

**NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED  
s-CONVEX FUNCTIONS IN THE SECOND SENSE**

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ABSTRACT. In this paper some new Ostrowski type inequalities for co-ordinated s-convex functions in the second sense are obtained.

1. INTRODUCTION

In 1938, A. Ostrowski proved the following interesting inequality [21]:

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$ .*

Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \quad (1.1)$$

for all  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible.

The inequality (1.1) can be rewritten in equivalent form as:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \|f'\|_\infty.$$

Since 1938 when A. Ostrowski proved his famous inequality, many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings, s-convex mappings and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [4, 5, 7, 8, 11, 12, 20, 23, 24, 25, 26] and the references therein.

Let us consider now a bidimensional interval  $\Delta =: [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ , a mapping  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex on  $\Delta$  if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w),$$

holds for all  $(x, y), (z, w) \in \Delta$  and  $\lambda \in [0, 1]$ . The mapping  $f$  is said to be concave on the co-ordinates on  $\Delta$  if the above inequality holds in reversed direction, for all  $(x, y), (z, w) \in \Delta$  and  $\lambda \in [0, 1]$ .

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A modification for convex (concave) functions on  $\Delta$ , which is also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [9, 13] as follows:

A function  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex (concave) on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$  are convex (concave) where defined for all  $x \in [a, b], y \in [c, d]$ .

A formal definition for co-ordinated convex (concave) functions may be stated in:

**Definition 1.** [18] *A mapping  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex on the co-ordinates on  $\Delta$  if the inequality*

$$\begin{aligned} & f(tx + (1-t)y, ru + (1-r)w) \\ & \leq trf(x, u) + t(1-r)f(x, w) + r(1-t)f(y, u) + (1-t)(1-r)f(y, w), \end{aligned} \quad (1.2)$$

*holds for all  $t, r \in [0, 1]$  and  $(x, u), (y, w) \in \Delta$ . The mapping  $f$  is concave on the co-ordinates on  $\Delta$  if the inequality (1.2) holds in reversed direction for all  $t, r \in [0, 1]$  and  $(x, y), (u, w) \in \Delta$ .*

Clearly, every convex (concave) mapping  $f : \Delta \rightarrow \mathbb{R}$  is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [9, 13]).

The main result proved concerning the co-ordinated convex function from [9, 13] is given in:

**Theorem 2.** [9] *Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex on  $\Delta$ . Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (1.3)$$

*The above inequalities are sharp. The inequalities in (1.3) hold in reverse direction if the mapping  $f$  is concave.*

The concept of  $s$ -convex functions on the co-ordinates in the second sense was introduced by Almoari and Darus in [1] as a generalization of the co-ordinated convexity in:

**Definition 2.** [1] *Consider the bidimensional interval  $\Delta =: [a, b] \times [c, d]$  in  $[0, \infty)^2$  with  $a < b$  and  $c < d$ . The mapping  $f : \Delta \rightarrow \mathbb{R}$  is  $s$ -convex in the second sense on*

$\Delta$  if

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w)$$

, holds for all  $(x, y), (z, w) \in \Delta$ ,  $\lambda \in [0, 1]$  with some fixed  $s \in (0, 1]$ .

A function  $f : \Delta =: [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  is called  $s$ -convex in the second sense on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow \mathbb{R}$ ,  $f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow \mathbb{R}$ ,  $f_x(v) = f(x, v)$ , are  $s$ -convex in the second sense for all  $y \in [c, d]$ ,  $x \in [a, b]$  and  $s \in (0, 1]$ , i.e., the partial mappings  $f_y$  and  $f_x$  are  $s$ -convex in the second sense with some fixed  $s \in (0, 1]$ .

A formal definition of co-ordinated  $s$ -convex function in second sense may be stated as follows:

**Definition 3.** A function  $f : \Delta =: [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  is called  $s$ -convex in the second sense on the co-ordinates on  $\Delta$  if

$$\begin{aligned} & f(tx + (1 - t)y, ru + (1 - r)w) \\ & \leq t^s r^s f(x, u) + t^s (1 - r)^s f(x, w) + r^s (1 - t)^s f(y, u) + (1 - t)^s (1 - r)^s f(y, w), \end{aligned} \quad (1.4)$$

holds for all  $t, r \in [0, 1]$  and  $(x, u), (y, w) \in \Delta$ , for some fixed  $s \in (0, 1]$ . The mapping  $f$  is concave on the co-ordinates on  $\Delta$  if the inequality (1.4) holds in reversed direction for all  $t, r \in [0, 1]$  and  $(x, y), (u, w) \in \Delta$  with some fixed  $s \in (0, 1]$ .

