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Like Chebyshev's Inequalities For Convex Functions

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Abstract. In this paper, we will show some new inequalities for convex functions, and we will also make a connection between them and Chebyshev's inequality, which implies the existence of new class of functions satisfying Chebyshev's inequality.

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1 Introduction and main results

A classic result due to Chebyshev (1882-1883) (see [2, 3, 6, 7, 9]) is stated in the following theorem.

Theorem A *Let $f, g : [a, b] \to \mathbb{R}$ and $p : [a, b] \to \mathbb{R}_+$ be integrable functions. If f and g are monotonic in the same direction, then*

$$\int_a^b p(x)\,dx \int_a^b p(x) f(x) g(x)\,dx \geqslant \int_a^b p(x) f(x)\,dx \int_a^b p(x) g(x)\,dx \tag{1.1}$$

provided that the integrals exist. If f and g are monotonic in opposite directions, then the reverse of the inequality in (1.1) *is valid. In both cases, equality in* (1.1) *holds if and only if either g or f is constant almost everywhere.*

There exist several results which show that Chebyshev inequality is valid under weaker conditions, for example the condition that the functions be monotonic can be replaced by the condition that they be similarly ordered. In this case Theorem A is a simple consequence of the following identity:

$$\int_a^b p(x)\,dx \int_a^b p(x)\,f(x)\,g(x)\,dx - \int_a^b p(x)\,f(x)\,dx \int_a^b p(x)\,g(x)\,dx$$

$$= \frac{1}{2}\int_a^b \int_a^b p(x)\,p(y)\,(f(x)-f(y))\,(g(x)-g(y))\,dxdy. \tag{1.2}$$

Note that the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to be similarly ordered if

$$(f(x)-f(y))\,(g(x)-g(y)) \geqslant 0 \text{ for every } x, y \in I \tag{1.3}$$

holds, and they are said to be oppositely ordered if the reverse inequality holds. Of course, the generalization of the identity in (1.2) for functions with several variables is also valid (see [1, 7]). Thus similar generalizations of Theorem A are valid for similarly ordered functions. The first such result is given by Hardy, Littlewood and Pólya ([3, p. 168].

Another modification of the conditions for Chebyshev inequality was given by Levin and Stečkin (see [4, 5]), and they proved the following result:

Theorem B *Let $p(x)$ be defined on $[0,1]$ and satisfying the conditions:*
(i) $p(x)$ *is increasing for* $x \in \left[0, \frac{1}{2}\right]$.
(ii) $p(x) = p(1-x)$ *for* $x \in [0,1]$.
Then for every continuous convex function ϕ we have

$$\int_0^1 p(x)\,\phi(x)\,dx \leqslant \int_0^1 p(x)\,dx \int_0^1 \phi(x)\,dx.$$

The aim of this paper is to prove new type of inequalities for convex functions, and we put a link between these inequalities and Chebyshev's inequality.

Theorem 1.1 *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be convex (or concave) functions and* $p : [a,b] \to \mathbb{R}_+$ *be integrable symmetric function about* $x = \frac{a+b}{2}$ *(i.e.,* $p(a+b-x) = p(x)$, *for all* $x \in [a,b]$*). Then*

$$\int_a^b p(x) f(x) g(x)\, dx + \int_a^b p(x) f(x) g(a+b-x)\, dx$$

$$\geqslant \frac{2}{\int_a^b p(x)\, dx} \int_a^b p(x) f(x)\, dx \int_a^b p(x) g(x)\, dx. \qquad (1.4)$$

If f *is convex (or concave) and* g *is concave (or convex) functions, then the inequality* (1.4) *is reversed, equality in* (1.4) *holds if and only if either* g *or* f *is constant almost everywhere.*

Corollary 1.1 *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be convex (or concave) functions. If* g *is symmetric function about* $x = \frac{a+b}{2}$, *then*

$$\int_a^b f(x) g(x)\, dx \geqslant \frac{1}{b-a} \int_a^b f(x)\, dx \int_a^b g(x)\, dx. \qquad (1.5)$$

If f *is convex (or concave) and* g *is concave (or convex) functions, then the inequality* (1.5) *is reversed, equality in* (1.5) *holds if and only if either* g *or* f *is constant almost everywhere.*

