Received 02/01/12

SOME INEQUALITIES OF KATO'S TYPE FOR SEQUENCES OF OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR[1,2]

Abstract. By the use of the celebrated Kato's inequality we obtain in this paper some new inequalities for n-tuples of bounded linear operators on a complex Hilbert space H. Natural applications for functions defined by power series of normal operators as well as different inequalities concerning the Euclidian norm, the Euclidian radius, the s-1-norms and s-1-radius of an n-tuple of operators are given as well.

1. Introduction

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$.

If P is a positive selfadjoint operator on H, i.e. $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle, \tag{1.1}$$

for any $x,y\in H$.

The following inequality is of interest as well, see [12, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2\leq\|P\|\langle Px,x\rangle \tag{1.2}$$

for any $x\in H$.

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [12, p. 240]: *If the operator $A\in B(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B:=\sqrt{A}\in B(H)$ such that $B^2=A$. If A is invertible, then so is B.*

If $A\in\mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A|:=\sqrt{A^*A}$.

In 1952, Kato [13] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator T on H:

$$|\langle Tx,y\rangle|^2\leq\langle(T^*T)^\alpha x,x\rangle\left\langle(TT^*)^{1-\alpha}y,y\right\rangle, \tag{1.3}$$

for any $x,y\in H$, $\alpha\in[0,1]$. Utilizing the modulus notation introduced before, we can write (1.3) as follows

$$|\langle Tx,y\rangle|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2(1-\alpha)}y,y\right\rangle \tag{1.4}$$

for any $x,y\in H$, $\alpha\in[0,1]$.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Bounded linear operators, Operator inequalities, Kato's inequality, Functions of normal operators, Euclidian norm and numerical radius.

It is useful to observe that, if $T = N$, a normal operator, i.e., we recall that $NN^* = N^*N$, then the inequality (1.4) can be written as

$$|\langle Nx, y\rangle|^2 \leq \left\langle |N|^{2\alpha} x, x\right\rangle \left\langle |N|^{2(1-\alpha)} y, y\right\rangle, \tag{1.5}$$

and in particular, for selfadjoint operators A we can state it as

$$|\langle Ax, y\rangle| \leq \||A|^{\alpha} x\| \left\||A|^{1-\alpha} y\right\| \tag{1.6}$$

for any $x, y \in H$, $\alpha \in [0, 1]$.

If $T = U$, a unitary operator, i.e., we recall that $UU^* = U^*U = 1_H$, then the inequality (1.4) becomes

$$|\langle Ux, y\rangle| \leq \|x\| \|y\|$$

for any $x, y \in H$, which provides a natural generalization for the Schwarz inequality in H.

The symetric powers in the inequalities above are natural to be considered, so if we choose in (1.4), (1.5) and in (1.6) $\alpha = 1/2$ then we get for any $x, y \in H$

$$|\langle Tx, y\rangle|^2 \leq \langle |T| x, x\rangle \langle |T^*| y, y\rangle, \tag{1.7}$$

$$|\langle Nx, y\rangle|^2 \leq \langle |N| x, x\rangle \langle |N| y, y\rangle, \tag{1.8}$$

and

$$|\langle Ax, y\rangle| \leq \left\||A|^{1/2} x\right\| \left\||A|^{1/2} y\right\| \tag{1.9}$$

respectively.

It is also worthwhile to observe that, if we take the supremum over $y \in H$, $\|y\| = 1$ in (1.4) then we get

$$\|Tx\|^2 \leq \|T\|^{2(1-\alpha)} \left\langle |T|^{2\alpha} x, x\right\rangle \tag{1.10}$$

for any $x \in H$, or in an equivalent form

$$\|Tx\| \leq \||T|^{\alpha} x\| \|T\|^{1-\alpha} \tag{1.11}$$

for any $x \in H$.

If we take $\alpha = 1/2$ in (1.10), then we get

$$\|Tx\|^2 \leq \|T\| \langle |T| x, x\rangle \tag{1.12}$$

for any $x \in H$, which in the particular case of $T = P$, a positive operator, provides the result from (1.2).