In [5], Alomari et al. also proved a variant of inequalities given above by (1.3) for  $s$ -convex functions in the second sense on the co-ordinates on a rectangle from the plane  $\mathbb{R}^2$ :

**Theorem 3.** [1] Suppose  $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty)$  is  $s$ -convex function in the second sense on the co-ordinates on  $\Delta$ . Then one has the inequalities:

$$\begin{aligned} & 4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq 2^{s-2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{2(s+1)} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_a^b [f(a, y) + f(b, y)] dy \right] \\ & \leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{(s+1)^2}. \end{aligned} \quad (1.5)$$

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1, 2, 3, 6, 9, 15, 17, 18, 19, 20, 22, 27] (see also the references therein). Alomari et al. [1]-[3], proved several Hermite-Hadamard type inequalities for co-ordinated  $s$ -convex functions. Bakula et. al [6], proved Jensen's inequality for convex functions on the co-ordinates from the rectangle from the plan  $\mathbb{R}^2$ . Dragomir [9], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [15],

also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated with the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [17]-[20], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, for product of two co-ordinated convex mappings, for co-ordinated  $h$ -convex mappings and also proved some Ostrowski type inequalities for co-ordinated convex mappings. Özdemir et. al [22], proved Hadamard's type inequalities for co-ordinated  $m$ -convex and  $(\alpha, m)$ -convex functions. Sarikaya, et. al [27] proved Hermite-Hadamard type inequalities for differentiable co-ordinated convex function. For further inequalities on co-ordinated convex functions see also the references in the above cited papers.

In the present paper, we establish new Ostrowski type inequalities for co-ordinated  $s$ -convex functions in second sense similar to those from [20].

## 2. MAIN RESULTS

To establish our main results we need the following identity:

**Lemma 1.** [20] *Let  $f : \Delta \rightarrow \mathbb{R}$  be a twice partial differentiable mapping on  $\Delta^\circ$ . If  $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$ , then the following identity holds:*

$$\begin{aligned} & f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \\ &= \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) dr dt \\ &- \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) dr dt \\ &- \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) dr dt \\ &+ \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) dr dt, \end{aligned} \quad (2.1)$$

for all  $(x, y) \in \Delta$ , where

$$A = \frac{1}{d-c} \int_c^d f(x, v) dv + \frac{1}{b-a} \int_a^b f(u, y) du.$$

We begin with the following result:

**Theorem 4.** *Let  $\Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  be a twice partial differentiable mapping on  $\Delta^\circ$  such that  $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|$  is  $s$ -convex in the second sense on the co-ordinates on  $\Delta$  with  $s \in (0, 1]$  and  $\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds:*

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{M}{(s+1)^2} \left[ \frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[ \frac{(y-c)^2 + (d-y)^2}{d-c} \right], \end{aligned} \quad (2.2)$$

for all  $(x, y) \in \Delta$ , where  $A$  is defined in Lemma 1.

*Proof.* By Lemma 1, we have that the following inequality holds:

$$\begin{aligned}
& \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) \, dvdu - A \right| \\
& \leq \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right| \, drdt \\
& + \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right| \, drdt \\
& + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right| \, drdt \\
& + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right| \, drdt, \quad (2.3)
\end{aligned}$$

for all  $(x, y) \in \Delta$ .

