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INEQUALITIES FOR POWER SERIES WITH POSITIVE COEFFICIENTS

Alawiah Ibrahim[1,3], S. S. Dragomir[1,2], P. Cerone[1], and Maslina Darus[3]

Abstract. Using some results for convex functions such as the Jensen and one of its reverses due to Dragomir and Ionescu, we derive new inequalities for functions defined by real power series with positive coefficients. Applications for particular functions of interest are provided as well.

1. Introduction

Let X be a real linear space, C be a convex subset of X. The function $f : C \subset X \to \mathbb{R}$ is called convex if for all $x, y \in C$, $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

The following inequality is well known in the literature as the Jensen inequality (see [**9**], [**10**], [**11**]):

$$f\left(\sum_{i=1}^{k} q_i x_i\right) \leq \sum_{i=1}^{k} q_i f(x_i) \tag{1.1}$$

which holds for all convex function $f : C \subset X \to \mathbb{R}$, $x_i \in I$ and q_i are probabilities, namely $q_i \geq 0$, $\sum_{i=1}^{k} q_i = 1$, $i \in \{1, 2, ..., k\}$.

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities. Since its discovery in 1906 [**9**], Jensen's inequality has been proven to be one of the most useful inequalities in mathematical analysis, because it implies many of the other classical inequalities [**8**]. For instance, the weighted arithmetic mean-geometric mean inequality, Cauchy's inequality, the Ky Fan, Hölder, Young and Minkowski inequalities, etc. can be obtained as particular cases of Jensen's inequality (1.1).

Also, the following weighted version of Jensen's inequality holds:

$$f\left(\frac{\sum_{i=1}^{k} w_i x_i}{\sum_{i=1}^{k} w_i}\right) \leq \frac{\sum_{i=1}^{k} w_i f(x_i)}{\sum_{i=1}^{k} w_i}, \tag{1.2}$$

where $x_i \in C$, $w_i \geq 0$, $i \in \{1, 2, \ldots, k\}$ and $\sum_{i=1}^{k} w_i > 0$.

If the function f is concave, then the inequality (1.2) reverses.

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In 1994, Dragomir and Ionescu [**2**] proved the following reverse of the Jensen's inequality, namely

$$(1.3)\qquad (0\leq)\frac{\sum_{i=1}^{k}w_{i}g\left(x_{i}\right)}{\sum_{i=1}^{k}w_{i}}-g\left(\frac{\sum_{i=1}^{k}w_{i}x_{i}}{\sum_{i=1}^{k}w_{i}}\right)\leq\frac{\sum_{i=1}^{k}w_{i}x_{i}g'\left(x_{i}\right)}{\sum_{i=1}^{k}w_{i}}-\frac{\sum_{i=1}^{k}w_{i}x_{i}}{\sum_{i=1}^{k}w_{i}}\cdot\frac{\sum_{i=1}^{k}w_{i}g'\left(x_{i}\right)}{\sum_{i=1}^{k}w_{i}}$$

provided $g:I\to\mathbb{R}$ is differentiable convex on the interval I, $x_i\in I$, $\mathrm{w}_i\geq 0$, $i\in\{1,\ldots,k\}$ and $\sum_{i=1}^{k}w_i>0$.

For other generalizations, refinements and applications of Jensen's inequality, see [**3**], [**4**], [**5**], [**6**], [**7**], [**8**] and the references cited therein.

Utilizing the celebrated Jensen's inequality (1.2) and its reverse (1.3) for particular real convex functions, we establish in this paper some interesting inequalities for functions f defined by power series $\sum_{n=0}^{\infty}a_nx^n$ with positive coefficients a_n. Some applications for particular functions of interest are also presented.

2. Some Results Via Convexity

First, we state the following results.