For various interesting generalizations, extension and Kato related results, see the papers [2]-[11], [16]-[20] and [22].

In this paper we pursue a different path than the avenues considered in the literature mentioned above. By the use of Kato's inequality (1.4) and by utilising only elementary techniques and tools such as Hölder and Cauchy-Bunyakowsky-Schwarz discrete inequalities we provide here some new inequalities for n-tuples of bounded linear operators on a complex Hilbert space H. Natural applications for functions defined by power series of normal operators as well as different inequalities concerning the *Euclidian norm, the Euclidian radius, the* s-1-*norms* and s-1-*radius* of an n-tuple of operators are given as well.

2. Vector Inequalities

The following vector inequality holds:

Theorem 1. *Let* $(T_1,...,T_n) \in \mathcal{B}(H) \times ... \times \mathcal{B}(H) := \mathcal{B}^{(n)}(H)$ *be an* n*-tuple of bounded linear operators on the Hilbert space* $(H;\langle\cdot,\cdot\rangle)$ *and* $(p_1,...,p_n) \in \mathbb{R}_+^{*n}$ *an* n*-tuple of nonnegative weights not all of them equal to zero. Then we have*

$$\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle|^2 \le \left\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha} \tag{2.1}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and* $\alpha \in [0,1]$.

Proof. We must prove the inequalities only in the case $\alpha \in (0,1)$, since the case $\alpha = 0$ or $\alpha = 1$ follows directly from the coresponding case of Kato's inequality.

Utilizing Kato's inequality for the operator $T_j, j \in \{1,...,n\}$ we have

$$\begin{aligned} \sum_{j=1}^{n} p_j |\langle T_j x, y\rangle|^2 &\le \sum_{j=1}^{n} p_j \left\langle |T_j|^{2\alpha} x, x \right\rangle \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y \right\rangle \\ &\le \sum_{j=1}^{n} p_j \left\langle |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha} \end{aligned} \tag{2.2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, where for the last inequality we have used the Hölder-McCarthy inequality $\langle P^r x, x\rangle \le \langle Px, x\rangle^r$ that holds for any positive operator P and any power $r \in (0,1)$.

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^{n} p_j a_j b_j \le \left(\sum_{j=1}^{n} p_j a_j^p \right)^{1/p} \left(\sum_{j=1}^{n} p_j b_j^q \right)^{1/q}, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1,...,a_n), (b_1,...,b_n) \in \mathbb{R}_+^n$, and choose $a_j = \left\langle |T_j|^2 x, x \right\rangle^{\alpha}, b_j = \left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha}$, $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ then we get

$$\begin{aligned} &\sum_{j=1}^{n} p_j \left\langle |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha} \\ &\le \left\{ \sum_{j=1}^{n} p_j \left[\left\langle |T_j|^2 x, x \right\rangle^{\alpha} \right]^{1/\alpha} \right\}^{\alpha} \left\{ \sum_{j=1}^{n} p_j \left[\left\langle \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha} \right]^{1/(1-\alpha)} \right\}^{1-\alpha} \\ &= \left\{ \sum_{j=1}^{n} p_j \left\langle |T_j|^2 x, x \right\rangle \right\}^{\alpha} \left\{ \sum_{j=1}^{n} p_j \left\langle \left|T_j^*\right|^2 y, y \right\rangle \right\}^{1-\alpha} \\ &= \left\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right|^2 y, y \right\rangle^{1-\alpha} \end{aligned} \tag{2.3}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Utilizing (2.2) and (2.3) we deduce the desired inequality (2.1). □

Remark 1. *The inequality (2.1) becomes for $y = x$ the following simpler result that is useful for deriving numerical radius inequalties:*

$$\sum_{j=1}^{n} p_j \left|\langle T_j x, x\rangle\right|^2 \leq \left\langle \sum_{j=1}^{n} p_j \left|T_j\right|^2 x, x \right\rangle^{\alpha} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right|^2 x, x \right\rangle^{1-\alpha} \leq \left\langle \sum_{j=1}^{n} p_j \left[\alpha \left|T_j\right|^2 + (1-\alpha)\left|T_j^*\right|^2\right] x, x \right\rangle \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