Using the co-ordinated  $s$ -convexity of  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|$ , we have that the following inequality holds:

$$\begin{aligned}
& \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right| \, drdt \\
& \leq \left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \int_0^1 \int_0^1 t^{s+1} r^{s+1} \, drdt + \left| \frac{\partial^2}{\partial r \partial t} f(x, c) \right| \int_0^1 \int_0^1 t^{s+1} r (1-r)^s \, drdt \\
& \quad + \left| \frac{\partial^2}{\partial r \partial t} f(a, y) \right| \int_0^1 \int_0^1 r^{s+1} t (1-t)^s \, drdt \\
& \quad + \left| \frac{\partial^2}{\partial r \partial t} f(a, c) \right| \int_0^1 \int_0^1 rt (1-t)^s (1-r)^s \, drdt. \quad (2.4)
\end{aligned}$$

Since

$$\begin{aligned}
& \int_0^1 \int_0^1 t^{s+1} r^{s+1} \, drdt = \frac{1}{(s+2)^2}, \\
& \int_0^1 \int_0^1 t^{s+1} r (1-r)^s \, drdt = \int_0^1 \int_0^1 r^{s+1} t (1-t)^s \, drdt = \frac{1}{(s+1)(s+2)^2}, \\
& \int_0^1 \int_0^1 rt (1-t)^s (1-r)^s \, drdt = \frac{1}{(s+1)^2 (s+2)^2}
\end{aligned}$$

and

$$\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M, \quad (x, y) \in \Delta,$$

where we have used the Euler Beta function defined by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad x, y > 0$$

and the properties

$$\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \quad \text{and} \quad \Gamma(x+1) = x\Gamma(x)$$

of Euler Beta function to evaluate the above integrals. Hence from (2.4), we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right| dr dt \\ & \leq \frac{M}{(s+2)^2} + \frac{2M}{(s+1)(s+2)^2} + \frac{M}{(s+1)^2(s+2)^2} = \frac{M}{(s+1)^2} \end{aligned} \quad (2.5)$$

Analogously, we also have

$$\int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right| dr dt \leq \frac{M}{(s+1)^2}, \quad (2.6)$$

$$\int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right| dr dt \leq \frac{M}{(s+1)^2} \quad (2.7)$$

and

$$\int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right| dr dt \leq \frac{M}{(s+1)^2}. \quad (2.8)$$

Now by making use of the inequalities (2.5)-(2.8) and the fact that

$$\begin{aligned} & (x-a)^2(y-c)^2 + (x-a)^2(d-y)^2 + (b-x)^2(y-c)^2 + (b-x)^2(d-y)^2 \\ & = \left[ (x-a)^2 + (b-x)^2 \right] \left[ (y-c)^2 + (d-y)^2 \right], \end{aligned}$$

we get the inequality (2.2). This completes the proof.  $\square$

The corresponding version for powers of the absolute value of the partial derivative is incorporated in the following result:

**Theorem 5.**  $\Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  be a twice partial differentiable mapping on  $\Delta^\circ$  such that  $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$  is  $s$ -convex in the second sense on the co-ordinates on  $\Delta$ ,  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds:

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{M}{(1+p)^{\frac{2}{p}}} \left( \frac{2}{s+1} \right)^{\frac{2}{q}} \left[ \frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[ \frac{(y-c)^2 + (d-y)^2}{d-c} \right], \end{aligned} \quad (2.9)$$

for all  $(x, y) \in \Delta$ , where  $A$  is defined in Lemma 1.

*Proof.* By Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
& \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \leq \left( \int_0^1 \int_0^1 r^p t^p dr dt \right)^{\frac{1}{p}} \\
& \times \left[ \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right|^q dr dt \right)^{\frac{1}{q}} \right. \\
& + \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q dr dt \right)^{\frac{1}{q}} \\
& + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, sy + (1-s)c) \right|^q dr dt \right)^{\frac{1}{q}} \\
& \left. + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q dr dt \right)^{\frac{1}{q}} \right], \tag{2.10}
\end{aligned}$$

for all  $(x, y) \in \Delta$ .