Theorem 1. *Assume that* $f(x)=\sum_{n=0}^{\infty}p_nx^n$ *is a function with nonnegative coefficients defined on* $(-R,R)$, $R>0$. *If* $a,b,c>0$ *are such that* $ac,bc\in(0,R)$, *then*

$$(2.1)\qquad (\ln a-\ln b)\,acf'(ac)\geq f(ac)-f(bc)\geq(\ln a-\ln b)\,bcf'(bc).$$

Proof. It is well known that if $f:I\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on $\mathring{I}$, the interior of I, then for any $x,y\in\mathring{I}$ we have

$$(2.2)\qquad f'(y)(y-x)\geq f(y)-f(x).$$

Now, if we apply the property (2.2) for the function $f(t)=-\ln t$, $t>0$, we get

$$(2.3)\qquad \frac{x}{y}-1\geq\ln x-\ln y,\qquad x,y>0.$$

If in (2.3) we choose $x=a^n$, $y=b^n$, $n\geq 0$, then we obtain

$$(2.4)\qquad a^n-b^n\geq nb^n\ln a-nb^n\ln b,\qquad a,b>0,\ n\geq 0.$$

If we multiply (2.4) by $p_nc^n\geq 0$ $(n\geq 0)$ and sum over n from 0 to k, we derive

$$(2.5)\qquad \sum_{n=0}^{k}p_nc^na^n-\sum_{n=0}^{k}p_nb^nc^n\geq\ln a\sum_{n=0}^{k}np_nb^nc^n-\ln b\sum_{n=0}^{k}np_nb^nc^n.$$

Since

$$\sum_{n=0}^{\infty}p_nc^na^n=f(ac),\qquad\sum_{n=0}^{\infty}p_nc^nb^n=f(cb)$$

and

$$\sum_{n=0}^{\infty}np_nb^nc^n=bc\sum_{n=1}^{\infty}np_nb^{n-1}c^{n-1}=bcf'(bc),$$

then by letting $k\to\infty$ in (2.5), we deduce

$$(2.6)\qquad f(ac)-f(bc)\geq(\ln a-\ln b)\,bcf'(bc).$$

Now, by replacing a with b in (2.6), we have

$$f(bc)-f(ac)\geq(\ln b-\ln a)\,acf'(ac),$$

which is equivalent with

$$f(ac)-f(bc)\leq(\ln a-\ln b)\,acf'(ac). \tag{2.7}$$

Utilizing (2.6) and (2.7) we derive the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 1 and if $a,b\in(0,R)$, then*

$$(\ln a-\ln b)\,af'(a)\geq f(a)-f(b)\geq(\ln a-\ln b)\,bf'(b). \tag{2.8}$$

Corollary 2. *With the assumptions of Theorem 1 and if $a,c>0$ and $a,ac\in(0,R)$, then*

$$ac\ln a\ f'(ac)\geq f(ac)-f(c)\geq\ln a\ cf'(c). \tag{2.9}$$

The above result (2.8) has some natural applications for particular functions of interest as follows:

(1) If we apply the inequality (2.8) for $f(x)=\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, $x\in(-1,1)$, then for any $a,b\in(0,1)$ we get

$$(\ln a-\ln b)\cdot\frac{a}{(1-a)^2}\geq\frac{1}{1-a}-\frac{1}{1-b}\geq(\ln a-\ln b)\cdot\frac{b}{(1-b)^2}.$$

Hence

$$\frac{(1-b)(a-b)}{b(1-a)}\geq\ln a-\ln b\geq\frac{(1-a)(a-b)}{a(1-b)}$$

for any $a,b\in(0,1)$.

(2) If we apply the inequality (2.8) for $f(x)=\dfrac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n$, $x\in(-1,1)$, then for any $a,b\in(0,1)$ we obtain that

$$\begin{aligned}(\ln a-\ln b)\cdot\frac{a(1+a)}{(1-a)^3}&\geq\frac{a}{(1-a)^2}-\frac{b}{(1-b)^2}\\&\geq(\ln a-\ln b)\cdot\frac{b(1+b)}{(1-b)^3},\end{aligned}$$

which implies that

$$\begin{aligned}\frac{(1-b)(1-ab)(a-b)}{b(1+b)(1-a)^2}&\geq\ln a-\ln b\\&\geq\frac{(1-a)(1-ab)(a-b)}{a(1+a)(1-b)^2}\end{aligned}$$

for any $a,b\in(0,1)$.

(3) If in (2.8) we choose the function $f(x)=-\ln(1-x)=\sum_{n=1}^{\infty}\dfrac{x^n}{n}$, $x\in(-1,1)$, then for any $a,b\in(0,1)$ we get the following inequality:

$$\ln\left(\frac{a}{b}\right)^{\frac{a}{1-a}}\geq\ln\left(\frac{1-b}{1-a}\right)\geq\ln\left(\frac{a}{b}\right)^{\frac{b}{1-b}}.$$

Hence

$$\left(\frac{a}{b}\right)^{\frac{a}{1-a}} \geq \frac{1-b}{1-a} \geq \left(\frac{a}{b}\right)^{\frac{b}{1-b}} \tag{2.10}$$

for any $a, b \in (0,1)$.

