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An Analogue of the Ostrowski Inequality and Applications

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Abstract. A new analogue of the Ostrowski inequality is introduced in three different cases for functions in $L^1[a,b]$ and $L^\infty[a,b]$ spaces and its application is given for deriving error bounds of some quadrature rules.

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1. Introduction

Let $L^p[a,b]$ $(1 \le p < \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\| f \|_p = \left(\int_a^b | f(t) |^p \, dt \right)^{1/p},$$

and $L^\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\| f \|_\infty = ess \sup_{x \in [a,b]} | f(x) | .$$

If $h \in L^1[a,b]$ and $g \in L^\infty[a,b]$, the following inequality is well known

$$\left| \int_a^b h(x)\, g(x)\, dx \right| \le \| h \|_1 \| g \|_\infty .$$

For two absolutely continuous functions $f, g : [a,b] \to \mathbf{R}$ such that $f, g, fg \in L^1[a,b]$, the Chebyshev functional is defined by

$$\mathbf{T}(f,g) = \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(x)\,dx \right)\left(g(x) - \frac{1}{b-a}\int_a^b g(x)\,dx \right) dx$$
$$= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right).$$

A well-known inequality in the literature, which is related to the Chebyshev functional, is the Ostrowski inequality [15]. If $f : [a,b] \to \mathbf{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left(\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right) \| f' \|_\infty \quad \text{for all } x \in [a,b]. \tag{1}$$

The above result has been extended for absolutely continuous functions and Lebesgue *p*-norms of the derivative f' in [3,4] as follows: Let $f : [a,b] \to \mathbf{R}$ be absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \begin{cases} \dfrac{1}{(p+1)^{\frac{1}{p}}}\left(\left(\dfrac{x-a}{b-a}\right)^{p+1} + \left(\dfrac{b-x}{b-a}\right)^{p+1} \right)(b-a)^{\frac{1}{q}} \| f' \|_q & (\frac{1}{p} + \frac{1}{q} = 1,\ p > 1), \\ \left(\dfrac{1}{2} + \dfrac{1}{b-a}\left| x - \dfrac{a+b}{2} \right| \right) \| f' \|_1 . \end{cases} \tag{2}$$

The constants $1/(p+1)^{1/p}$ and $1/2$ in (2) are respectively sharp in the sense that they cannot be replaced by a smaller constant.

The inequalities (1), (2) can also be obtained, in a slightly different form, as particular cases of some results established by Fink in [10] for *n*-time differentiable functions.

For other Ostrowski type inequalities concerning Lipschitzian type functions, see [5].

The cases of bounded variation functions and monotonic functions were considered in [6].

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables see the monograph [7] and the references therein. For related results see [1,8,11] and [13,14].

The Ostrowski inequality also plays an important role in numerical quadrature rules [9,12].

In this paper we introduce a new analogue of the Ostrowski inequality in three different cases and apply them for some quadrature rules. For this purpose, let us first consider the following well-known kernel on $[a,b]$:

$$K(x;t)=\begin{cases} t-a & t\in[a,x], \\ t-b & t\in(x,b], \end{cases} \tag{3}$$

After some computations, we can directly conclude that

$$\int_a^b \left|K(x;t)\right| dt = \int_a^x (t-a)\,dt - \int_x^b (t-b)\,dt = \frac{1}{2}\left((x-a)^2+(b-x)^2\right),$$

and also

$$\int_a^b f'(t)\,K(x;t)\,dt = (b-a)\,f(x) - \int_a^b f(x)\,dx\,. \tag{4}$$

2. Main Results

Theorem 1. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $\alpha(x)\le f'(x)\le\beta(x)$ *for any* $\alpha,\beta\in C[a,b]$ *and* $x\in[a,b]$ *then the following inequality holds*

$$\begin{aligned}\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\,dt+\int_x^b (t-b)\,\beta(t)\,dt\right) &\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt+\int_x^b (t-b)\,\alpha(t)\,dt\right).\end{aligned} \tag{5}$$

