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SIMPSON'S TYPE INEQUALITIES FOR FUNCTIONS WHOSE THIRD DERIVATIVES IN THE ABSOLUTE VALUES ARE $P-$CONVEX

A. BARANI[1,*], S. BARANI[2] AND S. S. DRAGOMIR[3,4]

[1] *Department of Mathematics, Lorestan University*
P. O. Box 465, Khoramabad, Iran
[2] *Department of Civil Engineering, Shahid Chamran University*
P. O. Box 135, Ahvaz, Iran
[3] *School of Engineering and Science, Victoria University*
PO Box 14428 Melbourne City, MC 8001, Australia.
[4] *School of Computational and Applied Mathematics*
University of the Witwatersrand, Private Bag 3
Wits 2050, Johannesburg, South Africa.

Abstract. In this paper, we establish some new inequalities of Simpson's type based on $p-$convexity. Some applications for Simpson's Formula are also given.

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1. Introduction

The following inequality is well known in the literature as Simpson's inequality:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left|\left|f^{(4)}\right|\right|_\infty . \tag{1}$$

where the function $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval (a,b) and for the fourth derivative to be bounded on (a,b), that is

$$\left|\left|f^{(4)}\right|\right|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty.$$

It is well known that if the mapping f is neither four times differentiable nor its fourth derivative $f^{(4)}$ bounded on (a,b), then we cannot apply the Simpson's inequality.

For recent results on Simpsons type inequalities see the papers [2, 3, 6, 7], [10] - [14].

In [3], Alomari and Hussain established two results connected with the Simpson's type inequalities for quasi-convex functions by using the following lemma

Lemma 1.1. *Let $f:I\to\mathbb{R}$ be a function such that f'' be absolutely continuous on I°, the interior of I. Assume that $a,b\in I^\circ$, with $a<b$ and $f'''\in L[a,b]$. Then, the following equality*

*Corresponding author.
E-mail addresses:
alibarani2000@yahoo.com (and barani.a@lu.ac.ir) (A. Barani), seebb86@yahoo.com(S. Barani) sever.dragomir@vu.edu.au(S.S. Dragomir).

holds,

$$\begin{aligned}&\int_a^b f(x)dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\&=(b-a)^4\int_0^1 p(t)f'''(ta+(1-t)b)dt,\end{aligned} \tag{2}$$

where

$$p(t)=\begin{cases}\frac{1}{6}t^2(t-\frac{1}{2}), & t\in[0,\frac{1}{2}],\\ \frac{1}{6}(t-1)^2(t-\frac{1}{2}), & t\in(\frac{1}{2},1].\end{cases}$$

The main results of [3] are given by the following theorems.

Theorem 1.1. *Let $f'':I\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous on I°. Assume that $a,b\in I^\circ$, with $a<b$ and $f'''\in L[a,b]$. If $|f'''|$ is quasi-convex function on $[a,b]$ then, the following inequality holds*

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\le\frac{(b-a)^4}{1152}\left[\max\left\{\left|f'''(a)+\left|f'''\left(\frac{a+b}{2}\right)\right|\right|\right\}+\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|+\left|f'''(b)\right|\right\}\right].\end{aligned} \tag{3}$$

Theorem 1.2. *Let $f'':I\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous on I°. Assume that $a,b\in I^\circ$, with $a<b$ and $f'''\in L[a,b]$. If $|f'''|^q, q=p/(p-1)$ for $p>1$ is quasi-convex on $[a,b]$ then, the following inequality holds*

$$\begin{aligned}&\left|\int_a^b f(x)dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\&\le\frac{2^{-\frac{1}{p}}(b-a)^4}{48}\left(\frac{\Gamma(1+p)\Gamma(1+2p)}{\Gamma(2+3p)}\right)^{1/p}\left[\left(\max\left\{|f'''(a)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right.\\&\left.+\left(\max\left\{|f'''(b)|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{1/q}\right].\end{aligned} \tag{4}$$

On the other hand, S.S. Dragomir et al. in [8] defined the following class of functions:

Definition 1.1. Let $I\subseteq\mathbb{R}$ be an interval. The function $f:I\to\mathbb{R}$ is said to be $P-$convex (or belong to the class $P(I)$) if it is nonnegative and, for all $x,y\in I$ and $\lambda\in[0,1]$, satisfies the inequality

$$f(\lambda x+(1-\lambda)y)\le f(x)+f(y). \tag{5}$$

Note that $P(I)$ contain all nonnegative convex and quasi-convex functions. The function $f(x):=4-x^2,\ x\in[-2,2]$ is $P-$convex but it is not quasi-convex.
Since then numerous articles have appeared in the literature reflecting further applications in this category, see [1, 4, 5, 8, 9, 12] and references therein. It has been proved in [8] that for a $P-$convex on $[a,b]$ the following inequality holds

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int_a^b f(x)dx\le 2(f(a)+f(b)). \tag{6}$$

In [12] M.E. Özdemir and C. Yıldız presented the following Simpson's type inequality in which twice differentiable $p-$convex functions are involved.

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $|f''|^q$, $q \geq 1$ is $P-$convex then, the following inequality holds*

$$\begin{aligned} &\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{(b-a)^2}{162}\left\{\left|f'''(a)\right|^q + \left|f'''(b)\right|^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{7}$$

The main purpose of this paper is to establish new estimations of the Simpson's inequality (1) for the functions whose third derivatives in absolute values are $P-$convex.

2. Main results

The following theorem is a generalization of Theorem 1.1 for $p-$convex functions.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a function such that f'' be absolutely continuous on I°. Assume that $a, b \in I^\circ$, with $a < b$ and $f''' \in L[a, b]$. If $|f'''|$ is a $P-$convex function on $[a, b]$ then, the following inequality holds*

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{1152}\left\{\left|f'''(a)\right| + 2\left|f'''\left(\frac{a+b}{2}\right)\right| + \left|f'''(b)\right|\right\}. \end{aligned} \tag{8}$$

Proof. Since $|f'''|$ is a $P-$convex function, by using Lemma 1.1 we get

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq (b-a)^4\int_0^1 |p(t)|\left|f''(ta+(1-t)b)\right|dt \\ &\leq \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}}\left|t^2(t-\frac{1}{2})\right|\left|f'''(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1\left|(t-1)^2(t-\frac{1}{2})\right|\left|f'''(ta+(1-t)b)\right|dt\right] \\ &= \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}} t^2(\frac{1}{2}-t)\left(\left|f'''(b)\right| + \left|f'''(\frac{a+b}{2})\right|\right)dt\right. \\ &\left.+ \int_{\frac{1}{2}}^1 (1-t)^2(t-\frac{1}{2})\left(\left|f'''(\frac{a+b}{2})\right| + \left|f'''(a)\right|\right)dt\right] \\ &= \frac{(b-a)^4}{1152}\left\{\left|f'''(a)\right| + 2\left|f'''\left(\frac{a+b}{2}\right)\right| + \left|f'''(b)\right|\right\}, \end{aligned} \tag{9}$$

and proof is completed. □

An immediate consequence of Theorem 2.1 is as follows.

Corollary 2.1. *Let f as in Theorem 2.1 then,*
(i) if $f'''\left(\frac{a+b}{2}\right) = 0$, then we have

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{1152}\left\{\left|f'''(a)\right| + \left|f'''(b)\right|\right\}. \end{aligned} \tag{10}$$

(ii) if $f'''(a) = f'''(b) = 0$, then we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{576}\left|f'''\left(\frac{a+b}{2}\right)\right|.\end{aligned} \tag{11}$$

(iii) there exits $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$, then we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{288}M.\end{aligned} \tag{12}$$

The corresponding version for powers of the absolute value of the third derivative is incorporated in the following theorem.