In particular, if in (2.10) we choose $b = 1-a$, then we obtain the simpler result:

$$\left(\frac{a}{1-a}\right)^{\frac{a}{1-a}} \geq \frac{a}{1-a} \geq \left(\frac{a}{1-a}\right)^{\frac{1-a}{a}}$$

for any $a \in (0,1)$.

(4) If in (2.8) we choose the function $f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)}$, $x \in (-1,1)$, then for any $a, b \in (0,1)$ we get the following inequality:

$$\begin{aligned}(\ln a - \ln b) \cdot \frac{a}{1-a^2} &\geq \frac{1}{2}\left[\ln\left(\frac{1+a}{1-a}\right) - \ln\left(\frac{1+b}{1-b}\right)\right] \\ &\geq (\ln a - \ln b) \cdot \frac{b}{1-b^2},\end{aligned}$$

which is equivalent to

$$\ln\left(\frac{a}{b}\right)^{\frac{a}{1-a^2}} \geq \ln\left(\frac{(1+a)(1-b)}{(1-a)(1+b)}\right)^{\frac{1}{2}} \geq \ln\left(\frac{a}{b}\right)^{\frac{b}{1-b}}.$$

Hence

$$\left(\frac{a}{b}\right)^{\frac{2a}{1-a^2}} \geq \frac{(1+a)(1-b)}{(1-a)(1+b)} \geq \left(\frac{a}{b}\right)^{\frac{2b}{1-b^2}}. \tag{2.11}$$

for any $a, b \in (0,1)$.

(5) If we apply the same inequality (2.8) for the function $f(x) = \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $x \in \mathbb{R}$, then for any $a, b \in (0,R)$ we obtain that

$$\ln\left(\frac{a}{b}\right)^{a\sinh(a)} \geq \cosh(a) - \cosh(b) \geq \ln\left(\frac{a}{b}\right)^{b\sinh(b)}$$

for any $a, b \in (0,R)$, $R > 0$.

(6) Now, if we apply the inequality (2.9) for the function $f(x) = \cosh(x)$, $x \in \mathbb{R}$, then for any $a, b \in \mathbb{R}$ we have the following inequality:

$$\ln a^{ab\sinh(ab)} \geq \cosh(ab) - \cosh(b) \geq \ln a^{b\sinh(b)}. \tag{2.12}$$

In particular, if in (2.12) we choose $b = 1$, then we obtain that

$$\ln a^{a\sinh(a)} \geq \cosh(a) - \cosh(1) \geq \ln a^{\sinh(1)}$$

for any $a \in \mathbb{R}$.

Now, we can prove the following Jensen type inequality.

THEOREM 2. *Assume that f is as in Theorem 1 and $R = 1$ or $R = +\infty$. If $a_i \in (0,R)$ and $p_i \geq 0$, $i \in \{1,\ldots,n\}$ with $\sum_{i=1}^{n} p_i = 1$, then we have the* *inequalities:*

$$\ln\left[\frac{\prod_{i=1}^{n} a_i^{p_i a_i f'(a_i)}}{\left(\prod_{j=1}^{n} a_j^{p_j}\right)^{\sum_{i=1}^{n} p_i a_i f'(a_i)}}\right] \geq \sum_{i=1}^{n} p_i f(a_i) - f\left(\prod_{j=1}^{n} a_j^{p_j}\right) \geq 0. \tag{2.13}$$

PROOF. We use the inequality (2.8) for the choices $a = a_i$ and $b = \prod_{j=1}^{n} a_j^{p_j}$ to get

$$\begin{aligned} &\left(\ln a_i - \ln\left(\prod_{j=1}^{n} a_j^{p_j}\right)\right) a_i f'(a_i) \\ &\geq f(a_i) - f\left(\prod_{j=1}^{n} a_j^{p_j}\right) \\ &\geq \left(\ln a_i - \ln\left(\prod_{j=1}^{n} a_j^{p_j}\right)\right)\left(\prod_{j=1}^{n} a_j^{p_j}\right) f'\left(\prod_{j=1}^{n} a_j^{p_j}\right) \end{aligned} \tag{2.14}$$

for any $i \in \{1, \dots, n\}$.

