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New Inequalities For Convex Sequences

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Abstract. In this paper, we will show some new inequalities for convex sequences, and we will also make a connection between them and Chebyshev's inequality, which implies the existence of new class of sequences satisfying Chebyshev's inequality.

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1 Introduction and main results

A classic result due to Chebyshev (1882-1883) (see [1, 2, 3, 6, 7, 9]) is stated in the following theorem.

Theorem A *Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two sequences of real numbers monotonic in the same direction, and $p = (p_1, p_2, ..., p_n)$ be a positive sequence. Then*

$$\left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i a_i b_i\right) \geqslant \left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right). \tag{1.1}$$

If a and b are monotonic in opposite directions, then the reverse of the inequality in (1.1) *holds. In either case equality holds if and only if either $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$.*

There exist several results which show that Chebyshev inequality is valid under weaker conditions, for example the condition that the sequences be monotonic can be replaced by the condition that they be similarly ordered. In this case Theorem A is a simple consequence of the following identity

$$\left(\sum_{i=1}^{n}p_i\right)\left(\sum_{i=1}^{n}p_ia_ib_i\right)-\left(\sum_{i=1}^{n}p_ia_i\right)\left(\sum_{i=1}^{n}p_ib_i\right)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j\left(a_i-a_j\right)\left(b_i-b_j\right). \tag{1.2}$$

Note that the sequences $a=(a_1,a_2,...,a_n)$ and $b=(b_1,b_2,...,b_n)$ are said to be similarly ordered if

$$\left(a_i-a_j\right)\left(b_i-b_j\right)\geqslant 0,\ 1\leqslant i,j\leqslant n \tag{1.3}$$

holds, and they are said to be oppositely ordered if the reverse inequality holds.

Considerable attention has been given to the study of convex sequences and their properties, and the corresponding inequalities with applications. In general, convex sequences as discrete versions of convex functions play an important role in mathematical analysis and in the theory of inequalities. Inequalities for convex sequences provided considerable interest in proving a large number of elegant results with applications (see Wu and Shi [11], Wu and Debnath [12] and Mercer [5]). In addition, several authors including Mitrinović and Vasić [8], Roberts and Varberg [10], and Mitrinović et al. [6] presented a large number of major results for convex sequences and related inequalities.

The aim of this paper is to prove new type of inequalities for convex sequences, and we put a link between these inequalities and Chebyshev's inequality. Before we state our results we give the following definition.

Definition A ([4]) *Let $a=(a_1,a_2,...,a_n)$ be a sequence of real numbers, a is a convex sequence if for all $i=1,...,n-2$, we have*

$$a_i+a_{i+2}\geqslant 2a_{i+1}.$$

If the above inequality reversed, then a is termed concave sequence.

Theorem 1.1 *Let $a = (a_1, a_2, ..., a_{2n})$ and $b = (b_1, b_2, ..., b_{2n})$ be two convex (concave) sequences, and $p = (p_1, p_2, ..., p_{2n})$ be a positive sequence symmetric about n (i. e., $p_k = p_{2n+1-k}$, for all $k = 1, ..., 2n$). Then*

$$\left(\sum_{i=1}^{2n} p_i a_i b_i\right) + \left(\sum_{i=1}^{2n} p_i a_i b_{2n+1-i}\right)$$

$$\geqslant \frac{2}{\left(\sum\limits_{i=1}^{2n} p_i\right)} \left(\sum_{i=1}^{2n} p_i a_i\right) \left(\sum_{i=1}^{2n} p_i b_i\right). \tag{1.4}$$

If a is convex (or concave) and b is concave (or convex) sequences, then the inequality (1.4) is reversed. In either case equality holds if and only if either $a_1 = a_2 = ... = a_{2n}$ or $b_1 = b_2 = ... = b_{2n}$.

Corollary 1.1 *Let $a = (a_1, a_2, ..., a_{2n})$ and $b = (b_1, b_2, ..., b_{2n})$ be two convex (concave) sequences. If either a or b is symmetric about n, then*

$$\sum_{i=1}^{2n} a_i b_i \geqslant \frac{1}{2n} \left(\sum_{i=1}^{2n} a_i\right) \left(\sum_{i=1}^{2n} b_i\right). \tag{1.5}$$

If a is convex (or concave) and b is concave (or convex) sequences, then the inequality (1.5) is reversed. In either case equality holds if and only if either $a_1 = a_2 = ... = a_{2n}$ or $b_1 = b_2 = ... = b_{2n}$.