Proof. By referring to the kernel (3) and identity (4) we first have

$$\begin{aligned}&\int_a^b K(x;t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt \\ &= (b-a)\,f(x)-\int_a^b f(t)\,dt-\frac{1}{2}\left(\int_a^b K(x;t)(\alpha(t)+\beta(t))\,dt\right) \\ &= (b-a)\,f(x)-\int_a^b f(t)\,dt-\frac{1}{2}\left(\int_a^x (t-a)(\alpha(t)+\beta(t))\,dt+\int_x^b (t-b)(\alpha(t)+\beta(t))\,dt\right).\end{aligned} \tag{6}$$

On the other hand, the given assumption $\alpha(t)\le f'(t)\le\beta(t)$ results in

$$\left|\,f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{\beta(t)-\alpha(t)}{2}\,. \tag{7}$$

Therefore, one can conclude from (11) and (12) that

$$\left|(b-a)\,f(x)-\int_a^b f(t)\,dt-\frac{1}{2}\left(\int_a^x (t-a)\,(\alpha(t)+\beta(t))\,dt+\int_x^b (t-b)\,(\alpha(t)+\beta(t))\,dt\right)\right|$$

$$=\left|\int_a^b K(x;t)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)dt\right|\le\int_a^b \left|K(x;t)\right|\frac{\beta(t)-\alpha(t)}{2}\,dt \tag{8}$$

$$=\frac{1}{2}\left(\int_a^x (t-a)\,(\beta(t)-\alpha(t))\,dt-\int_x^b (t-b)\,(\beta(t)-\alpha(t))\,dt\right).$$

After re-arranging (8), the main inequality (5) will be derived directly. ■

Theorem 1 is actually remarkable as it improves all previous results which made use of the Lebesgue norms of $f'(x)$ in (1) and (2). Moreover, a further advantage of this theorem is that necessary computations in bounds (5) are just in terms of the pre-assigned functions $\alpha(t), \beta(t)$ (not f').

Special case 1. Suppose that $f'(x)$ is bounded at two arbitrary linear functions, e.g. $\alpha(x)=\alpha_1 x+\alpha_0\neq 0$ and $\beta(x)=\beta_1 x+\beta_0\neq 0$. In this case, the main inequality (5) takes the form

$$\frac{(x-a)^2}{b-a}\left(\frac{\alpha_1}{3}(x-a)+\frac{\alpha_0+a\,\alpha_1}{2}\right)-\frac{(x-b)^2}{b-a}\left(\frac{\beta_1}{3}(x-b)+\frac{\beta_0+b\,\beta_1}{2}\right)$$

$$\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\le \tag{9}$$

$$\frac{(x-a)^2}{b-a}\left(\frac{\beta_1}{3}(x-a)+\frac{\beta_0+a\,\beta_1}{2}\right)-\frac{(x-b)^2}{b-a}\left(\frac{\alpha_1}{3}(x-b)+\frac{\alpha_0+b\,\alpha_1}{2}\right).$$

In this sense, Dragomir in [2] has recently obtained a special case of (5) for $\alpha(x)=\alpha_0\neq 0$ and $\beta(x)=\beta_0\neq 0$, which are equivalent to a special case of (9) for $\alpha_1=\beta_1=0$, as follows:

$$\frac{\alpha_0\,(x-a)^2-\beta_0\,(b-x)^2}{2\,(b-a)}\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{\beta_0\,(x-a)^2-\alpha_0\,(b-x)^2}{2\,(b-a)}.$$

Remark 1. Although $\alpha(x)\le f'(x)\le\beta(x)$ is a straightforward condition in theorem 1, however sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, we can make use of two analogue theorems. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $\alpha(x)\le f'(x)$ *for any* $\alpha\in C[a,b]$ *and* $x\in[a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\alpha(t)\,dt+\int_x^b (t-b)\alpha(t)\,dt-\max\{x-a,b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right)\right)$$
$$\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\le \tag{10}$$
$$\frac{1}{b-a}\left(\int_a^x (t-a)\alpha(t)\,dt+\int_x^b (t-b)\alpha(t)\,dt+\max\{x-a,b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right)\right).$$

