \lambda \\
& =\frac{1}{n!} L_n \int_{m-0}^M|\lambda-c|^n\left|\left\langle E_\lambda x, y\right\rangle\right| d \lambda,
\end{aligned}
$$

for any $x, y \in H$, and the first part of (4.1) is proved.

Finally, we observe that the bounds for the integral $\int_{m-0}^M|\lambda-c|^n\left|\left\langle E_\lambda x, y\right\rangle\right| d \lambda$ can be obtained in a similar manner to the ones from the proof of Theorem 6 and the details are omitted. □

The following result contains error bounds for the particular expansions considered in Corollary 1:

Corollary 5. *With the assumptions in Theorem 7 we have the inequalities*

$$
\begin{aligned}
(4.4)\qquad & \left|\left\langle T_n\left(f, m, M\right) x, y\right\rangle\right| \\
& \leq \frac{1}{n!} L_n \int_{m-0}^M(\lambda-m)^n\left|\left\langle E_\lambda x, y\right\rangle\right| d \lambda \\
& \leq \frac{1}{(n+1)!} L_n \\
& \times\begin{cases}
(M-m)^{n+1}\|x\|\|y\| ; \\
\left[\left\langle\left[(M-m)^{n+1} 1_H-\left(A-m 1_H\right)^{n+1}\right] x, x\right\rangle\right]^{1/2} \\
\times\left[\left\langle\left[(M-m)^{n+1} 1_H-\left(A-m 1_H\right)^{n+1}\right] y, y\right\rangle\right]^{1/2} ; \\
(n+1)(M-m)^n\left[\left\langle\left(M 1_H-A\right) x, x\right\rangle\left\langle\left(M 1_H-A\right) y, y\right\rangle\right]^{1/2} ;
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.5)\qquad & \left|\left\langle Y_n\left(f, m, M\right) x, y\right\rangle\right| \\
& \leq \frac{1}{n!} L_n \int_{m-0}^M(M-\lambda)^n\left|\left\langle E_\lambda x, y\right\rangle\right| d \lambda \\
& \leq \frac{1}{(n+1)!} L_n \\
& \times\begin{cases}
(M-m)^{n+1}\|x\|\|y\| ; \\
\left[\left\langle\left[\left(M 1_H-A\right)^{n+1}\right] x, x\right\rangle\right]^{1/2}\left[\left\langle\left[\left(M 1_H-A\right)^{n+1}\right] y, y\right\rangle\right]^{1/2} ; \\
(n+1)\left[(M-m)\right]^n\left[\left\langle\left(M 1_H-A\right) x, x\right\rangle\left\langle\left(M 1_H-A\right) y, y\right\rangle\right]^{1/2} ;
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.6)\quad & \left|\left\langle W_n\left(f,m,M\right)x,y\right\rangle\right| \\
& \leq \frac{1}{n!}L_n \int_{m-0}^{M}\left|\lambda-\frac{m+M}{2}\right|^n \left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda \leq \frac{1}{(n+1)!}L_n
\end{aligned}
$$

$$
\times\begin{cases}
\frac{(M-m)^{n+1}}{2^n}\left\|x\right\|\left\|y\right\|; \\
\left[\left\langle\left[\frac{(M-m)^{n+1}}{2^n}1_H - sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^{n+1}\right]x,x\right\rangle\right]^{1/2} \\
\times\left[\left\langle\left[\frac{(M-m)^{n+1}}{2^n}1_H - sgn\left(A-\frac{m+M}{2}1_H\right)\left|A-\frac{m+M}{2}1_H\right|^{n+1}\right]y,y\right\rangle\right]^{1/2}; \\
\frac{n+1}{2^n}\left(M-m\right)^n\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(M1_H-A\right)y,y\right\rangle\right]^{1/2};
\end{cases}
$$

for any $x,y\in H$, *respectively.*

5. Applications

In order to obtain various vectorial operator inequalities one can use the above results for particular elementary functions. We restrict ourself to only two examples of functions, namely the exponential and the logarithmic functions.

