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GENERALIZATION OF SĂLĂGEAN OPERATOR FOR A CERTAIN SUBCLASS OF UNIVALENT FUNCTIONS

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Abstract. For analytic functions f in the open unit disc U, a generalization operator $D^{\lambda} f(z)^{\alpha}$ of Sălăgean operator is introduced. Some properties for $D^{\lambda} f(z)^{\alpha}$ are discussed in the present paper.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disk U by S.

The following classes of functions are well known and have been studied repeatedly by many authors, namely Sălăgean[**6**], Abdulhalim [**1**], [**2**], Opoola [**7**], Babalola and Opoola [**3**], Darus [**4**] among others.

(i) $S_0 = \left\{ f(z) \in A : \mathrm{Re}\left(\frac{f(z)}{z}\right) > 0,\ z \in U \right\}$,

(ii) $B(\beta) = \left\{ f(z) \in A : \mathrm{Re}\left(\frac{f(z)}{z}\right) > \beta,\ 0 \le \beta < 1,\ z \in U \right\}$,

(iii) $\delta(\beta) = \{ f(z) \in A : \mathrm{Re}\,(f'(z)) > \beta, 0 \le \beta < 1,\ z \in U \}$,

(iv) $B_n(\beta) = \left\{ f(z) \in A : \mathrm{Re}\left(\frac{D^n f(z)^{\beta}}{z^{\beta}}\right) > 0,\ \beta > 0,\ z \in U \right\}$.

In 1994, Opoola [**7**] defined the class $T_n^{\alpha}(\beta)$ to be a subclass of A consisting of analytic functions satisfying the condition

$$\mathrm{Re}\left\{\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right\} > \beta,\ z \in U, n = 0, 1, 2, ..., \alpha > 0, 0 \le \beta < 1, \tag{1.2}$$

where D^n is the Sălăgean differential operator [**6**] defined as follows:

(i) $D^0 f(z) = f(z)$,

(ii) $D^1 f(z) = Df(z) = zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k$,

(iii) $D^n f(z) = D\left(D^{n-1} f(z)\right) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $(n = 1, 2, 3...)$.

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Now, by binomial expansion, we have

$$f(z)^{\alpha} = z^{\alpha} + \sum_{j=1}^{\infty} \alpha_j \left(a_2 z + a_3 z^2 + ...\right)^j,$$

where $\alpha_j = \binom{\alpha}{j} = \dfrac{\alpha!}{j!\,(\alpha - j)!}$.

Then, we can write

$$f(z)^{\alpha} = z^{\alpha} + \sum_{j=1}^{\infty} A_k(\alpha)\, z^{\alpha+k-1},$$

where $A_k(\alpha), k = 2, 3, ...$ depends on the coefficients a_k of $f(z)$ given by (1.1) and the index $\alpha, \alpha > 0$.

In view of the Sălăgean operator, we introduce

$$D^{\lambda} f(z)^{\alpha} = \alpha^{\lambda} \left(z^{\alpha} + \sum_{j=1}^{\infty} A_k(\alpha) \left(\frac{\alpha + k - 1}{\alpha}\right)^{\lambda} z^{\alpha+k-1} \right) \tag{1.3}$$

for $f \in A$. Then for any real $\lambda \in \mathbb{R}$ we see that

$$\begin{aligned} D^{\lambda+1} f(z)^{\alpha} &= \alpha^{\lambda+1} \left(z^{\alpha} + \sum_{j=1}^{\infty} A_k(\alpha) \left(\frac{\alpha + k - 1}{\alpha}\right)^{\lambda+1} z^{\alpha+k-1} \right) \\ &= z\left(D^{\lambda} f(z)^{\alpha}\right)' \end{aligned}$$

and

$$\begin{aligned} D^{\lambda-1} f(z)^{\alpha} &= \alpha^{\lambda-1} \left(z^{\alpha} + \sum_{j=1}^{\infty} A_k(\alpha) \left(\frac{\alpha + k - 1}{\alpha}\right)^{\lambda-1} z^{\alpha+k-1} \right) \\ &= \int_0^z \frac{D^{\lambda} f(t)^{\alpha}}{t} dt. \end{aligned}$$

It is easy to see that

$$D^{\lambda_1+\lambda_2} f(z)^{\alpha} = D^{\lambda_1}\left(D^{\lambda_2} f(z)^{\alpha}\right) = D^{\lambda_2}\left(D^{\lambda_1} f(z)^{\alpha}\right)$$

for any real λ_1 and λ_2.

2. Properties of the operator $D^{\lambda} f(z)^{\alpha}$

To discuss our new problem, we have to recall the following lemmas.

LEMMA 1. [**5**], *Let $w(z)$ be non-constant and analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0 \in U$, then we have $z_0 w(z_0)' = k w(z_0)$ where $k \geq 1$.*

LEMMA 2. [**3**] *Let $f \in A$ and $\zeta > 0$ be real. If for $z \in U$, $\dfrac{D^{n+1} f(z)^{\zeta}}{D^n f(z)^{\zeta}}$ is independent of n, then*

$$\frac{D^{n+1} f(z)^{\zeta}}{D^n f(z)^{\zeta}} = \zeta \frac{D^{n+1} f(z)}{D^n f(z)}.$$

Our first result for the operator $D^{\lambda} f(z)^{\alpha}$ is contained in the following theorem.