Proof. Since

$$\int_a^b K(x;t)\left(f'(t)-\alpha(t)\right)dt$$
$$=(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^b K(x;t)\alpha(t)\,dt\right)$$
$$=(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^x (t-a)\alpha(t)\,dt+\int_x^b (t-b)\alpha(t)\,dt\right),$$

so we have

$$\left|(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^x (t-a)\alpha(t)\,dt+\int_x^b (t-b)\alpha(t)\,dt\right)\right|$$
$$=\left|\int_a^b K(x;t)\left(f'(t)-\alpha(t)\right)dt\right|\le\int_a^b \left|K(x;t)\right|\left(f'(t)-\alpha(t)\right)dt \tag{11}$$
$$\le\max_{t\in[a,b]}\left|K(x;t)\right|\int_a^b\left(f'(t)-\alpha(t)\right)dt=\max\{x-a,b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\,dt\right).$$

After re-arranging (11), the main inequality (10) will be derived. ■

Special case 2. If $\alpha(x)=\alpha_0\neq 0$ then (10) becomes

$$\alpha_0\left(x-\frac{a+b}{2}\right)-\max\{x-a,b-x\}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right)\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\le\alpha_0\left(x-\frac{a+b}{2}\right)+\max\{x-a,b-x\}\left(\frac{f(b)-f(a)}{b-a}-\alpha_0\right).$$

1.3. Theorem 3. *Let* $f:\mathbf{I}\to\mathbf{R}$, *where* $\mathbf{I}$ *is an interval, be a function differentiable in the interior* $\mathbf{I}^0$ *of* $\mathbf{I}$, *and let* $[a,b]\subset\mathbf{I}^0$. *If* $f'(x)\le\beta(x)$ *for any* $\beta\in C[a,b]$ *and* $x\in[a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt+\int_x^b (t-b)\,\beta(t)\,dt-\max\{x-a,b-x\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right)\right)$$

$$\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\le \tag{12}$$

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt+\int_x^b (t-b)\,\beta(t)\,dt+\max\{x-a,b-x\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right)\right).$$

Proof. Since

$$\int_a^b K(x;t)\left(f'(t)-\beta(t)\right)dt$$
$$=(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^b K(x;t)\,\beta(t)\,dt\right)$$
$$=(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^x (t-a)\,\beta(t)\,dt+\int_x^b (t-b)\,\beta(t)\,dt\right),$$

so we have

$$\left|(b-a)\,f(x)-\int_a^b f(t)\,dt-\left(\int_a^x (t-a)\,\beta(t)\,dt+\int_x^b (t-b)\,\beta(t)\,dt\right)\right|$$
$$=\left|\int_a^b K(x;t)\left(f'(t)-\beta(t)\right)dt\right|\le\int_a^b\left|K(x;t)\right|\left(\beta(t)-f'(t)\right)dt \tag{13}$$
$$\le\max_{t\in[a,b]}\left|K(x;t)\right|\int_a^b\left(\beta(t)-f'(t)\right)dt=\max\{x-a,b-x\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right).$$

After re-arranging (13), the main inequality (12) will be derived. ■

Special case 3. If $\beta(x)=\beta_0\neq 0$ then (12) becomes

$$\beta_0\left(x-\frac{a+b}{2}\right)-\max\{x-a,b-x\}\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right)\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\le\beta_0\left(x-\frac{a+b}{2}\right)+\max\{x-a,b-x\}\left(\beta_0-\frac{f(b)-f(a)}{b-a}\right).$$

3. Applications in numerical integration

A general ($n+1$)-point weighted quadrature formula is denoted by

$$\int_a^b w(x)\,f(x)\,dx=\sum_{k=0}^{n} w_k\;f(x_k)+R_{n+1}[f], \tag{14}$$

where $w(x)$ is a positive weight function on $[a,b]$, $\{x_k\}_{k=0}^n$ and $\{w_k\}_{k=0}^n$ are respectively nodes and weight coefficients and $R_{n+1}[f]$ is the corresponding error [16].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most *d*. The quadrature formula (14) has degree of exactness *d* if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}[p] = 0$. In addition, if $R_{n+1}[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (14) has precise degree of exactness *d*.