Now, if we multiply (2.14) by $p_i \geq 0$ and sum over i from 1 to n, we get

$$\begin{aligned} &\ln\left(\prod_{i=1}^{n} a_i^{p_i a_i f'(a_i)}\right) - \sum_{i=1}^{n} p_i a_i f'(a_i) \cdot \ln\left(\prod_{j=1}^{n} a_j^{p_j}\right) \\ &\geq \sum_{i=1}^{n} p_i f(a_i) - f\left(\prod_{j=1}^{n} a_j^{p_j}\right) \geq 0, \end{aligned}$$

which is clearly equivalent with the desired result (2.13). □

REMARK 1. *The second inequality in (2.13) shows that*

$$\lambda f(x) + (1-\lambda) f(y) \geq f\left(x^{\lambda} y^{1-\lambda}\right) \tag{2.15}$$

for any $\lambda \in [0,1]$ *and* $x, y \in (0, R)$ $(R = 1 \ \ or \ \ R = \infty)$, *i.e.,* f *is an* $GA-$*convex function in the sense of terminology introduced in* [**1**].

The following result also holds.

THEOREM 3. *Assume that* f, a_i *and* p_i *are as in Theorem 2. Then we have the inequalities:*

$$\begin{aligned} &\ln\left[\left(\prod_{i=1}^{n} a_i^{p_i a_i f'(a_i)}\right) \cdot \left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)^{\sum_{i=1}^{n} p_i a_i f'(a_i)}\right] \\ &\geq \sum_{i=1}^{n} p_i f(a_i) - f\left[\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right)^{-1}\right] \\ &\geq \ln\left[\left(\prod_{i=1}^{n} a_i^{p_i}\right) \cdot \left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)\right]^{f'\left[\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right)^{-1}\right]\left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)^{-1}} \geq 0. \end{aligned} \tag{2.16}$$

Proof. From the inequality (2.8) we have for the choices $a = a_i$, $b = \frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}}$ that

$$\begin{aligned} (2.17) \quad & \left(\ln a_i - \ln\left(\frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}}\right)\right) a_i f'(a_i) \\ & \geq f(a_i) - f\left(\frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}}\right) \\ & \geq \left(\ln a_i - \ln\left(\frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}}\right)\right) \frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}} f'\left(\frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}}\right), \end{aligned}$$

for any $i \in \{1, \ldots, n\}$.

Now, if we multiply (2.17) by $p_i \geq 0$ and sum over i from 1 to n, then we deduce

$$\begin{aligned} \ln\left[\frac{\prod_{i=1}^{n} a_i^{p_i a_i f'(a_i)}}{\left(\prod_{j=1}^{n} a_j^{p_j}\right)^{-\sum_{i=1}^{n} p_i a_i f'(a_i)}}\right] & \geq \sum_{i=1}^{n} p_i f(a_i) - f\left(\frac{1}{\sum_{j=1}^{n} \frac{p_j}{a_j}}\right) \\ & \geq \ln\left[\frac{\prod_{i=1}^{n} a_i^{p_i}}{\left(\sum_{j=1}^{n} \frac{p_j}{a_j}\right)^{-1}}\right]^{\frac{f'\left(1/\sum_{j=1}^{n} \frac{p_j}{a_j}\right)}{\sum_{j=1}^{n} \frac{p_j}{a_j}}} \\ & \geq 0 \end{aligned}$$

giving the desired result (2.16). □

The following result also holds.

Theorem 4. *With the assumptions in Theorem 1, we have the inequality:*

$$(2.18) \qquad f\left(\left(\prod_{i=1}^{n} a_i^{p_i a_i f'(a_i)}\right)^{\frac{1}{\sum_{i=1}^{n} p_i a_i f'(a_i)}}\right) \geq \sum_{i=1}^{n} p_i f(a_i).$$

Proof. From the inequality (2.8) we have

$$(2.19) \qquad f(a) - f(a_i) \geq (\ln a - \ln a_i) a_i f'(a_i)$$

for any $i \in \{1, \ldots, n\}$. Now, if we multiply (2.19) by $p_i \geq 0$ and sum over i from 1 to n, we get

$$f(a) - \sum_{i=1}^{n} p_i f(a_i) \geq \ln a \cdot \sum_{i=1}^{n} p_i a_i f'(a_i) - \sum_{i=1}^{n} p_i a_i f'(a_i) \ln a_i.$$

