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A COMPANION OF GRÜSS TYPE INEQUALITY FOR RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. In this paper we derive a new companion of Grüss' type inequality for Riemann–Stieltjes integral. Applications to the approximation problem of the Riemann-Stieltjes are also pointed out.

1. Introduction

In 1935, G. Grüss proved the following famous inequality regarding the integral of the product of two functions and the product of the integrals:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma) \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller one.

In [16], Dragomir and Fedotov, have established the following functional:

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \tag{1.2}$$

provided that the Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

In the same paper [16], the authors have proved the following inequality:

Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K>0$. Then we have*

$$|\mathcal{D}(f;u)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Also, in [7], Dragomir has obtained the following inequality:

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Theorem 2. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(y)-u(x)|\le L|x-y|,\forall x,y\in[a,b],\ \ (L>0)$$

and f is Riemann integrable on $[a,b]$.

If $m,M\in\mathbb{R}$, are such that $m\le f(x)\le M$, for any $x\in[a,b]$, then the inequality

$$|\mathcal{D}(f;u)|\le\frac{1}{2}L(M-m)(b-a) \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

For other recent inequalities for the Riemann–Stieltjes integral, see [1]–[7], [9]–[16], [18] and the references therein.

Motivated by [17], S.S. Dragomir in [10] has proved the following companion of the Ostrowski inequality for mappings of bounded variation:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b(f), \tag{1.5}$$

for any $x\in[a,\frac{a+b}{2}]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

The aim of this paper, is to study a companion functional of (1.2). Namely, we introduce the functional

$$\mathcal{GS}(f;u):=\int_a^{\frac{a+b}{2}}\frac{f(x)+f(a+b-x)}{2}du(x)-\frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt,$$

provided that the Stieltjes integral $\int_a^b\frac{f(x)+f(a+b-x)}{2}du(x)$, and the Riemann integral $\int_a^b f(t)\,dt$ exist. Therefore, several bounds for $\mathcal{GS}(f;u)$ are obtained. More specifically, the integrand f is assumed to be of r-H–Hölder type and the integrator u is to be of bounded variation, Lipschitzian and monotonic.

2. The case of bounded variation integrators

The following result holds:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, where r and $H>0$ are given, and $u:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then the inequality*

$$|\mathcal{GS}(f;u)|\le\frac{H}{r+1}(b-a)^r\cdot\bigvee_a^{\frac{a+b}{2}}(u), \tag{2.1}$$

holds.

Proof. It is well-known that for a continuous function $p:[a,b]\to\mathbb{R}$ and a function $\nu:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right|\le\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu).$$

Therefore, as u is of bounded variation on $[a,b]$, we have

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \\
&= \left|\int_a^{\frac{a+b}{2}} \left[\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right] du(x)\right| \\
&\le \sup_{x\in\left[a,\frac{a+b}{2}\right]} \left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \cdot \bigvee_a^{\frac{a+b}{2}}(u) \\
&= \frac{1}{b-a}\sup_{x\in\left[a,\frac{a+b}{2}\right]} \left|\int_a^b \left[\frac{f(x)+f(a+b-x)}{2} - f(t)\right] dt\right| \cdot \bigvee_a^{\frac{a+b}{2}}(u) \qquad (2.2)
\end{aligned}$$

As f is of r-H-Hölder type, then we have

$$\begin{aligned}
\left|\int_a^b \left[\frac{f(x)+f(a+b-x)}{2} - f(t)\right] dt\right| &= \left|\int_a^b \frac{f(x)-f(t)+f(a+b-x)-f(t)}{2} dt\right| \\
&\le \frac{1}{2}\int_a^b |f(x)-f(t)|\,dt + \frac{1}{2}\int_a^b |f(a+b-x)-f(t)|\,dt \\
&\le \frac{H}{2}\left[\int_a^b |x-t|^r\,dt + \int_a^b |a+b-x-t|^r\,dt\right] \\
&= \frac{H}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right] \qquad (2.3)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{x\in\left[a,\frac{a+b}{2}\right]} \left|\int_a^b \left[\frac{f(x)+f(a+b-x)}{2} - f(t)\right] dt\right| &\le \frac{H}{r+1}\cdot \sup_{x\in\left[a,\frac{a+b}{2}\right]} \left[(x-a)^{r+1}+(b-x)^{r+1}\right] \\
&\le \frac{H}{r+1}(b-a)^{r+1}, \qquad (2.4)
\end{aligned}$$

combining (2.2) and (2.4), we get the desired result in (2.1). □

Remark 1. *We remark that if* $\bigvee_a^{\frac{a+b}{2}}(u) = \bigvee_{\frac{a+b}{2}}^b(u)$, *then (2.1) becomes*

$$|\mathcal{GS}(f;u)| \le \frac{H}{2(r+1)}(b-a)^r \cdot \bigvee_a^b(u) \qquad (2.5)$$