The convergence order of quadrature rule (14) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^n$ we can always achieve a degree of exactness $d = n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_{n+1}(x) = \prod_{k=0}^{n} (x - x_k),$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} f(x_k)\, L(x; x_k) + r_{n+1}(f; x),$$

where

$$L(x; x_k) = \frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x - x_k)} \qquad (k = 0,1,\ldots,n),$$

we obtain (14), with

$$w_k = \frac{1}{\Psi'_{n+1}(x_k)} \int_a^b \frac{\Psi_{n+1}(x)\, w(x)}{x - x_k} dx \qquad (k = 0,1,\ldots,n),$$

and

$$R_{n+1}[f] = \int_a^b r_{n+1}(f; x)\, w(x)\, dx.$$

We should note that for each $f \in \mathbf{\Pi}_n$ we have $r_{n+1}(f; x) = 0$ and therefore $R_{n+1}[f] = 0$.

Quadrature formulae obtained in this way are known as interpolatory. If a quadrature is not of the interpolatory type, i.e. if it does not follow the concept of the degree of exactness, then it would be a nonstandard quadrature rule.

Usually the simplest interpolatory quadrature formula of type (14) with pre-determined nodes $\{x_k\}_{k=0}^n \in [a,b]$ is called a weighted Newton-Cotes formula. For $w(x) = 1$ and the equidistant nodes $\{x_k\}_{k=0}^n = \{a + kh\}_{k=0}^n$ with $h = (b-a)/n$, the classical Newton-Cotes formulas including

the midpoint rule for $n=0$ and $w(x)=1$, the trapezoidal rule for $n=1$ and $w(x)=1$ and so on are derived.
In this section we use theorems 1, 2 and 3 to obtain error bounds for midpoint rule and six further nonstandard quadratures as follows:

$$I_1(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(\frac{a+b}{2}),$$

$$I_2(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(a),$$

$$I_3(f): \quad \int_a^b f(x)\,dx \cong (b-a)\, f(b),$$

$$I_4(f): \quad \int_a^b f(t)\,dt \cong \frac{b-a}{2}\left(-f(a)+2f(\frac{a+b}{2})+f(b)\right),$$

$$I_5(f): \quad \int_a^b f(t)\,dt \cong \frac{b-a}{2}\left(f(a)+2f(\frac{a+b}{2})-f(b)\right),$$

$$I_6(f): \quad \int_a^b f(x)\,dx \cong (b-a)\big(2f(a)-f(b)\big),$$

$$I_7(f): \quad \int_a^b f(x)\,dx \cong (b-a)\big(-f(a)+2f(b)\big).$$

Corollary 1. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x=\frac{a+b}{2} \in [a,b]$ in (5), the error of midpoint rule $I_1(f)$ can be bounded as

$$\int_a^{\frac{a+b}{2}} (t-a)\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\beta(t)\,dt \le (b-a)\, f(\frac{a+b}{2}) - \int_a^b f(t)\,dt$$
$$\le \int_a^{\frac{a+b}{2}} (t-a)\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\alpha(t)\,dt. \qquad (15)$$

For instance, if $\alpha(x)=\alpha_1 x+\alpha_0 \ne 0$ and $\beta(x)=\beta_1 x+\beta_0 \ne 0$ in (15) then we have

$$\frac{(b-a)^2}{4}\left(\frac{b-a}{6}(\alpha_1+\beta_1)+\frac{\alpha_0+a\alpha_1-(\beta_0+b\beta_1)}{2}\right) \le (b-a)\, f(\frac{a+b}{2}) - \int_a^b f(t)\,dt$$
$$\le \frac{(b-a)^2}{4}\left(\frac{b-a}{6}(\alpha_1+\beta_1)+\frac{\beta_0+a\beta_1-(\alpha_0+b\alpha_1)}{2}\right),$$

provided that $\alpha_1 t+\alpha_0 \le f'(t) \le \beta_1 t+\beta_0 \quad \forall t \in [a,b]$.