Now, if we choose a so that

$$\ln a = \frac{\sum_{i=1}^{n} p_i a_i f'(a_i) \ln a_i}{\sum_{i=1}^{n} p_i a_i f'(a_i)} = \ln\left(\prod_{i=1}^{n} a_i^{p_i a_i f'(a_i)}\right)^{\frac{1}{\sum_{i=1}^{n} p_i a_i f'(a_i)}},$$

we get the desired result (2.18). □

3. Further Results Via Jensen Type Inequalities

In this section, we shall make use (1.2) and also the reverse of the Jensen inequality due to Dragomir and Ionescu [**2**], i.e. the inequality (1.3) to derive some interesting inequalities for power series with positive coefficients.

Theorem 5. *Assume that the function* $f(x) = \sum_{n=0}^{\infty} p_n x^n$ *is defined on the interval* $(-R, R)$ *and has positive coefficients, i.e.,* $p_n \geq 0$ *for* $n \geq 0$. *If* $a, b > 0$ *are such that* $a, ab \in (0, R)$, *then*

$$b^{\frac{af'(a)}{f(a)}} \leq \frac{f(ab)}{f(a)} \leq b^{\frac{abf'(ab)}{f(ab)}}. \tag{3.1}$$

Proof. If we apply Jensen's inequality (1.2) for the convex function $g(t) = -\ln t$, $t > 0$, then we have

$$\ln\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) \geq \frac{\sum_{n=0}^{k} p_n a^n \ln b^n}{\sum_{n=0}^{k} p_n a^n} = \frac{\ln b \sum_{n=0}^{k} n p_n a^n}{\sum_{n=0}^{k} p_n a^n} = \frac{a \ln b \sum_{n=1}^{k} n p_n a^{n-1}}{\sum_{n=0}^{k} p_n a^n} \tag{3.2}$$

for $a, b > 0$ with $a, ab \in (0, R)$.

Taking the limit over $k \to \infty$ and since $\sum_{n=0}^{\infty} p_n a^n b^n = f(ab)$, $\sum_{n=0}^{\infty} p_n a^n = f(a)$ and $\sum_{n=1}^{\infty} n p_n a^{n-1} = f'(a)$, then from (3.2) we have

$$\ln\left(\frac{f(ab)}{f(a)}\right) \geq \frac{a \ln b f'(a)}{f(a)}$$

for $a, b > 0$ with $a, ab \in (0, R)$, which is equivalent with the first inequality in (3.1).

Now, if we apply Jensen's inequality (1.2) for the convex function $g(t) = t \ln t$, $t > 0$, we can also state that

$$\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) \cdot \ln\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) \leq \frac{\sum_{n=0}^{k} p_n a^n b^n \ln b^n}{\sum_{n=0}^{k} p_n a^n} = \frac{\ln b \sum_{n=0}^{k} n p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n} = \frac{ab \ln b \sum_{n=1}^{k} n p_n a^{n-1} b^{n-1}}{\sum_{n=0}^{k} p_n a^n} \tag{3.3}$$

for $a, b > 0$ with $a, ab \in (0, R)$.

Taking the limit over $k \to \infty$ and since $\sum_{n=1}^{\infty} n p_n a^{n-1} b^{n-1} = f'(ab)$, then from (3.3) we have

$$\frac{f(ab)}{f(a)} \ln\left(\frac{f(ab)}{f(a)}\right) \leq \frac{ab \ln b f'(ab)}{f(a)},$$

which is clearly equivalent with the second inequality in (3.1). □

Corollary 3. *With the assumptions in Theorem 5, and if* $a, c \in (0, R)$, *then*

$$\left(\frac{c}{a}\right)^{\frac{af'(a)}{f(a)}} \leq \frac{f(c)}{f(a)} \leq \left(\frac{c}{a}\right)^{\frac{cf'(c)}{f(c)}}. \tag{3.4}$$

PROOF. Follows from Theorem 5 on choosing $c = ab$. □

Now, we provide some applications of the inequality (3.4) for particular functions of interest.