Corollary 1. *Let u as in Theorem 4 and $f:[a,b]\to\mathbb{R}$ be an L–Lipschitzian mapping on $[a,b]$. Then the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{1}{2}L(b-a)\cdot \bigvee_a^{\frac{a+b}{2}}(u) \qquad (2.6)$$

holds.

Corollary 2. *Assume f as in Theorem 4. Let $u \in C^{(1)}[a,b]$. Then we have the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{H}{r+1}(b-a)^r \cdot \|u'\|_{1,\left[a,\frac{a+b}{2}\right]} \tag{2.7}$$

where $\|\cdot\|_1$ is the L_1 norm, namely $\|u'\|_{1,\left[a,\frac{a+b}{2}\right]} := \int_a^{\frac{a+b}{2}} |u'(t)|\,dt$.

Corollary 3. *Assume f as in Theorem 4. Let $u : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{LH}{2(r+1)}(b-a)^{r+1}. \tag{2.8}$$

Corollary 4. *Assume f as in Theorem 4. Let $u : [a,b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{H}{r+1}(b-a)^r \cdot \left|u\left(\frac{a+b}{2}\right) - u(a)\right|. \tag{2.9}$$

Remark 2. *For the last three inequalities, one may deduce several inequalities for L-Lipschitzian mappings by setting $r = 1$ and replace H by L. We left the details to the reader.*

Remark 3. *In Theorem 4, if $f(x)$ is assumed to be symmetric over $[a, \frac{a+b}{2}]$, i.e., $f(x) = f(a+b-x)$, then we have*

(2.10)

$$\left|\int_a^{\frac{a+b}{2}} f(x)\,du(x) - \frac{u\left(\frac{a+b}{2}\right) - u(a)}{b-a}\int_a^b f(t)\,dt\right| \le \frac{H}{r+1}(b-a)^r \cdot \bigvee_a^{\frac{a+b}{2}}(u).$$

3. The case of Lipschitzian integrators

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, where r and $H, L > 0$ are given. Then the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{LH}{(r+1)(r+2)}(b-a)^{r+1} \tag{3.1}$$

holds.

Proof. It is well-known that for a Riemann integrable function $p : [a,b] \to \mathbb{R}$ and L–Lipschitzian function $\nu : [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le L\int_a^b |p(t)|\,dt.$$

Therefore, as u is L-Lipschitzian on $[a,b]$, we have

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \\
&= \left|\int_a^{\frac{a+b}{2}} \left[\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right] du(x)\right| \\
&\le L\int_a^{\frac{a+b}{2}} \left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| dx \\
&= \frac{L}{b-a}\int_a^{\frac{a+b}{2}} \left|\int_a^b \left[\frac{f(x)+f(a+b-x)}{2} - f(t)\right] dt\right| dx
\end{aligned}$$

As f is of r-H-Hölder type, by (2.3) we get

$$\left|\int_a^b \left[\frac{f(x)+f(a+b-x)}{2} - f(t)\right] dt\right| \le \frac{H}{r+1}\left[(x-a)^{r+1} + (b-x)^{r+1}\right]$$

It follows that

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{L}{b-a}\int_a^{\frac{a+b}{2}} \left|\int_a^b \left[\frac{f(x)+f(a+b-x)}{2} - f(t)\right] dt\right| dx \\
&\le \frac{L}{b-a}\cdot\frac{H}{r+1}\int_a^{\frac{a+b}{2}} \left[(x-a)^{r+1} + (b-x)^{r+1}\right] dx \\
&= \frac{LH}{(r+1)(r+2)}(b-a)^{r+1}
\end{aligned}$$

and the theorem is proved. □

Corollary 5. *Let u as in Theorem 5 and $f:[a,b]\to\mathbb{R}$ be an K–Lipschitzian mapping on $[a,b]$. Then the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{1}{6}LK(b-a)^2 \tag{3.2}$$

holds.