Let $(N_1, ..., N_n) \in \mathcal{B}^{(n)}(H)$ be an n-tuple of normal operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Then from the above Theorem 1 we have the following result that can be utilized in obtaining various inequalities for functions of normal operators defined by power series, namely:

$$\sum_{j=1}^{n} p_j \left|\langle N_j x, y\rangle\right|^2 \leq \left\langle \sum_{j=1}^{n} p_j \left|N_j\right|^2 x, x \right\rangle^{\alpha} \left\langle \sum_{j=1}^{n} p_j \left|N_j\right|^2 y, y \right\rangle^{1-\alpha} \tag{2.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, $\alpha \in [0,1]$ and any n-tuple weights $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$.

In particular, we get from (2.5) the following inequality for modulus of normal operators

$$\sum_{j=1}^{n} p_j \left|\langle N_j x, x\rangle\right|^2 \leq \left\langle \sum_{j=1}^{n} p_j \left|N_j\right|^2 x, x \right\rangle \tag{2.6}$$

for any $x \in H$ with $\|x\| = 1$.

The following result provides upper bounds for the sum $\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right|$ and has important consequences in refining the fundamental traiangle inequality for operator norm.

Theorem 2. *With the assumptions in Theorem 1 we have*

$$\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| \leq \left\langle \sum_{j=1}^{n} p_j \left|T_j\right|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{2.7}$$

for any $x, y \in H$.

Proof. From Kato's inequality for the operator $T_j, j \in \{1, ..., n\}$ we have

$$\sum_{j=1}^{n} p_j \left|\langle T_j x, y\rangle\right| \leq \sum_{j=1}^{n} p_j \left\langle \left|T_j\right|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \left|T_j^*\right|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{2.8}$$

for any $x, y \in H$.

Now, on making use of the weighted Cauchy-Bunyakovsky-Schwarz discrete inequality

$$\sum_{j=1}^{n} p_j a_j b_j \leq \left(\sum_{j=1}^{n} p_j a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j b_j^2\right)^{1/2}$$

where $(a_1,...,a_n),(b_1,...,b_n)\in\mathbb{R}_+^n$, and choose $a_j=\left\langle |T_j|^{2\alpha}x,x\right\rangle^{1/2}$ and $b_j=\left\langle \left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle^{1/2}$, then we get

$$\sum_{j=1}^n p_j\left\langle |T_j|^{2\alpha}x,x\right\rangle^{1/2}\left\langle \left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{2.9}$$
$$\le\left\{\sum_{j=1}^n p_j\left[\left\langle |T_j|^{2\alpha}x,x\right\rangle^{1/2}\right]^2\right\}^{1/2}\left\{\sum_{j=1}^n p_j\left[\left\langle \left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle^{1/2}\right]^2\right\}^{1/2}$$
$$=\left\{\sum_{j=1}^n p_j\left\langle |T_j|^{2\alpha}x,x\right\rangle\right\}^{1/2}\left\{\sum_{j=1}^n p_j\left\langle \left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle\right\}^{1/2}$$
$$=\left\langle\sum_{j=1}^n p_j|T_j|^{2\alpha}x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^n p_j\left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle^{1/2}$$

for any $x,y\in H$. □

Remark 2. *The one vector version of (2.7) is as follows:*

$$\sum_{j=1}^n p_j|\langle T_jx,x\rangle|\le\left\langle\sum_{j=1}^n p_j|T_j|^{2\alpha}x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^n p_j\left|T_j^*\right|^{2(1-\alpha)}x,x\right\rangle^{1/2} \tag{2.10}$$
$$\le\left\langle\sum_{j=1}^n p_j\left[\frac{|T_j|^{2\alpha}+\left|T_j^*\right|^{2(1-\alpha)}}{2}\right]x,x\right\rangle$$

for any $x\in H$.

Remark 3. *The symetric case for powers, namely the case $\alpha=\frac{1}{2}$ in (2.7) is of interest since will produce the simpler result*

$$\sum_{j=1}^n p_j|\langle T_jx,y\rangle|\le\left\langle\sum_{j=1}^n p_j|T_j|x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^n p_j\left|T_j^*\right|y,y\right\rangle^{1/2} \tag{2.11}$$

for any $x,y\in H$.