Since  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$  is  $s$ -convex in the second sense on the co-ordinates on  $\Delta$  and  $\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , we have

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(tx + (1-t)a, ry + (1-r)c) \right|^q dr dt \\
& \leq \left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right|^q \int_0^1 \int_0^1 t^s r^s dr dt + \left| \frac{\partial^2}{\partial r \partial t} f(a, c) \right|^q \int_0^1 \int_0^1 (1-t)^s (1-r)^s dr dt \\
& + \left| \frac{\partial^2}{\partial r \partial t} f(a, y) \right|^q \int_0^1 \int_0^1 r^s (1-t)^s dr dt + \left| \frac{\partial^2}{\partial r \partial t} f(x, c) \right|^q \int_0^1 \int_0^1 t^s (1-r)^s dr dt \\
& = \frac{4M^q}{(s+1)^2}.
\end{aligned}$$

Similarly, we also have the following inequalities:

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q dr dt \leq \frac{4M^q}{(s+1)^2}, \\
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q dr dt \leq \frac{4M^q}{(s+1)^2}
\end{aligned}$$

and

$$\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q dr dt \leq \frac{4M^q}{(s+1)^2}.$$

Using the fact

$$\int_0^1 \int_0^1 r^p t^p dr dt = \frac{1}{(1+p)^{\frac{2}{p}}}$$

and the above inequalities in (2.10), we get (2.9). This completes the proof of the theorem.  $\square$

A different approach leads us to the following result:

**Theorem 6.** Let  $\Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  be a twice partial differentiable mapping on  $\Delta^\circ$  such that  $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$  is  $s$ -convex on the co-ordinates on  $\Delta$ ,  $q \geq 1$  and  $\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , then the following inequality holds:

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) \, dv \, du - A \right| \\ & \leq \frac{M}{4} \left( \frac{2}{s+1} \right)^{\frac{2}{q}} \left[ \frac{(x-a)^2 - (b-x)^2}{b-a} \right] \left[ \frac{(y-c)^2 - (d-y)^2}{d-c} \right], \end{aligned} \quad (2.11)$$

for all  $(x, y) \in \Delta$ , where  $A$  is defined in Lemma 1.

*Proof.* Suppose  $q \geq 1$ . From Lemma 1 and using the power mean inequality for double integrals, we have

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) \, dv \, du - A \right| \leq \left( \int_0^1 \int_0^1 r t \, dr \, dt \right)^{1-\frac{1}{q}} \\ & \times \left[ \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 t r \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \right. \\ & + \frac{(x-a)^2 (y-d)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 t r \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \\ & + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 t r \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \\ & \left. + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 t r \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \right], \end{aligned} \quad (2.12)$$

for all  $(x, y) \in \Delta$ .

By similar argument as in Theorem 5 that  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$  is  $s$ -convex on the co-ordinates on  $\Delta$  in the second sense and  $\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M$ ,  $(x, y) \in \Delta$ , we have

$$\begin{aligned} & \int_0^1 \int_0^1 t r \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right|^q \, dr \, dt \\ & \leq \left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right|^q \int_0^1 \int_0^1 t^{s+1} r^{s+1} \, dr \, dt + \left| \frac{\partial^2}{\partial r \partial t} f(x, c) \right|^q \int_0^1 \int_0^1 t^{s+1} r (1-r)^s \, dr \, dt \\ & \quad + \left| \frac{\partial^2}{\partial r \partial t} f(a, y) \right|^q \int_0^1 \int_0^1 t (1-t)^s r^{s+1} \, dr \, dt \\ & \quad + \left| \frac{\partial^2}{\partial r \partial t} f(a, c) \right|^q \int_0^1 \int_0^1 t (1-t)^s r (1+r)^s \, dr \, dt \\ & = \frac{M^q}{(s+2)^2} + \frac{M^q}{(s+1)(s+2)^2} + \frac{M^q}{(s+1)(s+2)^2} + \frac{M^q}{(s+1)^2(s+2)^2} = \frac{M^q}{(s+1)^2}. \end{aligned}$$



In a similar way, we also have that the following inequalities:

$$\int_0^1 \int_0^1 tr \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q dr dt \leq \frac{M^q}{(s+1)^2}$$

$$\int_0^1 \int_0^1 tr \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q dr dt \leq \frac{M^q}{(s+1)^2}$$

and

$$\int_0^1 \int_0^1 tr \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q dr dt \leq \frac{M^q}{(s+1)^2}.$$

Now using the above inequalities and

$$\int_0^1 \int_0^1 rtdr dt = \frac{1}{4}$$

in (2.12), we get the desired inequality (2.11). This completes the proof.  $\square$

**Remark 1.** Since  $(1+p)^{\frac{1}{p}} < 2$ ,  $p > 1$  and accordingly, we have

$$\frac{1}{2} < \frac{1}{(1+p)^{\frac{1}{p}}}, p > 1$$

which gives

$$\frac{1}{4} < \frac{1}{(1+p)^{\frac{2}{p}}}, p > 1.$$

This reveals that the the inequality (2.11) gives tighter estimate than that of the inequality (2.9).