Remark 4. *In Theorem 5, if $f(x)$ is assumed to be symmetric over $[a,\frac{a+b}{2}]$, i.e., $f(x)=f(a+b-x)$, then we have*

$$\left|\int_a^{\frac{a+b}{2}} f(x)\,du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \le \frac{LH}{(r+1)(r+2)}(b-a)^{r+1}. \tag{3.3}$$

4. The case of monotonic integrators

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$, where r and $H>0$ are given. Then the inequality*

$$|\mathcal{GS}(f;u)| \le \frac{H}{r+1}\left(1+\frac{1}{2^{r+1}}\right)(b-a)^r\left[u\left(\frac{a+b}{2}\right)-u(a)\right] \tag{4.1}$$

holds.

Proof. It is well-known that for a monotonic non-decreasing function $\nu:[a,b]\to\mathbb{R}$ and continuous function $p:[a,b]\to\mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le \int_a^b |p(t)|\,d\nu(t).$$

Therefore, as u is monotonic non-decreasing on $[a,b]$, we have

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \\
&= \left|\int_a^{\frac{a+b}{2}}\left[\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right]du(x)\right| \\
&= \frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}}\left[\int_a^b\left(\frac{f(x)+f(a+b-x)}{2}-f(t)\right)dt\right]du(x)\right| \\
&\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left|\int_a^b\left[\frac{f(x)+f(a+b-x)}{2}-f(t)\right]dt\right|du(x)
\end{aligned}$$

As f is of r-H-Hölder type, by (2.3) we get

$$\left|\int_a^b\left[\frac{f(x)+f(a+b-x)}{2}-f(t)\right]dt\right| \le \frac{H}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]$$

It follows that

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left|\int_a^b\left[\frac{f(x)+f(a+b-x)}{2}-f(t)\right]dt\right|du(x) \\
&\le \frac{1}{b-a}\cdot\frac{H}{r+1}\int_a^{\frac{a+b}{2}}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]du(x). \qquad (4.2)
\end{aligned}$$

Now, using Riemann–Stieltjes integral we have

$$\int_a^{\frac{a+b}{2}}(x-a)^{r+1}du(x) = \frac{(b-a)^{r+1}}{2^{r+1}}u\left(\frac{a+b}{2}\right) - (r+1)\int_a^{\frac{a+b}{2}}(x-a)^r u(x)\,dx$$

and

$$\int_a^{\frac{a+b}{2}} (b-x)^{r+1}\,du(x) = \frac{(b-a)^{r+1}}{2^{r+1}} u\left(\frac{a+b}{2}\right) - (b-a)^{r+1} u(a) + (r+1)\int_a^{\frac{a+b}{2}} (b-x)^r u(x)\,dx.$$

Adding the above equalities,we get

$$\int_a^{\frac{a+b}{2}} \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x). \tag{4.3}$$

$$= (b-a)^{r+1}\left[\frac{1}{2^r}u\left(\frac{a+b}{2}\right) - u(a)\right] + (r+1)\int_a^{\frac{a+b}{2}} \left[(b-x)^r - (x-a)^r\right] u(x)\,dx.$$

Now, by the monotonicity property of u we have

$$\int_a^{\frac{a+b}{2}} (x-a)^r u(x)\,dx \geq u(a)\int_a^{\frac{a+b}{2}} (x-a)^r\,dx = \frac{(b-a)^{r+1}}{2^{r+1}(r+1)} u(a)$$

and

$$\int_a^{\frac{a+b}{2}} (b-x)^r u(x)\,dx \leq u\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} (b-x)^r\,dx = \frac{\left(2^{r+1}-1\right)}{2^{r+1}(r+1)}(b-a)^{r+1} u\left(\frac{a+b}{2}\right)$$

which gives that

$$\int_a^{\frac{a+b}{2}} \left[(b-x)^r - (x-a)^r\right] u(x)\,dx = \frac{(b-a)^{r+1}}{2^{r+1}(r+1)}\left[\left(2^{r+1}-1\right) u\left(\frac{a+b}{2}\right) - u(a)\right]. \tag{4.4}$$