In particular, from (2.10) we derive

$$\sum_{j=1}^n p_j|\langle T_jx,x\rangle|\le\left\langle\sum_{j=1}^n p_j|T_j|x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^n p_j\left|T_j^*\right|x,x\right\rangle^{1/2} \tag{2.12}$$
$$\le\left\langle\sum_{j=1}^n p_j\left[\frac{|T_j|+\left|T_j^*\right|}{2}\right]x,x\right\rangle$$

for any $x\in H$.

Let $(N_1,...,N_n)\in\mathcal{B}^{(n)}(H)$ be an n-tuple of normal operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Then from the above Theorem 2 we have

$$\sum_{j=1}^n p_j|\langle N_jx,y\rangle|\le\left\langle\sum_{j=1}^n p_j|N_j|^{2\alpha}x,x\right\rangle^{1/2}\left\langle\sum_{j=1}^n p_j|N_j|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{2.13}$$

for any $x, y \in H$.

In particular, we have

$$\sum_{j=1}^{n} p_j \left|\langle N_j x, x\rangle\right| \leq \left\langle \sum_{j=1}^{n} p_j \left|N_j\right|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left|N_j\right|^{2(1-\alpha)} x, x \right\rangle^{1/2} \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{\left|N_j\right|^{2\alpha} + \left|N_j\right|^{2(1-\alpha)}}{2}\right] x, x \right\rangle \tag{2.14}$$

for any $x \in H$.

3. Functional Inequalities

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power sereis will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_A = f$.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n = \ln \frac{1}{1+z}, \ z \in D(0,1); \\ g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \ z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \ z \in \mathbb{C}; \\ l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \ z \in D(0,1); \end{aligned} \tag{3.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_A(z) &= \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z}, \ z \in D(0,1); \\ g_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C}; \\ h_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C}; \\ l_A(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1). \end{aligned} \tag{3.2}$$

The following result is a functional generalization of Kato's inequality for normal operators from (1.5).

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If N is a normal* *operator on the Hilbert space H and for $\alpha \in (0,1)$ we have that $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then we have the inequalities*

$$|\langle f(N)x,y\rangle| \le \left\langle f_A\left(|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\langle f_A\left(|N|^{2(1-\alpha)}\right)y,y\right\rangle^{1/2} \tag{3.3}$$

for any $x,y \in H$.

In particular, if $\|N\| < R$, then

$$|\langle f(N)x,y\rangle| \le \langle f_A(|N|)x,x\rangle^{1/2}\langle f_A(|N|)y,y\rangle^{1/2} \tag{3.4}$$

for any $x,y \in H$.

Proof. If N is a normal operator, then for any $j \in \mathbb{N}$ we have that

$$\left|N^j\right|^2 = (N^*N)^j = |N|^{2j}.$$

Now, utilising the inequality (2.13) we can write that

$$\begin{aligned} &\left|\left\langle\sum_{j=0}^{n} a_j N^j x, y\right\rangle\right| \\ &\le \sum_{j=0}^{n}|a_j|\left|\left\langle N^j x,y\right\rangle\right| \\ &\le \left\langle\sum_{j=0}^{n}|a_j|\left|N^j\right|^{2\alpha}x,x\right\rangle^{1/2}\left\langle\sum_{j=0}^{n}|a_j|\left|N^j\right|^{2(1-\alpha)}y,y\right\rangle^{1/2} \\ &= \left\langle\sum_{j=0}^{n}|a_j|\left(|N|^{2\alpha}\right)^j x,x\right\rangle^{1/2}\left\langle\sum_{j=0}^{n}|a_j|\left(|N|^{2(1-\alpha)}\right)^j y,y\right\rangle^{1/2} \end{aligned} \tag{3.5}$$

for any $x,y \in H$ and $n \in \mathbb{N}$.