Theorem 2.2. *Let $f : I \to \mathbb{R}$ be a function such that f'' be absolutely continuous on I°. Assume that $a, b \in I^\circ$, with $a < b$ and $f''' \in L[a,b]$. If $|f'''|^q, q = p/(p-1)$ for $p > 1$ is a $P-$convex function on $[a,b]$ then, the following inequality holds*

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq \frac{2^{-\frac{1}{p}}(b-a)^4}{96}\left(\frac{\Gamma(1+p)\Gamma(1+2p)}{\Gamma(2+3p)}\right)^{1/p}\left\{\left(\left|f'''(a)\right|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\ &\left.+ \left(\left|f'''(b)\right|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right\}.\end{aligned} \tag{13}$$

Proof. By assumption, Lemma 1.1 and Hölder's inequality, we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\leq (b-a)^4\int_0^1 |p(t)|\left|f'''(ta+(1-t)b)\right|dt \\ &\leq \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}}\left|t^2(t-\frac{1}{2})\right|\left|f'''(ta+(1-t)b)\right|dt\right. \\ &\left.+\int_{\frac{1}{2}}^1\left|(t-1)^2(t-\frac{1}{2})\right|\left|f'''(ta+(1-t)b)\right|dt\right] \\ &\leq \frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}}\left[t^2\left(\frac{1}{2}-t\right)\right]^p dt\right)^{1/p}\left(\int_0^{\frac{1}{2}}\left|f'''(ta+(1-t)b)\right|^q dt\right)^{1/q}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1\left[(t-1)^2\left(t-\frac{1}{2}\right)\right]^p dt\right)^{1/p}\left(\int_{\frac{1}{2}}^1\left|f''(ta+(1-t)b)\right|^q dt\right)^{1/q}\right\}.\end{aligned} \tag{14}$$

On the other hand, since $|f'''|^q$, is a $P-$convex function on $[a,b]$, by using (6) we get

$$\begin{aligned}&\int_0^{\frac{1}{2}} |f'''(ta+(1-t)b)|^q\,dt \\ &= \frac{1}{2}\left(\frac{1}{\frac{a+b}{2}-b}\right)\int_b^{\frac{a+b}{2}} |f'''(x)|^q\,dt \le \frac{1}{2}\left(|f'''(b)|^q + \left|f'''(\frac{a+b}{2})\right|^q\right).\end{aligned} \tag{15}$$

Similarly

$$\int_{\frac{1}{2}}^1 |f'''(ta+(1-t)b)|^q\,dt \le \frac{1}{2}\left(\left|f'''(\frac{a+b}{2})\right|^q + |f'''(a)|^q\right). \tag{16}$$

By combining (14), (15) and (16) we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ &\le \frac{2^{-\frac{1}{p}}(b-a)^4}{96}\left(\beta(1+p,1+2p)\right)^{1/p}\left\{\left(|f'''(a)|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right. \\ &\left.+\left(|f'''(b)|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right\}.\end{aligned} \tag{17}$$

Using (17) and the property

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

show that the inequality (13) holds and proof is completed. □

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.2. *Let f as in Theorem 2.2, if in addition*
(i) $f'''(\frac{a+b}{2}) = 0$, then we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ &\le \frac{2^{-\frac{1}{p}}(b-a)^4}{96}\left(\frac{\Gamma(1+p)\Gamma(1+2p)}{\Gamma(2+3p)}\right)^{1/p}\left\{|f'''(a)| + |f'''(b)|\right\}.\end{aligned} \tag{18}$$

(ii) $f'''(a) = f''(b) = 0$, then we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ &\le \frac{2^{\frac{-1}{p}}(b-a)^4}{96}\left(\frac{\Gamma(1+p)\Gamma(1+2p)}{\Gamma(2+3p)}\right)^{1/p}\left\{\left|f'''(\frac{a+b}{2})\right|\right\}.\end{aligned} \tag{19}$$

(iii) there exits $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$, then we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{2^{\frac{-1}{p}}(b-a)^4}{28}\left(\frac{\Gamma(1+p)\Gamma(1+2p)}{\Gamma(2+3p)}\right)^{1/p} M.
\end{aligned} \tag{20}$$

Another similar result may be extended in the following theorem, which is a generalization of the Theorem 1.3 to three times differentiable functions.