Therefore, by (4.3) and (4.4), we have

(4.5)

$$\int_a^{\frac{a+b}{2}} \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x).$$

$$= (b-a)^{r+1}\left[\frac{1}{2^r}u\left(\frac{a+b}{2}\right) - u(a)\right] + \frac{(b-a)^{r+1}}{2^{r+1}}\left[\left(2^{r+1}-1\right) u\left(\frac{a+b}{2}\right) - u(a)\right].$$

$$= \left(1 + \frac{1}{2^{r+1}}\right)(b-a)^{r+1}\left[u\left(\frac{a+b}{2}\right) - u(a)\right].$$

Combining (4.2) and (4.5), we get

$$\left|\int_a^{\frac{a+b}{2}} \frac{f(x) + f(a+b-x)}{2} du(x) - \frac{u\left(\frac{a+b}{2}\right) - u(a)}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq \frac{H}{r+1}\left(1 + \frac{1}{2^{r+1}}\right)(b-a)^r\left[u\left(\frac{a+b}{2}\right) - u(a)\right],$$

which is required. □

Corollary 6. *Let $f:[a,b]\to\mathbb{R}$ be an K–Lipschitzian mapping on $[a,b]$, and $u:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$, where $L>0$ is given. Then the inequality*

$$|\mathcal{GS}(f;u)|\le\frac{5K}{8}(b-a)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]\tag{4.6}$$

holds.

5. A Numerical quadrature formula for the Riemann–Stieltjes integral

In this section, we use Theorems 4–6, to approximate the Riemann–Stieltjes integral $\int_a^{\frac{a+b}{2}}\left[\frac{f(x)+f(a+b-x)}{2}\right]du(x)$, in terms of the Riemann integral $\int_a^b f(t)\,dt$.

Theorem 7. *Let f,u be as in Theorem 4 and consider*

$$I_h:=\{a=x_0<x_1<\ldots<x_{n-1}<x_n=b\},$$

be a partition of $[a,b]$. Denote $h_i=x_{i+1}-x_i$, $i=1,2,\cdots n-1$. Then we have

$$\int_a^{\frac{a+b}{2}}\frac{f(x)+f(a+b-x)}{2}du(x)=A_n(f,u,I_h)+R_n(f,u,I_h)\tag{5.1}$$

where,

$$A_n(f,u,I_h)=\sum_{i=0}^{n-1}\frac{u\left(\frac{x_{i+1}+x_i}{2}\right)-u(x_i)}{h_i}\times\int_{x_i}^{\frac{x_{i+1}+x_i}{2}}f(t)\,dt\tag{5.2}$$

and the Remainder $R_n(f,u,I_h)$ satisfies the estimation

$$|R_n(f,u,I_h)|\le\frac{H}{r+1}\cdot[\nu(h)]^r\cdot\bigvee_a^{\frac{a+b}{2}}(u)\tag{5.3}$$

where, $\nu(h)=\max\limits_{i=\overline{0,n-1}}\{h_i\}$.

Proof. Applying Theorem 4 on the intervals $[x_i,x_{i+1}]$, $i=1,2,\cdots n-1$, we get

$$\left|\int_{x_i}^{\frac{x_{i+1}+x_i}{2}}\frac{f(x)+f(a+b-x)}{2}du(x)-\frac{u\left(\frac{x_{i+1}+x_i}{2}\right)-u(x_i)}{h_i}\int_{x_i}^{\frac{x_{i+1}+x_i}{2}}f(t)\,dt\right|$$
$$\le\frac{H}{r+1}\cdot h_i^r\cdot\bigvee_{x_i}^{\frac{x_{i+1}+x_i}{2}}(u).$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned}\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - A_n(f,u,I_h)\right| &\le \frac{H}{r+1}\cdot\sum_{i=0}^{n-1} h_i^r \cdot \bigvee_{x_i}^{\frac{x_{i+1}+x_i}{2}}(u) \\ &\le \frac{H}{r+1}\cdot \max_{i=\overline{0,n-1}}\{h_i^r\}\cdot\sum_{i=0}^{n-1}\bigvee_{x_i}^{\frac{x_{i+1}+x_i}{2}}(u) \\ &= \frac{H}{r+1}\cdot\left[\max_{i=\overline{0,n-1}}\{h_i\}\right]^r\cdot\bigvee_a^{\frac{a+b}{2}}(u) \\ &= \frac{H}{r+1}\cdot[\nu(h)]^r\cdot\bigvee_a^{\frac{a+b}{2}}(u),\end{aligned}$$