Theorem 1.2 *Let* $f, g : [a,b] \longrightarrow \mathbb{R}$ *be convex (or concave) functions.*
(i) *If* f *and* g *are similarly ordered, then*

$$\int_a^b f(x) g(x)\, dx \geqslant \frac{1}{2} \left(\int_a^b f(x) g(x)\, dx + \int_a^b f(x) g(a+b-x)\, dx \right)$$

$$\geqslant \frac{1}{b-a} \int_a^b f(x)\, dx \int_a^b g(x)\, dx. \qquad (1.6)$$

(ii) *If f and g are oppositely ordered, then*

$$\int_a^b f(x)\,g(a+b-x)\,dx \geqslant \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$\geqslant \int_a^b f(x)\,g(x)\,dx. \tag{1.7}$$

Theorem 1.3 *Let $f,g : [a,b] \longrightarrow \mathbb{R}$, where f is convex function and g decreasing on $\left[a,\frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2},b\right]$. Then the inequality* (1.4) *holds.*

2 Some lemmas

Lemma 2.1 *Let $f : [a,b] \longrightarrow \mathbb{R}$ be convex (or concave) function. Then*

$$F(x) = f(x) + f(a+b-x)$$

is decreasing (increasing) on $\left[a,\frac{a+b}{2}\right]$ and increasing (decreasing) on $\left[\frac{a+b}{2},b\right]$.

Proof. Suppose that f is a convex function. Since f is a convex function then either f is monotonic on $]a,b[$, or there exists an $\zeta \in\ ]a,b[$ such that f is decreasing on the interval $[a,\zeta]$ and increasing on the interval $[\zeta,b]$ (for more information see [8, p. 26]). Since F is convex function on $[a,b]$, and symmetric about $x = \dfrac{a+b}{2}$, then we can deduce easily that F must be decreasing on the interval $\left[a,\frac{a+b}{2}\right]$ and increasing on the interval $\left[\frac{a+b}{2},b\right]$. Now, if f is a concave function, then by using the same proof as above we obtain the result.

Lemma 2.2 *Let $f,g : [a,b] \longrightarrow \mathbb{R}$ be two integrable functions. If f and g are similarly ordered, then*

$$\int_a^b f(x)\,g(x)\,dx \geqslant \int_a^b f(x)\,g(a+b-x)\,dx. \tag{2.1}$$

If f and g are oppositely ordered, then the inequality (2.1) *is reversed.*

Proof. Since f and g are similarly ordered, then we have for all $x \in [a, b]$

$$(f(x) - f(a+b-x))(g(x) - g(a+b-x)) \geqslant 0, \tag{2.2}$$

which implies that

$$f(x) g(x) + f(a+b-x) g(a+b-x) \geqslant f(x) g(a+b-x)$$

$$+g(x) f(a+b-x). \tag{2.3}$$

By integrating both sides of inequality (2.3) over the segment $[a, b]$ we obtain (2.1).

3 Proof of Theorems

Proof of Theorem 1.1: First, we suppose that f and g are convex functions and we denote by F and G the following functions

$$F(x) = f(x) + f(a+b-x), \quad G(x) = g(x) + g(a+b-x).$$

Since f and g are convex functions, by using Lemma 2.1 we deduce that F and G have the same variation. By applying Chebyshev's inequality in $\left[a, \frac{a+b}{2}\right]$, we obtain

$$\int_a^{\frac{a+b}{2}} p(x) F(x) G(x) dx \geqslant \frac{1}{\int_a^{\frac{a+b}{2}} p(x)} \int_a^{\frac{a+b}{2}} p(x) F(x) dx \int_a^{\frac{a+b}{2}} p(x) G(x) dx, \tag{3.1}$$

where $p : [a, b] \to \mathbb{R}_+$ is integrable symmetric function. Then

$$\int_a^{\frac{a+b}{2}} p(x) [f(x) g(x) dx + f(a+b-x) g(a+b-x)] dx$$

$$+\int_a^{\frac{a+b}{2}} p(x)\left[f(x)\,g(a+b-x)\,dx+f(a+b-x)\,g(x)\right]dx$$

$$\geqslant \left(\frac{1}{\int_a^{\frac{a+b}{2}} p(x)}\right)\left(\int_a^{\frac{a+b}{2}} p(x)\left[f(x)+f(a+b-x)\right]dx\right)$$

$$\times\left(\int_a^{\frac{a+b}{2}} p(x)\left[g(x)+g(a+b-x)\right]dx\right). \tag{3.2}$$