Corollary 2. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = a \in [a,b]$ in (5), the error of nonstandard quadrature $I_2(f)$ can be bounded as

$$\int_a^b (t-b)\,\beta(t)\,dt \le (b-a)\,f(a) - \int_a^b f(t)\,dt \le \int_a^b (t-b)\,\alpha(t)\,dt. \tag{16}$$

For instance, if $\alpha(x) = \alpha_1 x + \alpha_0 \ne 0$ and $\beta(x) = \beta_1 x + \beta_0 \ne 0$ then (16) takes the form

$$(b-a)^2 \left(\frac{\beta_1}{3}(b-a) - \frac{\beta_0 + b\,\beta_1}{2} \right) \le (b-a)\,f(a) - \int_a^b f(t)\,dt \le (b-a)^2 \left(\frac{\alpha_1}{3}(b-a) - \frac{\alpha_0 + b\,\alpha_1}{2} \right).$$

provided that $\alpha_1\, t + \alpha_0 \le f'(t) \le \beta_1\, t + \beta_0 \quad \forall t \in [a,b]$.

Corollary 3. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then by replacing $x = b \in [a,b]$ in (5), the error of nonstandard quadrature $I_3(f)$ can be bounded as

$$\int_a^b (t-a)\,\alpha(t)\,dt \le (b-a)\,f(b) - \int_a^b f(t)\,dt \le \int_a^b (t-a)\,\beta(t)\,dt. \tag{17}$$

For instance, if $\alpha(x) = \alpha_1 x + \alpha_0 \ne 0$ and $\beta(x) = \beta_1 x + \beta_0 \ne 0$ in (17) then

$$(b-a)^2 \left(\frac{\alpha_1}{3}(b-a) + \frac{\alpha_0 + a\,\alpha_1}{2} \right) \le (b-a)\,f(b) - \int_a^b f(t)\,dt \le (b-a)^2 \left(\frac{\beta_1}{3}(b-a) + \frac{\beta_0 + a\,\beta_1}{2} \right).$$

provided that $\alpha_1\, t + \alpha_0 \le f'(t) \le \beta_1\, t + \beta_0 \quad \forall t \in [a,b]$.

Corollary 4. If $\alpha(x) \le f'(x) \le \beta(x)$ for any $x \in [a,b]$ and $\alpha, \beta \in C[a,b]$ then the error of nonstandard quadrature $I_4(f)$ can be bounded as

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} (t-a)\,\alpha(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\alpha(t)\,dt + \frac{b-a}{2}\int_a^b \alpha(t)\,dt \\
&\le \frac{b-a}{2}\left(-f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right) - \int_a^b f(t)\,dt \le \\
&\int_a^{\frac{a+b}{2}} (t-a)\,\beta(t)\,dt + \int_{\frac{a+b}{2}}^b (t-b)\,\beta(t)\,dt + \frac{b-a}{2}\int_a^b \beta(t)\,dt.
\end{aligned} \tag{18}$$

Proof. To prove (18) we need to use the results of both theorems 2 and 3 simultaneously such that by replacing $x=(a+b)/2$ in (10) we first obtain