(1) If we apply the inequality (3.4) for function $f(x) = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, $x \in (-1,1)$, then for any $a, c \in (0,1)$ we get that
$$\left(\frac{c}{a}\right)^{-\frac{a}{(1-a)\ln(1-a)}} \leq \frac{\ln(1-c)}{\ln(1-a)} \leq \left(\frac{c}{a}\right)^{-\frac{c}{(1-c)\ln(1-c)}}.$$

(2) If in (3.4) we choose the power series $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$, then we can state that for any $a, c \in (0,R)$ $(R > 0)$,
$$\left(\frac{c}{a}\right)^a \leq \exp(c-a) \leq \left(\frac{c}{a}\right)^c.$$

(3) If we apply the same inequality (3.4) for the function $f(x) = \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}$, $x \in \mathbb{R}$, then for any $a, c \in (0,R)$ $(R > 0)$, we obtain
$$\left(\frac{c}{a}\right)^{a\coth(a)} \leq \frac{\sinh(c)}{\sinh(a)} \leq \left(\frac{c}{a}\right)^{c\coth(c)}$$
for any $a, c \in (0,R)$ $(R > 0)$.

In order to state our second result, we use the reverse of the Jensen inequality (1.3). Then, the following result holds.

THEOREM 6. *Let f be as in Theorem 5. If $a, b > 0$ are such that a, ab and $\frac{a}{b} \in (0,R)$, then*
$$\frac{f(ab)}{f(a)} \leq b^{\frac{af'(a)}{f(a)}} \cdot \exp\left[\frac{f(ab)\, f\left(\frac{a}{b}\right)}{f^2(a)} - 1\right]. \tag{3.5}$$

PROOF. Utilizing the inequality (1.3), for the convex function $g(t) = -\ln t$, $t > 0$, we can write that
$$\ln\left(\frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n}\right) - \frac{\sum_{n=0}^{k} p_n a^n \ln b^n}{\sum_{n=0}^{k} p_n a^n} \leq \frac{\sum_{n=0}^{k} p_n a^n b^n}{\sum_{n=0}^{k} p_n a^n} \cdot \frac{\sum_{n=0}^{k} \frac{p_n a^n}{b^n}}{\sum_{n=0}^{k} p_n a^n} - 1 \tag{3.6}$$
for $a, b > 0$ with $a, ab, \frac{a}{b} \in (0,R)$.

Taking the limit over $k \to \infty$ and since $\sum_{n=0}^{\infty} p_n \left(\frac{a}{b}\right)^n = f\left(\frac{a}{b}\right)$, then by (3.6) we deduce
$$\ln\left(\frac{f(ab)}{f(a)}\right) - \frac{a \ln b f'(a)}{f(a)} \leq \frac{f(ab)\, f\left(\frac{a}{b}\right)}{f^2(a)} - 1,$$
which is clearly equivalent with the desired result (3.5). □

The particular case of Theorem 6 can be stated as well.

Corollary 4. *With the assumptions in Theorem 6 and if* $a, c \in (0, R)$ *with* $\frac{a^2}{c} \in (0, R)$, *then we have the inequality:*

$$\frac{f(c)}{f(a)} \le \left(\frac{c}{a}\right)^{\frac{af'(a)}{f(a)}} \exp\left[\frac{f(c) f\left(\frac{a^2}{c}\right)}{f^2(a)} - 1\right]. \tag{3.7}$$

If $\frac{c^2}{a} \in (0, R)$, *then we also have:*

$$\exp\left[1 - \frac{f(a) f\left(\frac{c^2}{a}\right)}{f^2(c)}\right] \left(\frac{c}{a}\right)^{\frac{cf'(c)}{f(c)}} \le \frac{f(c)}{f(a)}. \tag{3.8}$$

Some examples for particular functions that are generated by power series with positive coefficients are as follows.

(1) If we apply the inequality (3.7) for $f(x) = \frac{1}{1-x}$, $x \in (-1, 1)$, then for any $a, c \in (0, 1)$ we get

$$\frac{1-a}{1-c} \le \left(\frac{c}{a}\right)^{\frac{a}{1-a}} \exp\left[\frac{c(1-a)^2}{(1-c)(c-a^2)} - 1\right].$$

(2) If in (3.7) we choose $f(x) = e^x$, $x \in \mathbb{R}$, then for any $a, c \in (0, R)$ we can state that

$$\exp(c-a) \le \left(\frac{c}{a}\right)^a \exp\left[e^{\frac{(c-a)^2}{c}} - 1\right].$$