Using the identities

$$\int_a^{\frac{a+b}{2}} f(x)\,dx=\int_{\frac{a+b}{2}}^{b} f(a+b-x)\,dx, \tag{3.3}$$

$$\int_a^{\frac{a+b}{2}} p(x)\,dx=\frac{1}{2}\int_a^{b} p(x)\,dx, \tag{3.4}$$

and

$$\int_a^{\frac{a+b}{2}} f(x)\,g(a+b-x)\,dx=\int_{\frac{a+b}{2}}^{b} f(a+b-x)\,g(x)\,dx, \tag{3.5}$$

we obtain

$$\int_a^{\frac{a+b}{2}} p(x)\,f(x)\,g(x)\,dx+\int_{\frac{a+b}{2}}^{b} p(x)\,f(x)\,g(x)\,dx$$

$$+\int_{a}^{\frac{a+b}{2}} p(x) f(x) g(a+b-x)\, dx + \int_{\frac{a+b}{2}}^{b} p(x) f(x) g(a+b-x)\, dx$$

$$\geqslant \left(\frac{2}{\int_a^b p(x)} \right) \left(\int_{a}^{\frac{a+b}{2}} p(x) f(x)\, dx + \int_{\frac{a+b}{2}}^{b} p(x) f(x)\, dx \right)$$

$$\times \left(\int_{a}^{\frac{a+b}{2}} p(x) g(x)\, dx + \int_{\frac{a+b}{2}}^{b} p(x) g(x)\, dx \right). \tag{3.6}$$

Now, if f and g are concave functions, then by using the same proof as above we obtain the result.

Proof of Theorem 1.2: (i) Since f and g are convex functions and similarly ordered, then by Lemma 2.2 we have

$$\int_a^b f(x) g(x)\, dx \geqslant \int_a^b f(x) g(a+b-x)\, dx, \tag{3.7}$$

which we can write

$$2\int_a^b f(x) g(x)\, dx \geqslant \int_a^b f(x) g(a+b-x)\, dx + \int_a^b f(x) g(x)\, dx. \tag{3.8}$$

By Theorem 1.1 and (3.8), we have

$$2\int_a^b f(x) g(x)\, dx \geqslant \int_a^b \left[f(x) g(a+b-x) + f(x) g(x) \right] dx$$

$$\geqslant \frac{2}{b-a} \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right).$$

So,

$$\int_a^b f(x)\,g(x)\,dx \geqslant \frac{1}{2}\int_a^b \left[f(x)\,g(a+b-x)+f(x)\,g(x)\right]dx$$

$$\geqslant \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right). \tag{3.9}$$

(ii) Since f and g are convex functions, then by Theorem 1.1

$$\int_a^b f(x)\,g(a+b-x)\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$\geqslant \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) - \int_a^b f(x)\,g(x)\,dx. \tag{3.10}$$

On the other hand, we have

$$\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) \geqslant \int_a^b f(x)\,g(x)\,dx, \tag{3.11}$$

because f and g are oppositely ordered. By (3.10) and (3.11), we get

$$\int_a^b f(x)\,g(a+b-x)\,dx \geqslant \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$\geqslant \int_a^b f(x)\,g(x)\,dx, \tag{3.12}$$

and the proof of Theorem 1.2 is complete.

Proof of Theorem 1.3: We denote by F and G the following functions

$$F(x) = f(x) + f(a+b-x),$$

$$G(x) = g(x) + g(a+b-x).$$

Since f is convex function, then by Lemma 2.1, F is decreasing on $\left[a, \frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2}, b\right]$. In order to prove (1.4) we need to prove that G is decreasing on $\left[a, \frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2}, b\right]$. Let $x, y \in \left[a, \frac{a+b}{2}\right]$ and set $x^* = a+b-x$, $y^* = a+b-y$ where $x^*, y^* \in \left[\frac{a+b}{2}, b\right]$. It's clear that if $x \leqslant y$, then $x^* \geqslant y^*$. Since g is decreasing on $\left[a, \frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2}, b\right]$, then we have

$$g(x) \geqslant g(y),$$

and

$$g(x^*) \geqslant g(y^*).$$

Then

$$G(x) = g(x) + g(x^*) \geqslant g(y) + g(y^*) = G(y),$$

which implies that G is decreasing on $\left[a, \frac{a+b}{2}\right]$, by the same method we can prove easily that G is increasing on $\left[\frac{a+b}{2}, b\right]$. Then F and G have the same variation, and by applying Theorem A with $p : [a,b] \to \mathbb{R}_+$ is integrable symmetric function about $x = \frac{a+b}{2}$, we obtain inequality (1.4).

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