If we apply Corollary 3 for the exponential function, we can state the following result:

Proposition 1. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$ *and* $\{E_\lambda\}_\lambda$ *be its spectral family. Then we have*

$$
\begin{aligned}
(5.1)\quad & \left|\left\langle e^A x,y\right\rangle - e^M\sum_{k=0}^{n}\frac{(-1)^k}{k!}\left\langle\left(M1_H-A\right)^k x,y\right\rangle\right| \\
& \leq \frac{1}{(n-1)!}\int_{m-0}^{M}\left(M-\lambda\right)^{n-1}\left(e^M-e^\lambda\right)\left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda \\
& \leq \frac{1}{(n-1)!}\left(e^M-e^m\right)\int_{m-0}^{M}\left(M-\lambda\right)^{n-1}\left|\left\langle E_\lambda x,y\right\rangle\right| d\lambda \\
& \leq \frac{1}{n!}\left(e^M-e^m\right) \\
& \quad\times\begin{cases}
\left(M-m\right)^n\left\|x\right\|\left\|y\right\|; \\
\left[\left\langle\left(M1_H-A\right)^n x,x\right\rangle\right]^{1/2}\left[\left\langle\left(M1_H-A\right)^n y,y\right\rangle\right]^{1/2} \\
n\left(M-m\right)^{n-1}\left[\left\langle\left(M1_H-A\right)x,x\right\rangle\left\langle\left(M1_H-A\right)y,y\right\rangle\right]^{1/2}
\end{cases}
\end{aligned}
$$

for any $x,y\in H$.

If we use Corollary 5 then we can provide other bounds as follows:

Proposition 2. *With the assumptions of Proposition 1 we have*

$$
\begin{aligned}
(5.2)\qquad & \left|\left\langle e^{A}x,y\right\rangle - e^{M}\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}\left\langle (M1_{H}-A)^{k}x,y\right\rangle\right| \\
&\leq \frac{1}{n!}e^{M}\int_{m-0}^{M}(M-\lambda)^{n}\left|\langle E_{\lambda}x,y\rangle\right|d\lambda \\
&\leq \frac{1}{(n+1)!}e^{M} \\
&\times\begin{cases}
(M-m)^{n+1}\left\|x\right\|\left\|y\right\|; \\[2ex]
\left[\left\langle\left[(M1_{H}-A)^{n+1}\right]x,x\right\rangle\right]^{1/2}\left[\left\langle\left[(M1_{H}-A)^{n+1}\right]y,y\right\rangle\right]^{1/2}; \\[2ex]
(n+1)\left[(M-m)\right]^{n}\left[\langle (M1_{H}-A)x,x\rangle\langle (M1_{H}-A)y,y\rangle\right]^{1/2};
\end{cases}
\end{aligned}
$$

Finally, the Corollaries 3 and 5 produce the following results for the logarithmic function:

Proposition 3. *Let A be a positive definite operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]\subset(0,\infty)$ and $\{E_{\lambda}\}_{\lambda}$ be its spectral family, then*

$$
\begin{aligned}
(5.3)\qquad & \left|\langle \ln Ax,y\rangle - \langle x,y\rangle\ln M + \sum_{k=1}^{n}\frac{\left\langle (M1_{H}-A)^{k}x,y\right\rangle}{kM^{k}}\right| \\
&\leq \int_{m-0}^{M}(M-\lambda)^{n-1}\frac{M^{n}-\lambda^{n}}{M^{n}\lambda^{n}}\left|\langle E_{\lambda}x,y\rangle\right|d\lambda \\
&\leq \frac{M^{n}-m^{n}}{M^{n}m^{n}}\int_{m-0}^{M}(M-\lambda)^{n-1}\left|\langle E_{\lambda}x,y\rangle\right|d\lambda \\
&\leq \frac{M^{n}-m^{n}}{nM^{n}m^{n}} \\
&\times\begin{cases}
(M-m)^{n}\left\|x\right\|\left\|y\right\|; \\[2ex]
\left[\langle (M1_{H}-A)^{n}x,x\rangle\right]^{1/2}\left[\langle (M1_{H}-A)^{n}y,y\rangle\right]^{1/2} \\[2ex]
n(M-m)^{n-1}\left[\langle (M1_{H}-A)x,x\rangle\langle (M1_{H}-A)y,y\rangle\right]^{1/2}
\end{cases}
\end{aligned}
$$

and

$$\left|\langle \ln Ax, y\rangle - \langle x, y\rangle \ln M + \sum_{k=1}^{n} \frac{\left\langle (M1_H - A)^k x, y\right\rangle}{kM^k}\right| \tag{5.4}$$

$$\leq \frac{1}{m^{n+1}} \int_{m-0}^{M} (M-\lambda)^n \left|\langle E_\lambda x, y\rangle\right| d\lambda$$

$$\leq \frac{1}{(n+1)\, m^{n+1}}$$

$$\times \begin{cases} (M-m)^{n+1} \|x\| \|y\| ; \\ \\ \left[\left\langle (M1_H - A)^{n+1} x, x\right\rangle\right]^{1/2} \left[\left\langle (M1_H - A)^{n+1} y, y\right\rangle\right]^{1/2}; \\ \\ (n+1)\left[(M-m)\right]^n \left[\langle (M1_H - A)\, x, x\rangle \langle (M1_H - A)\, y, y\rangle\right]^{1/2}; \end{cases}$$

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