Since $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then it follows that the series $\sum_{j=0}^{\infty}|a_j|\left(|N|^{2\alpha}\right)^j$ and $\sum_{j=0}^{\infty}|a_j|\left(|N|^{2(1-\alpha)}\right)^j$ are absolute convergent in $\mathcal{B}(H)$, and by taking the limit over $n \to \infty$ in (3.5) we deduce the desired result (3.3). □

Remark 4. *Assume that f, R, N and α are as in Theorem 3. If we take the supremum in (3.3) over $y \in H, \|y\| = 1$, then we get*

$$\|f(N)x\| \le \left\langle f_A\left(|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\|f_A\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2} \tag{3.6}$$

for any $x \in H$, which produces the operator norm inequality

$$\|f(N)\| \le \left\|f_A\left(|N|^{2\alpha}\right)\right\|^{1/2}\left\|f_A\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2}. \tag{3.7}$$

If we take $y = x$ in (3.3), then we get

$$\begin{aligned} |\langle f(N)x,x\rangle| &\le \left\langle f_A\left(|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\langle f_A\left(|N|^{2(1-\alpha)}\right)x,x\right\rangle^{1/2} \\ &\le \left\langle\left[\frac{f_A\left(|N|^{2\alpha}\right)+f_A\left(|N|^{2(1-\alpha)}\right)}{2}\right]x,x\right\rangle \end{aligned} \tag{3.8}$$

for any $x \in H$. This produces the following inequalities for the numerical radius

$$w(f(N)) \le \begin{cases} \left\| f_A\left(|N|^{2\alpha}\right)\right\|^{1/2} \left\| f_A\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2}; \\ \left\| \frac{f_A\left(|N|^{2\alpha}\right)+f_A\left(|N|^{2(1-\alpha)}\right)}{2}\right\|. \end{cases} \tag{3.9}$$

Making use of the examples in (3.1) and (3.2) we can state the vector inequalities:

$$\begin{aligned} &\left|\left\langle \ln(1_H+N)^{-1} x, y\right\rangle\right| \\ &\le \left\langle \ln\left(1_H - |N|^{2\alpha}\right)^{-1} x, x\right\rangle^{1/2} \left\langle \ln\left(1_H - |N|^{2\alpha}\right)^{-1} y, y\right\rangle^{1/2}, \|N\| < 1; \end{aligned}$$

$$\begin{aligned} &\left|\left\langle (1_H+N)^{-1} x, y\right\rangle\right| \\ &\le \left\langle \left(1_H - |N|^{2\alpha}\right)^{-1} x, x\right\rangle^{1/2} \left\langle \left(1_H - |N|^{2\alpha}\right)^{-1} y, y\right\rangle^{1/2}, \|N\| < 1; \end{aligned}$$

$$\begin{aligned} &|\langle \sin(N) x, y\rangle| \\ &\le \left\langle \sinh\left(|N|^{2\alpha}\right) x, x\right\rangle^{1/2} \left\langle \sinh\left(|N|^{2(1-\alpha)}\right) y, y\right\rangle^{1/2}, \text{ for any } N; \end{aligned}$$

$$\begin{aligned} &|\langle \cos(N) x, y\rangle| \\ &\le \left\langle \cosh\left(|N|^{2\alpha}\right) x, x\right\rangle^{1/2} \left\langle \cosh\left(|N|^{2(1-\alpha)}\right) y, y\right\rangle^{1/2}, \text{ for any } N; \end{aligned}$$

for any $x, y \in H$.

We have, for instance, the following norm inequalities as well:

$$\begin{aligned} \|\sin(N)\| &\le \left\|\sinh\left(|N|^{2\alpha}\right)\right\|^{1/2} \left\|\sinh\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2}; \\ \|\cos(N)\| &\le \left\|\cosh\left(|N|^{2\alpha}\right)\right\|^{1/2} \left\|\cosh\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2} \end{aligned}$$

for any normal operator N and

$$\left\|\ln(1_H+N)^{-1}\right\| \le \left\|\ln\left(1_H - |N|^{2\alpha}\right)^{-1}\right\|^{1/2} \left\|\ln\left(1_H - |N|^{2\alpha}\right)^{-1}\right\|^{1/2}$$

for N with $\|N\| < 1$.