and the theorem is proved. □

Theorem 8. *Let f,u be as in Theorem 5. Let I_h as above. Then we have*

$$\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) = A_n(f,u,I_h) + R_n(f,u,I_h) \tag{5.4}$$

where, $A_n(f,u,I_h)$ is defined in (5.2) and the Remainder $R_n(f,u,I_h)$ satisfies the estimation

$$|R_n(f,u,I_h)| \le \frac{LH}{(r+1)(r+2)}\cdot[\nu(h)]^r\cdot(b-a) \tag{5.5}$$

where, $\nu(h) = \max_{i=\overline{0,n-1}}\{h_i\}$.

Proof. Applying Theorem 5 on the intervals $[x_i,x_{i+1}]$, $i=1,2,\cdots n-1$, we get

$$\left|\int_{x_i}^{\frac{x_{i+1}+x_i}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{x_{i+1}+x_i}{2}\right)-u(x_i)}{h_i}\int_{x_i}^{\frac{x_{i+1}+x_i}{2}} f(t)\,dt\right| \le \frac{LH}{(r+1)(r+2)}\cdot h_i^{r+1}.$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned}\left|\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - A_n(f,u,I_h)\right| &\le \frac{LH}{(r+1)(r+2)}\cdot\sum_{i=0}^{n-1} h_i^{r+1} \\ &\le \frac{LH}{(r+1)(r+2)}\cdot\left[\max_{i=\overline{0,n-1}}\{h_i\}\right]^r\cdot\sum_{i=0}^{n-1} h_i \\ &\le \frac{LH}{(r+1)(r+2)}\cdot[\nu(h)]^r\cdot(b-a),\end{aligned}$$

and the theorem is proved. □

Theorem 9. *Let f, u be as in Theorem 6. Let I_h as above. Then we have*

$$\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) = A_n(f,u,I_h) + R_n(f,u,I_h) \tag{5.6}$$

where, $A_n(f,u,I_h)$ is defined in (5.2) and the Remainder $R_n(f,u,I_h)$ satisfies the estimation

$$|R_n(f,u,I_h)| \le \frac{H}{r+1}\left(1+\frac{1}{2^{r+1}}\right) \cdot [\nu(h)]^r \cdot \left[u\left(\frac{a+b}{2}\right) - u(a)\right] \tag{5.7}$$

where, $\nu(h) = \max\limits_{i=\overline{0,n-1}} \{h_i\}$.

Proof. Applying Theorem 6 on the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \cdots n-1$, we get

$$\left| \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - \frac{u\left(\frac{x_{i+1}+x_i}{2}\right) - u(x_i)}{h_i} \int_{x_i}^{\frac{x_{i+1}+x_i}{2}} f(t)\,dt \right|$$
$$\le \frac{H}{r+1}\left(1+\frac{1}{2^{r+1}}\right) \cdot h_i^r \cdot \left[u\left(\frac{x_i+x_{i+1}}{2}\right) - u(x_i)\right].$$

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\left| \int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2} du(x) - A_n(f,u,I_h) \right|$$
$$\le \frac{H}{r+1}\left(1+\frac{1}{2^{r+1}}\right) \cdot \sum_{i=0}^{n-1} h_i^r \left[u\left(\frac{x_i+x_{i+1}}{2}\right) - u(x_i)\right]$$
$$\le \frac{H}{r+1}\left(1+\frac{1}{2^{r+1}}\right) \cdot \left[\max_{i=\overline{0,n-1}} \{h_i\}\right]^r \cdot \sum_{i=0}^{n-1} \left[u\left(\frac{x_i+x_{i+1}}{2}\right) - u(x_i)\right]$$
$$\le \frac{H}{r+1}\left(1+\frac{1}{2^{r+1}}\right) \cdot [\nu(h)]^r \cdot \left[u\left(\frac{a+b}{2}\right) - u(a)\right],$$

and the theorem is proved. □

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