THEOREM 1. *If $f \in A$ satisfies*

$$\left|\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}-\alpha\right|^{\mu}\left|z\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)'\right|^{\beta}<\left(\frac{1}{2\alpha}\right)^{\beta}\qquad(z\in U)$$

for some real α, β, μ with $\alpha>0$, $\mu+2\beta\geq 0$ and for any real λ, then

$$\mathrm{Re}\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)>0\qquad(z\in U).$$

PROOF. Let us defined $w(z)$ by

$$\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}=\frac{1+w(z)}{1-w(z)}\qquad(w(z)\neq 1).$$

Then, $w(z)$ is analytic in U and $w(0)=0$. Thus, by Lemma (2), we can write

$$\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}=\alpha\left(\frac{1+w(z)}{1-w(z)}\right)\qquad(w(z)\neq 1).$$

Since

$$\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}-\alpha=\frac{2\alpha w(z)}{1-w(z)}$$

and

$$\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)'=\frac{2\alpha w'(z)}{(1-w(z))^{2}},$$

we obtain that

$$\left|\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}-\alpha\right|^{\mu}\left|z\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)'\right|^{\beta}$$

$$=\left|\frac{2\alpha w(z)}{1-w(z)}\right|^{\mu}\left|\frac{2\alpha z w(z)'}{(1-w(z))^{2}}\right|^{\beta}<\left(\frac{1}{2\alpha}\right)^{\beta}$$

for all $z\in U$.

If there exists a point $z_0\in U$ such that $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$, then Lemma 1 gives us that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=ke^{i\theta}$, $k\geq 1$.

This implies that

$$\left|\frac{D^{\lambda+1}f(z_0)^{\alpha}}{D^{\lambda}f(z_0)^{\alpha}}-\alpha\right|^{\mu}\left|z_0\left(\frac{D^{\lambda+1}f(z_0)^{\alpha}}{D^{\lambda}f(z_0)^{\alpha}}\right)'\right|^{\beta}$$

$$=\left|\frac{2\alpha e^{i\theta}}{1-e^{i\theta}}\right|^{\mu}\left|\frac{2\alpha k e^{i\theta}}{(1-e^{i\theta})^{2}}\right|^{\beta}=\frac{2^{\mu+\beta}\alpha^{\mu+\beta}k^{\beta}}{|1-e^{i\theta}|^{\alpha+2\beta}}$$

$$\geq\left(\frac{k}{2\alpha}\right)^{\beta}\geq\left(\frac{1}{2\alpha}\right)^{\beta}$$

for all $z\in U$, which contradicts the condition of the theorem. This show that there is no $z_0\in U$ such that $|w(z_0)|=1$. Therefore $|w(z)|<1$ for all $z\in U$ which implies that

$$\mathrm{Re}\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)>0\qquad(z\in U).$$

This completes the proof of the Theorem 1. □

Next, we prove the following theorem.

THEOREM 2. *If* $f \in A$ *satisfies*

$$\left|\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}-\alpha\right|^{\mu}\left|z\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)'\right|^{\beta}$$

$$<\left(\frac{1}{2\alpha}\right)^{\beta}(1-\gamma)^{\mu+\beta} \qquad (z\in U)$$

for some real $\alpha, \beta, \mu, \gamma$ *with* $\alpha>0$, $\mu+2\beta\geq 0$ *and* $0\leq\gamma<1$, *then*

$$\operatorname{Re}\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)>\gamma \qquad (z\in U).$$

PROOF. Defining the function $w(z)$ by

$$\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}=\alpha\left(\frac{1+(1-2\gamma)w(z)}{1-w(z)}\right) \qquad (w(z)\neq 1),$$

we see that $w(z)$ is analytic in U and $w(0)=0$.

Note that

$$\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}-\alpha=\frac{2\alpha(1-\gamma)w(z)}{1-w(z)}$$

and

$$z\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)'=\frac{2\alpha(1-\gamma)zw'(z)}{(1-w(z))^{2}}.$$

Thus, we have that

$$\left|\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}-\alpha\right|^{\mu}\left|z\left(\frac{D^{\lambda+1}f(z)^{\alpha}}{D^{\lambda}f(z)^{\alpha}}\right)'\right|^{\beta}$$

$$=\left|\frac{2\alpha(1-\gamma)w(z)}{1-w(z)}\right|^{\mu}\left|\frac{2\alpha(1-\gamma)zw'(z)}{(1-w(z))^{2}}\right|^{\beta}$$

$$\leq\left(\frac{1}{2\alpha}\right)^{\beta}(1-\gamma)^{\mu+\beta} \qquad (z\in U).$$

If there exists a point $z_0\in U$ such that $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$, then $w(z)$ satisfies $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=ke^{i\theta}$ $(k\geq 1)$ by Lemma 1.