Theorem 1.2 *Let $a = (a_1, a_2, ..., a_{2n})$ and $b = (b_1, b_2, ..., b_{2n})$ be two convex (concave) sequences.*
(i) *If a and b are similarly ordered, then*

$$\sum_{i=1}^{2n} a_i b_i \geqslant \frac{1}{2} \left(\sum_{i=1}^{2n} a_i b_i + \sum_{i=1}^{2n} a_{2n+1-i} b_i\right) \geqslant \frac{1}{2n} \left(\sum_{i=1}^{2n} a_i\right) \left(\sum_{i=1}^{2n} b_i\right). \tag{1.6}$$

(ii) *If a and b are oppositely ordered, then*

$$\sum_{i=1}^{2n} a_{2n+1-i} b_i \geqslant \frac{1}{2n} \left(\sum_{i=1}^{2n} a_i\right) \left(\sum_{i=1}^{2n} b_i\right) \geqslant \sum_{i=1}^{2n} a_i b_i. \tag{1.7}$$

Theorem 1.3 *Let* $a = (a_1, a_2, ..., a_{2n})$ *and* $b = (b_1, b_2, ..., b_{2n})$ *be two sequences of real numbers where* a *is a convex sequence and* b *decreasing for all* $k = 1, ..., n$ *and increasing for all* $k = n, ..., 2n$. *Then the inequality* (1.4) *holds.*

Here, we obtain the discrete case of the Hermite-Hadamard inequalities.

Theorem 1.4 *Let* $a = (a_1, a_2, ..., a_n)$ *be a convex sequence of real numbers and let* $N = \left[\frac{n+1}{2}\right]$. *Then*

$$\frac{a_N + a_{n+1-N}}{2} \leqslant \frac{1}{n}\sum_{i=1}^{n} a_i \leqslant \frac{a_1 + a_n}{2}. \tag{1.8}$$

If $a = (a_1, a_2, ..., a_n)$ *is concave sequence, then the inequality* (1.8) *is reversed.*

2 Some lemmas

Lemma 2.1 *Let* $a = (a_1, a_2, ..., a_n)$ *be convex (or concave) sequence of real numbers. Then the sequence* $c = (c_1, c_2, ..., c_n)$, *where*

$$c_k = a_k + a_{n+1-k} \tag{2.1}$$

is decreasing (increasing) for all $k = 1, ..., \left[\frac{n+1}{2}\right]$ *and increasing (decreasing) for all* $k = \left[\frac{n+1}{2}\right], ..., n$.

Proof. Suppose that a is convex sequence. Since c is a symmetric sequence about $\left[\frac{n+1}{2}\right]$, then we need only to prove that c is decreasing for all $k = 1, ..., \left[\frac{n+1}{2}\right]$. We have

$$c_k - c_{k+1} = (a_k + a_{n+1-k}) - (a_{k+1} + a_{n-k})$$

$$= (a_k + a_{k+1} - a_{k+1} + ... + a_{n-k} - a_{n-k} + a_{n+1-k})$$

$$-(a_{k+1} + a_{k+2} - a_{k+2} + ... + a_{n-k-1} - a_{n-k-1} + a_{n-k})$$

$$= (a_k + a_{k+2} - 2a_{k+1}) + (a_{k+1} + a_{k+3} - 2a_{k+2})$$

$$+ ... + (a_{n-1-k} + a_{n+1-k} - 2a_{n-k}) \tag{2.2}$$

for all $k = 1, ..., \left[\frac{n+1}{2}\right]$. By using mathematical induction and (2.2), we obtain

$$c_k - c_{k+1} = \sum_{i=k}^{n-1-k} (a_i + a_{i+2} - 2a_{i+1}) \geqslant 0. \tag{2.3}$$

If a is a concave sequence, then by using similar proof we obtain the result.