**Remark 2.** From the inequalities proved above in Theorem 4-Theorem 6, one can get several midpoint type inequalities by setting  $x = \frac{a+b}{2}$  and  $y = \frac{c+d}{2}$ . However the details are left to the interested reader.

Now we drive some results with co-ordinated  $s$ -concavity property instead of co-ordinated  $s$ -convexity.

**Theorem 7.**  $\Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$  be a twice partial differentiable mapping on  $\Delta^\circ$  such that  $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$ . If  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$  is concave on the co-ordinates on  $\Delta$  and  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then the inequality

$$\begin{aligned} & \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\ & \leq \frac{4^{s-1}}{(1+p)^{\frac{2}{p}}(b-a)(d-c)} \left[ (x-a)^2 (y-c)^2 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{x+a}{2}, \frac{d+y}{2} \right) \right|^q \right. \\ & \quad + (x-a)^2 (d-y)^2 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{x+a}{2}, \frac{d+y}{2} \right) \right|^q \\ & \quad + (b-x)^2 (y-c)^2 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{b+a}{2}, \frac{y+c}{2} \right) \right|^q \\ & \quad \left. + (b-x)^2 (d-y)^2 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{b+x}{2}, \frac{d+y}{2} \right) \right|^q \right], \quad (2.13) \end{aligned}$$

holds for all  $(x, y) \in \Delta$ , where  $A$  is defined in Lemma 1.

*Proof.* From Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
& \left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) dv du - A \right| \\
& \leq \left( \int_0^1 \int_0^1 r^p t^p dr dt \right)^{\frac{1}{p}} \\
& \times \left[ \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right|^q dr dt \right)^{\frac{1}{q}} \right. \\
& + \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q dr dt \right)^{\frac{1}{q}} \\
& + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q dr dt \right)^{\frac{1}{q}} \\
& \left. + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q dr dt \right)^{\frac{1}{q}} \right], \tag{2.14}
\end{aligned}$$

for all  $(x, y) \in \Delta$ .

Since  $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$  is  $s$ -concave on the co-ordinates on  $\Delta$ , so an application of (1.3) with inequalities in reversed direction, gives us the following inequalities:

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right|^q dr dt \\
& \leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( tx + (1-t)a, \frac{y+c}{2} \right) \right|^q dt \right. \\
& \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{x+a}{2}, ry + (1-r)c \right) \right|^q dr \right] \\
& \leq 4^{s-1} \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{x+a}{2}, \frac{y+c}{2} \right) \right|^q, \tag{2.15}
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q ds dt \\
& \leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( tx + (1-t)a, \frac{d+y}{2} \right) \right|^q dt \right. \\
& \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{x+a}{2}, ry + (1-r)c \right) \right|^q dr \right] \\
& \leq 4^{s-1} \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{x+a}{2}, \frac{d+y}{2} \right) \right|^q, \tag{2.16}
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q dr dt \\
& \leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f\left(tx + (1-t)a, \frac{y+c}{2}\right) \right|^q dt \right. \\
& \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f\left(\frac{b+x}{2}, sy + (1-s)c\right) \right|^q dr \right] \\
& \leq 4^{s-1} \left| \frac{\partial^2}{\partial r \partial t} f\left(\frac{b+a}{2}, \frac{y+c}{2}\right) \right|^q \quad (2.17)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q dr dt \\
& \leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f\left(tx + (1-t)b, \frac{d+y}{2}\right) \right|^q dt \right. \\
& \quad \left. + \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f\left(\frac{b+x}{2}, ry + (1-r)d\right) \right|^q dr \right] \\
& \leq 4^{s-1} \left| \frac{\partial^2}{\partial r \partial t} f\left(\frac{b+x}{2}, \frac{d+y}{2}\right) \right|^q. \quad (2.18)
\end{aligned}$$

By making use of (2.15)-(2.18) in (2.14), we obtain (2.13). Thus the proof of the theorem is complete.  $\square$

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