This gives us that

$$\left|\frac{D^{\lambda+1}f(z_0)^{\alpha}}{D^{\lambda}f(z_0)^{\alpha}}-\alpha\right|^{\mu}\left|z_0\left(\frac{D^{\lambda+1}f(z_0)^{\alpha}}{D^{\lambda}f(z_0)^{\alpha}}\right)'\right|^{\beta}$$

$$=\left|\frac{2\alpha(1-\gamma)e^{i\theta}}{1-e^{i\theta}}\right|^{\mu}\left|\frac{2\alpha(1-\gamma)ke^{i\theta}}{(1-e^{i\theta})^{2}}\right|^{\beta}$$

$$\geq\frac{2^{\mu+\beta}\alpha^{\mu+\beta}k^{\beta}(1-\gamma)^{\mu+\beta}}{|1-e^{i\theta}|^{\mu+2\beta}}\geq\left(\frac{k}{2\alpha}\right)^{\beta}(1-\gamma)^{\mu+\beta}$$

$$\geq\left(\frac{1}{2\alpha}\right)^{\beta}(1-\gamma)^{\alpha+\beta} \qquad (z\in U)$$

which contradicts the condition of the theorem. This show that there is no $z_0 \in U$ such that $|w(z_0)| = 1$. Therefore $|w(z)| < 1$ for all $z \in U$.

Thus we conclude that

$$\operatorname{Re}\left(\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}}\right) > \gamma$$

for $z \in U$, $\alpha > 0$, $0 \leq \gamma < 1$. □

Finally, we derive the following theorem:

THEOREM 3. *If $f \in A$ satisfies*

$$\left|\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}} - \alpha^{\gamma}\right|^{\mu} \left|z\left(\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}}\right)'\right|^{\beta} < \left(\frac{\alpha\gamma}{2}\right)^{\beta} \qquad (z \in U)$$

for some real α, β, μ and $\gamma = \dfrac{\beta}{\mu+\beta}$, then

$$\operatorname{Re}\left(\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}}\right)^{\frac{1}{\gamma}} > 0 \qquad (z \in U).$$

PROOF. Defining the function $w(z)$ by

$$\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}} = \alpha^{\gamma}\left(\frac{1+w(z)}{1-w(z)}\right)^{\gamma} \qquad (w(z) \neq 1),$$

with $\gamma = \dfrac{\beta}{\mu+\beta}$, we see that $w(z)$ is analytic in U and $w(0) = 0$.

Noting that

$$z\left(\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}}\right)' = \frac{2\gamma\alpha^{\gamma} z w'(z)}{(1-w(z))^2}\left(\frac{1+w(z)}{1-w(z)}\right)^{\gamma-1},$$

and then, we have that

$$\begin{aligned}
&\left|\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}} - \alpha^{\gamma}\right|^{\mu} \left|z\left(\frac{D^{\lambda+1} f(z)^{\alpha}}{D^{\lambda} f(z)^{\alpha}}\right)'\right|^{\beta} \\
&= \alpha^{\mu\gamma}\left|\frac{1+w(z)}{1-w(z)}\right|^{\mu\gamma+\beta(\gamma-1)} \left|\frac{2\gamma\alpha^{\gamma} z w'(z)}{(1-w(z))^2}\right|^{\beta} \\
&= \gamma^{\beta}\alpha^{\mu\gamma+\beta\gamma}\left|\frac{2zw'(z)}{(1-w(z))^2}\right|^{\beta} \leq \left(\frac{\alpha\gamma}{2}\right)^{\beta} \qquad (z \in U),
\end{aligned}$$

since $\gamma = \dfrac{\beta}{\mu+\beta}$.

Now suppose that exists a point $z_0 \in U$ such that $\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$. Then, by Lemma 1, we have that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = ke^{i\theta}$ $(k \geq 1)$.

This gives us that

$$\begin{aligned}
&\left|\frac{D^{\lambda+1}f(z_0)^\alpha}{D^\lambda f(z_0)^\alpha}-\alpha^\gamma\right|^\mu \left|z_0\left(\frac{D^{\lambda+1}f(z_0)^\alpha}{D^\lambda f(z_0)^\alpha}\right)'\right|^\beta \\
&=\gamma^\beta\alpha^{\mu\gamma+\beta\gamma}\left|\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right|^\beta \geq \frac{2^\beta\gamma^\beta\alpha^\beta k^\beta}{|(1-e^{i\theta})|^{2\beta}} \\
&\geq \left(\frac{k\alpha\gamma}{2}\right)^\beta \geq \left(\frac{\alpha\gamma}{2}\right)^\beta \qquad (z\in U)
\end{aligned}$$

which contradicts the condition of the theorem. This show that there is no $z_0 \in U$ such that $|w(z_0)| = 1$. Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, that is, that

$$\operatorname{Re}\left(\frac{D^{\lambda+1}f(z_0)^\alpha}{D^\lambda f(z_0)^\alpha}\right)^{\frac{1}{\gamma}} > 0 \qquad (z\in U).$$

□

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