Lemma 2.2 *Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ be two sequences of real numbers. If a and b are similarly ordered, then*

$$\sum_{i=1}^{n} a_i b_i \geqslant \sum_{i=1}^{n} a_i b_{n+1-i}. \tag{2.4}$$

If a and *b are oppositely ordered, then the inequality* (2.4) *is reversed.*

Proof. Since a and b are similarly ordered, then we have for all $i = 1, ..., n$

$$(a_i - a_{n+1-i})(b_i - b_{n+1-i}) \geqslant 0 \tag{2.5}$$

which implies that

$$a_i b_i + a_{n+1-i} b_{n+1-i} \geqslant a_i b_{n+1-i} + a_{n+1-i} b_i. \tag{2.6}$$

Then

$$2\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (a_i b_i + a_{n+1-i} b_{n+1-i})$$

$$\geqslant \sum_{i=1}^{n} (a_i b_{n+1-i} + a_{n+1-i} b_i) = 2\sum_{i=1}^{n} a_i b_{n+1-i}.$$

It follows that

$$\sum_{i=1}^{n} a_i b_i \geqslant \sum_{i=1}^{n} a_i b_{n+1-i}.$$

If a and b are oppositely ordered, then by using similar proof we obtain the result.

3 Proof of Theorems

Proof of Theorem 1.1. First, we suppose that a and b are convex sequences and we denote by U and V the following sequences

$$U_i = a_i + a_{2n+1-i}, \quad V_i = b_i + b_{2n+1-i}.$$

Since a and b are convex sequences, then by using Lemma 2.1 we deduce that U and V having the same direction of monotony. By applying Chebyshev's inequality for all $i = 1, ..., n$, we obtain

$$\left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i U_i V_i\right) \geqslant \left(\sum_{i=1}^{n} p_i U_i\right)\left(\sum_{i=1}^{n} p_i V_i\right), \tag{3.1}$$

where $p = (p_1, p_2, ..., p_{2n})$ is a positive sequence and symmetric about n. Then

$$\sum_{i=1}^{n} p_i \left(a_i b_i + a_{2n+1-i} b_{2n+1-i}\right) + \sum_{i=1}^{n} p_i \left(a_i b_{2n+1-i} + a_{2n+1-i} b_i\right)$$

$$\geqslant \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)} \left(\sum_{i=1}^{n} p_i \left(a_i + a_{2n+1-i}\right)\right)\left(\sum_{i=1}^{n} p_i \left(b_i + b_{2n+1-i}\right)\right). \tag{3.2}$$

Using the identities

$$\sum_{i=1}^{n} p_i a_{2n+1-i} b_{2n+1-i} = \sum_{i=1}^{n} p_{2n+1-i} a_{2n+1-i} b_{2n+1-i} = \sum_{i=n+1}^{2n} p_i a_i b_i, \tag{3.3}$$

$$\sum_{i=1}^{n} p_i a_{2n+1-i} b_i = \sum_{i=n+1}^{2n} p_i a_i b_{2n+1-i}, \ \sum_{i=1}^{n} p_i = \frac{1}{2}\sum_{i=1}^{2n} p_i \tag{3.4}$$

and

$$\sum_{i=1}^{n} p_i a_{2n+1-i} = \sum_{i=n+1}^{2n} p_i a_i, \ \sum_{i=1}^{n} p_i b_{2n+1-i} = \sum_{i=n+1}^{2n} p_i b_i \tag{3.5}$$

we obtain

$$\sum_{i=1}^{n} p_i a_i b_i + \sum_{i=n+1}^{2n} p_i a_i b_i + \sum_{i=1}^{n} p_i a_i b_{2n+1-i} + \sum_{i=n+1}^{2n} p_i a_i b_{2n+1-i}$$

$$\geqslant \frac{2}{\sum_{i=1}^{2n} p_i} \left(\sum_{i=1}^{n} p_i a_i + \sum_{i=n+1}^{2n} p_i a_i \right) \left(\sum_{i=1}^{n} p_i b_i + \sum_{i=n+1}^{2n} p_i b_i \right). \qquad (3.6)$$

So, from (3.6) we obtain inequality (1.4). Now, if f and g are concave sequences, then by using similar proof as above we obtain the result.

Proof of Theorem 1.2. (i) Since a and b are convex sequences and similarly ordered, then by Lemma 2.2 we have

$$\sum_{i=1}^{2n} a_i b_i \geqslant \sum_{i=1}^{2n} a_i b_{2n+1-i} \qquad (3.7)$$

which we can write

$$2\sum_{i=1}^{2n} a_i b_i \geqslant \sum_{i=1}^{2n} a_i b_{2n+1-i} + \sum_{i=1}^{2n} a_i b_i. \qquad (3.8)$$

By Theorem 1.1 and (3.8), we have

$$2\sum_{i=1}^{2n} a_i b_i \geqslant \sum_{i=1}^{2n} \left(a_i b_{2n+1-i} + a_i b_i \right) \geqslant \frac{2}{2n} \left(\sum_{i=1}^{2n} a_i \right) \left(\sum_{i=1}^{2n} b_i \right).$$