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a function such that f'' be absolutely continuous on I°. Assume that $a, b \in I^{\circ}$, with $a < b$ and $f''' \in L[a, b]$. If $|f'''|^q$, for $q \geq 1$ is a $P-$convex function on $[a, b]$ then, the following inequality holds*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^4}{1152}\left\{\left(|f'''(a)|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q} + \left(|f'''(b)|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right\}.
\end{aligned} \tag{21}$$

Proof. By Lemma 1.1 and using well known power mean inequality, we have

$$
\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\le (b-a)^4\int_0^1 |p(t)|\left|f'''(ta+(1-t)b)\right|dt \\
&\le \frac{(b-a)^4}{6}\left[\int_0^{\frac{1}{2}}\left|t^2(t-\frac{1}{2})\right|\left|f'''(ta+(1-t)b)\right|dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|(t-1)^2(t-\frac{1}{2})\right|\left|f'''(ta+(1-t)b)\right|dt\right] \\
&\le \frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)dt\right)^{1-1/q}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left|f''(ta+(1-t)b)\right|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)^{1-1/q}\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)\left|f'''(ta+(1-t)b)\right|^q dt\right)^{1/q}\right\} \\
&\le \frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)dt\right)^{1-1/q}\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)\left(\left|f'''(b)\right|^q+\left|f'''(\frac{a+b}{2})\right|^q\right)dt\right)^{1/q}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)^{1-1/q}\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)\left(\left|f'''(\frac{a+b}{2})\right|^q+\left|f'''(a)\right|^q\right)dt\right)^{1/q}\right\} \\
&= \frac{(b-a)^4}{6}\left\{\left(\int_0^{\frac{1}{2}} t^2\left(\frac{1}{2}-t\right)dt\right)\left(\left|f'''(b)\right|^q+\left|f'''(\frac{a+b}{2})\right|^q\right)^{1/q}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 (t-1)^2\left(t-\frac{1}{2}\right)dt\right)\left(\left|f'''(\frac{a+b}{2})\right|^q+\left|f'''(a)\right|^q\right)^{1/q}\right\} \\
&= \frac{(b-a)^4}{1152}\left\{\left(\left|f'''(a)\right|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}+\left(\left|f'''(b)\right|^q+\left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right\}.
\end{aligned}
\tag{22}
$$

□

Corollary 2.3. *Let f as in Theorem 2.3, if in addition*
(i) $f'''(\frac{a+b}{2})=0$, then (10) holds.
(ii) $f'''(a)=f'''(b)=0$, (11) holds.
(iii) there exits $M>0$ such that $|f(x)|\le M$ for all $x\in[a,b]$, then (12) holds.

3. Applications to Simpson's Formula

Let Δ be a division of the interval $[a,b]$ that is,

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b,$$

$h_i = \frac{(x_{i+1}-x_i)}{2}$ and consider the Simpson's formula

$$S(f,\Delta) = \sum_{i=0}^{n-1}\frac{f(x_i)+4f(x_i+h)+f(x_{i+1})}{6}(x_{i+1}-x_i).$$

It is well known that if the function $f : [a,b] \to \mathbb{R}$, is differentiable such that $f^{(4)}(x)$ exists on (a,b) and

$$M = \sup_{x\in(a,b)} \left| f^{(4)}(x) \right| < \infty,$$

then

$$I = \int_a^b f(x)dx = S(f,\Delta) + E_S(f,\Delta), \tag{23}$$

where the approximation error $E_S(f,\Delta)$ of the integral I, by the Simpson's formula $S(f,\Delta)$ satisfies

$$|E_S(f,\Delta)| \le \frac{M}{2880} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5.$$

However, if the mapping f is not fourth differentiable or the fourth derivative is not bounded on (a,b), then (23) cannot be applied. In the following we give a new estimation for the remainder term $E_S(f,\Delta)$ in terms of the third derivative.