Theorem 7. *With the assumptions of Theorem 5 for the function* f *and if* $x_i \in (0, R)$ *and* $p_i \ge 0$ *with* $\sum_{i=1}^n p_i = 1$, *then we have the inequality:*

$$\sum_{i=1}^n p_i \left(\frac{x_i}{\sum_{j=1}^n p_j x_j}\right)^{\frac{\sum_{j=1}^n p_j x_j f'\left(\sum_{j=1}^n p_j x_j\right)}{f\left(\sum_{j=1}^n p_j x_j\right)}} \le \frac{\sum_{j=1}^n p_j f(x_j)}{f\left(\sum_{j=1}^n p_j x_j\right)} \le \sum_{i=1}^n p_i \left(\frac{x_i}{\sum_{j=1}^n p_j x_j}\right)^{\frac{x_i f'(x_i)}{f(x_i)}}. \tag{3.9}$$

Proof. Is obvious by the inequality (3.4) on noticing that for $c = x_i$ and $a = \sum_{j=1}^n p_j x_j$, we have

$$\left(\frac{x_i}{\sum_{j=1}^n p_j x_j}\right)^{\frac{\sum_{j=1}^n p_j x_j f'\left(\sum_{j=1}^n p_j x_j\right)}{f\left(\sum_{j=1}^n p_j x_j\right)}} \le \frac{f(x_i)}{f\left(\sum_{j=1}^n p_j x_j\right)} \le \left(\frac{x_i}{\sum_{j=1}^n p_j x_j}\right)^{\frac{x_i f'(x_i)}{f(x_i)}} \tag{3.10}$$

for any $i \in \{1, \dots, n\}$.

On multiplying the inequality (3.10) by p_i and summing over i from 1 to n, we get the desired result. □

A more natural result of Jensen type is incorporated in the following theorem.

THEOREM 8. *With the assumptions of Theorem 7, we have*

$$1 \leq \frac{G_p\left(f\left(x\right)\right)}{f\left(G_p\left(x\right)\right)} \leq \frac{G_p\left(x^{\frac{xf'(x)}{f(x)}}\right)}{\left[G_p\left(x\right)\right] A_p\left(\frac{xf'(x)}{f(x)}\right)}, \tag{3.11}$$

where $A_p\left(h\left(x\right)\right) := \sum p_i h\left(x_i\right)$ *and* $G_p\left(h\left(x\right)\right) := \prod_{i=1}^{n} h^{p_i}\left(x_i\right)$, $h : (0,\infty) \to (0,\infty)$.

PROOF. From the inequality (3.4) we have

$$\left(\frac{x_i}{G_p\left(x\right)}\right)^{\frac{G_p(x)f'(G_p(x))}{f(G_p(x))}} \leq \frac{f\left(x_i\right)}{f\left(G_p\left(x\right)\right)} \leq \left(\frac{x_i}{G_p\left(x\right)}\right)^{\frac{x_i f'(x_i)}{f(x_i)}},$$

which, by taking the power p_i gives that

$$\left(\frac{x_i}{G_p\left(x\right)}\right)^{p_i A} \leq \left[\frac{f\left(x_i\right)}{f\left(G_p\left(x\right)\right)}\right]^{p_i} \leq \left(\frac{x_i}{G_p\left(x\right)}\right)^{\frac{p_i x_i f'(x_i)}{f(x_i)}},$$

with

$$A := \frac{G_p\left(x\right) f'\left(G_p\left(x\right)\right)}{f\left(G_p\left(x\right)\right)}.$$

Now, multiplying the inequalities over i from 1 to n, we get

$$1 \leq \frac{\prod_{i=1}^{n} f^{p_i}\left(x_i\right)}{f\left(G_p\left(x\right)\right)} \leq \frac{\prod_{i=1}^{n} x_i^{\frac{p_i x_i f'(x_i)}{f(x_i)}}}{\left[G_p\left(x\right)\right]^{\sum_{i=1}^{n} \frac{p_i x_i f'(x_i)}{f(x_i)}}}$$

and the proof is completed. □

REMARK 2. *The reader can obtain other particular inequalities by choosing appropriate examples of power series. However the details are not presented here.*

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[1]Mathematics, School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: alawiah.ibrahim@live.vu.edu.au

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag-3, Wits-2050, Johannesburg, South Africa.

E-mail address: sever.dragomir@vu.edu.au

URL: http://www.staff.vu.edu.au/rgmia/dragomir/

E-mail address: pietro.cerone@vu.edu.au

URL: http://www.staff.vu.edu.au/rgmia/cerone/

[3]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia.

E-mail address: maslina@ukm.my