It follows that

$$\sum_{i=1}^{2n} a_i b_i \geqslant \frac{1}{2} \sum_{i=1}^{2n} \left(a_i b_{2n+1-i} + a_i b_i \right) \geqslant \frac{1}{2n} \left(\sum_{i=1}^{2n} a_i \right) \left(\sum_{i=1}^{2n} b_i \right). \qquad (3.9)$$

(ii) Since a and b are convex sequences, then by Theorem 1.1

$$\sum_{i=1}^{2n} a_i b_{2n+1-i} - \frac{1}{2n} \left(\sum_{i=1}^{2n} a_i \right) \left(\sum_{i=1}^{2n} b_i \right)$$

$$\geqslant \frac{1}{2n}\left(\sum_{i=1}^{2n} a_i\right)\left(\sum_{i=1}^{2n} b_i\right) - \sum_{i=1}^{2n} a_i b_i. \tag{3.10}$$

On the other hand, we have

$$\frac{1}{2n}\left(\sum_{i=1}^{2n} a_i\right)\left(\sum_{i=1}^{2n} b_i\right) \geqslant \sum_{i=1}^{2n} a_i b_i \tag{3.11}$$

because a and b are oppositely ordered. By (3.10) and (3.11), we get

$$\sum_{i=1}^{2n} a_i b_{2n+1-i} \geqslant \frac{1}{2n}\left(\sum_{i=1}^{2n} a_i\right)\left(\sum_{i=1}^{2n} b_i\right) \geqslant \sum_{i=1}^{2n} a_i b_i. \tag{3.12}$$

Now, if a and b are concave sequences, then by using similar proof as above we obtain the result.

Proof of Theorem 1.3: We denote by U and V the following sequences

$$U_i = a_i + a_{2n+1-i}, \tag{3.13}$$

$$V_i = b_i + b_{2n+1-i}. \tag{3.14}$$

Since U is convex sequence, then by Lemma 2.1, U is decreasing for all $i = 1, ..., n$ and increasing for all $i = n, ..., 2n$. In order to prove (1.4) we need to prove that V is decreasing for all $i = 1, ..., n$ and increasing for all $i = n, ..., 2n$. Let $1 \leqslant i \leqslant n$, and we denote by $j = 2n + 1 - i$ $(n \leqslant j \leqslant 2n)$. Then

$$V_i - V_{i+1} = (b_i + b_{2n+1-i}) - (b_{i+1} + b_{2n-i})$$

$$= (b_i - b_{i+1}) + (b_{2n+1-i} - b_{2n-i})$$

$$= (b_i - b_{i+1}) + (b_j - b_{j-1}) \geqslant 0 \tag{3.15}$$

because b is decreasing for all $i = 1, ..., n$ and increasing for all $i = n, ..., 2n$, by the same method we can prove easily that V is increasing for all $i = n, ..., 2n$.

Then U and V have the same direction of monotony, and by applying Theorem A with $p = (p_1, p_2, ..., p_{2n})$ is a positive sequence symmetric about n, we obtain inequality (1.4) .

Proof of Theorem 1.4. Suppose that $a = (a_1, a_2, ..., a_n)$ is a convex sequence, by applying Lemma 2.1 for the sequence $v_k = a_k + a_{n+1-k}$ we obtain the following inequalities

$$v_N \leqslant v_k \leqslant v_1, \text{ for all } k = 1, ..., N \tag{3.16}$$

and

$$v_N \leqslant v_k \leqslant v_n, \text{ for all } k = N, ..., n. \tag{3.17}$$

By (3.16) and (3.17) we deduce that

$$a_N + a_{n+1-N} \leqslant a_k + a_{n+1-k} \leqslant a_1 + a_n, \ k = 1, ..., n \tag{3.18}$$

which implies

$$n\left(a_N + a_{n+1-N}\right) \leqslant \sum_{k=1}^{n} \left(a_k + a_{n+1-k}\right) = 2\sum_{k=1}^{n} a_k \leqslant n\left(a_1 + a_n\right).$$

So,

$$\frac{a_N + a_{n+1-N}}{2} \leqslant \frac{1}{n}\sum_{k=1}^{n} a_k \leqslant \frac{a_1 + a_n}{2}.$$

For the case of concave sequence, we use similar proof.

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