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}(t-a)\,\alpha(t)\,dt+\int_{\frac{a+b}{2}}^b(t-b)\,\alpha(t)\,dt+\frac{b-a}{2}\int_a^b\alpha(t)\,dt\\&\leq\frac{b-a}{2}\left(-f(a)+2f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt,\end{aligned}\tag{19}$$

provided that $\alpha(t)\leq f'(t)\quad\forall t\in[a,b]$. On the other hand, replacing $x=(a+b)/2$ in (12) gives

$$\begin{aligned}&\frac{b-a}{2}\left(-f(a)+2f(\frac{a+b}{2})+f(b)\right)-\int_a^b f(t)\,dt\leq\\&\int_a^{\frac{a+b}{2}}(t-a)\,\beta(t)\,dt+\int_{\frac{a+b}{2}}^b(t-b)\,\beta(t)\,dt+\frac{b-a}{2}\int_a^b\beta(t)\,dt,\end{aligned}\tag{20}$$

provided that $f'(t)\leq\beta(t)\quad\forall t\in[a,b]$. Now by combining two latter results (19) and (20) the inequality (18) is derived.

Corollary 5. If $\alpha(x)\leq f'(x)\leq\beta(x)$ for any $x\in[a,b]$ and $\alpha,\beta\in C[a,b]$ then the error of nonstandard quadrature $I_5(f)$ can be bounded as

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}(t-a)\,\beta(t)\,dt+\int_{\frac{a+b}{2}}^b(t-b)\,\beta(t)\,dt-\frac{b-a}{2}\int_a^b\beta(t)\,dt\\&\leq\frac{b-a}{2}\left(f(a)+2f(\frac{a+b}{2})-f(b)\right)-\int_a^b f(t)\,dt\leq\\&\int_a^{\frac{a+b}{2}}(t-a)\,\alpha(t)\,dt+\int_{\frac{a+b}{2}}^b(t-b)\,\alpha(t)\,dt-\frac{b-a}{2}\int_a^b\alpha(t)\,dt.\end{aligned}\tag{21}$$

Proof. The proof of (21) is similar to that of corollary 4 if one replaces $x=(a+b)/2$ in respectively (10) and (12) and then combines them together.

Corollary 6. If $\alpha(x)\leq f'(x)\leq\beta(x)$ for any $x\in[a,b]$ and $\alpha,\beta\in C[a,b]$ then the error of nonstandard quadrature $I_6(f)$ can be bounded as

$$\int_a^b(t+a-2b)\,\beta(t)\,dt\leq(b-a)\big(2f(a)-f(b)\big)-\int_a^b f(t)\,dt\leq\int_a^b(t+a-2b)\,\alpha(t)\,dt.\tag{22}$$

Proof. Again, to prove (22) we need to use the results of both theorems 2 and 3 simultaneously such that by replacing $x=a$ in (10) we first obtain

$$(b-a)\left(2f(a)-f(b)\right)-\int_a^b f(t)\,dt \le \int_a^b (t+a-2b)\,\alpha(t)\,dt\,, \tag{23}$$

provided that $\alpha(t)\le f'(t)\quad \forall t\in[a,b]$. On the other hand, replacing $x=a$ in (12) gives

$$\int_a^b (t+a-2b)\,\beta(t)\,dt \le (b-a)\left(2f(a)-f(b)\right)-\int_a^b f(t)\,dt\,, \tag{24}$$

provided that $f'(t)\le\beta(t)\quad \forall t\in[a,b]$. Therefore, combining two latter results (23) and (24) approves (22).

Corollary 7. If $\alpha(x)\le f'(x)\le\beta(x)$ for any $x\in[a,b]$ and $\alpha,\beta\in C[a,b]$ then the error of nonstandard quadrature $I_7(f)$ can be bounded as

$$\int_a^b (t-2a+b)\,\alpha(t)\,dt \le (b-a)\left(-f(a)+2f(b)\right)-\int_a^b f(t)\,dt \le \int_a^b (t-2a+b)\,\beta(t)\,dt\,. \tag{25}$$

Proof. The proof of (25) is similar to that of corollary 6 if one replaces $x=b$ in respectively (10) and (12) and then combines them together.

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