Proposition 3.1. *Let $f : I \to \mathbb{R}$ be a function such that f'' be absolutely continuous on I°. Assume that $a,b \in I^\circ$, with $a < b$ and $f''' \in L[a,b]$. If $|f'''|$ is a $P-$convex function on $[a,b]$ then, for every division Δ of $[a,b]$ the following inequality holds*

$$|E_S(f,\Delta)| \le \frac{1}{1152} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left\{ \left| f'''(x_i) \right| + 2 \left| f''' \left(\frac{x_i + x_{i+1}}{2} \right) \right| + \left| f'''(x_{i+1}) \right| \right\}. \tag{24}$$

Proof. Since f is a $P-$convex function by using Theorem 2.1 on subinterval $[x_i, x_{i+1}], (i = 0,1,...,n-1)$ of the division Δ we get

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(x)dx - \frac{(x_{i+1} - x_i)}{6} \left[f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2} \right) + f(x_{i+1}) \right] \right| \\ &\le \frac{1}{1152} \left\{ \left| f'''(x_i) \right| + 2 \left| f''' \left(\frac{x_i + x_{i+1}}{2} \right) \right| + \left| f'''(x_{i+1}) \right| \right\}. \end{aligned} \tag{25}$$

Summing over $i = 1,...,n-1$ in (25) implies that

$$\begin{aligned} &\left| \int_a^b f(x)dx - S(f,\Delta) \right| \\ &\le \frac{1}{1152} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left\{ \left| f'''(x_i) \right| + 2 \left| f''' \left(\frac{x_i + x_{i+1}}{2} \right) \right| + \left| f'''(x_{i+1}) \right| \right\}, \end{aligned} \tag{26}$$

and proof is completed. □

References

[1] A.O. Akdemir and M.E. Ozdemir, Some Hadamard-Hadamard type inequalities for coordinated $P-$convex functions and Godunova-Levin functions, AIP Conference Proceedings, 1309 (2010), 7-15.

[2] M. Alomari, M. Darus, and S.S. Dragomir, New inequalities of Simpsons type for s-convex functions with applications, RGMIA Res. Rep. Coll., 12 (4) (2009), Article 9.

[3] M. Alomari, S. Hussain, Two inequalities of Simpson type for quasiconvex functions and applications, Appl. Math. E-Notes 11 (2011), 110-117.

[4] A. Barani, S. Barani and S.S. Dragomir, Hermite- Hadamard type inequalities for functions whose second derivatives absolute values are $P-$convex, RGMIA Res. Rep. Coll. 14 (2011) Article 73.

[5] A. Barani and S. Barani, Hermite-Hadamard type inequalities for functions when a power of the absolute value of the first derivative is $P-$convex (accepted to Bull. Austral. Math. Soc.).

[6] S.S. Dragomir, R.P. Agarwal and P. Cerone, On Simpsons inequality and applications, J. Inequal. Appl. 5 (2000) 533-579.
[7] S.S. Dragomir, On Hadamard's inequalities for convex functions, Mat. Balkanica 6 (1992), 215-222.
[8] S.S. Dragomir, J.E. Pecaric and L.E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (1995), 335-341.
[9] S.S. Dragomir C.E.M. Pearce, Quasiconvex functions and Hadamard's inequality, Bull. Austral. Math. Soc. 57 (1998) 377-385.
[10] B.Z. Liu, An inequality of Simpson type, Proc. R. Soc. A, 461 (2005) 2155-2158.
[11] M. Emin Özdemir, A. Ocak Akdemir, Havva Kavuramci and Merve Avci, On the Simpson's inequality for co-ordinated convex functions, ArXiv: 1101. 0075v2 [math.CA] 4 Jan 2011.
[12] M.E. Özdemir and C. Yildiz, New inequalities for Hermite-Hadamard and simpson type and applications, arXiv:1103.1965v1 [math.CA] 10 Mar 2011.
[13] N. Ujevic, Double integral inequalities of Simpson type and applications, J. Appl. Math. and Computing, 14 (2004) 213-223.
[14] L. Zhongxue, On sharp inequalities of Simpson type and Ostrowski type in two independent variables, Comp. and Math. with Appl., 56 (2008) 2043-2047.