If we utilise the following function as power series representations with nonnegative coefficients:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D(0,1);\tag{3.10}$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\qquad z\in D(0,1);$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D(0,1)$$

$$ {}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is the *Gamma function*, then we can state the following vector inequalities:

$$|\langle\exp(N)\,x,y\rangle|$$
$$\leq\left\langle\exp\left(|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\langle\exp\left(|N|^{2(1-\alpha)}\right)y,y\right\rangle^{1/2};\tag{3.11}$$

$$\left|\left\langle\ln\left(\frac{1_H+N}{1_H-N}\right)x,y\right\rangle\right|$$
$$\leq\left\langle\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)x,x\right\rangle^{1/2}\left\langle\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)y,y\right\rangle^{1/2};$$

$$\left|\left\langle\sin^{-1}(N)\,x,y\right\rangle\right|$$
$$\leq\left\langle\sin^{-1}\left(|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\langle\sin^{-1}\left(|N|^{2(1-\alpha)}\right)y,y\right\rangle^{1/2};$$

$$\left|\left\langle\tanh^{-1}(N)\,x,y\right\rangle\right|$$
$$\leq\left\langle\tanh^{-1}\left(|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\langle\tanh^{-1}\left(|N|^{2(1-\alpha)}\right)y,y\right\rangle^{1/2};$$

$$|\langle {}_2F_1(\alpha,\beta,\gamma,N)\,x,y\rangle|$$
$$\leq\left\langle {}_2F_1\left(\alpha,\beta,\gamma,|N|^{2\alpha}\right)x,x\right\rangle^{1/2}\left\langle {}_2F_1\left(\alpha,\beta,\gamma,|N|^{2(1-\alpha)}\right)y,y\right\rangle^{1/2};$$

for any $x,y\in H$. The first inequality in (3.11) holds for any normal operator N while the other ones request the assumption $\|N\|<1$.

We also have the norm inequalities

$$\|\exp(N)\| \le \left\|\exp\left(|N|^{2\alpha}\right)\right\|^{1/2} \left\|\exp\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2};$$
$$\|\cosh(N)\| \le \left\|\cosh\left(|N|^{2\alpha}\right)\right\|^{1/2} \left\|\cosh\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2};$$
$$\|\sinh(N)\| \le \left\|\sinh\left(|N|^{2\alpha}\right)\right\|^{1/2} \left\|\sinh\left(|N|^{2(1-\alpha)}\right)\right\|^{1/2};$$

for any normal operator N and

$$\left\|\ln\left(\frac{1_H+N}{1_H-N}\right)\right\| \le \left\|\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)\right\|^{1/2} \left\|\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right\|^{1/2}$$

for N with $\|N\| < 1$.

A similar result is the following one:

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If N is a normal operator on the Hilbert space H, $z \in \mathbb{C}$ such that $|z|^2, |z|\,\|N\|, \|N\|^2 < R$, then we have the inequalities*

$$|\langle f(zN)x, y\rangle|^2 \le f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x, x\right\rangle^{\alpha}\left\langle f_A\left(|N|^2\right)y, y\right\rangle^{1-\alpha} \tag{3.12}$$

for any $x, y \in H$ and $\alpha \in [0,1]$.

In particular, we have

$$|\langle f(zN)x, y\rangle|^2 \le f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x, x\right\rangle^{1/2}\left\langle f_A\left(|N|^2\right)y, y\right\rangle^{1/2}. \tag{3.13}$$

Proof. By the Cauchy-Bunyakowsky-Schwarz inequality we have

$$\left|\left\langle \sum_{j=0}^{n} a_j z^j N^j x, y\right\rangle\right|^2 \le \sum_{j=0}^{n} |a_j|\,|z|^{2j} \sum_{j=0}^{n} |a_j|\,\left|\left\langle N^j x, y\right\rangle\right|^2 \tag{3.14}$$

for any $n \in \mathbb{N}$ and $x, y \in H$.

Utilising (2.5) we also have

$$\begin{aligned}\sum_{j=0}^{n} |a_j|\,\left|\left\langle N^j x, y\right\rangle\right|^2 &\le \left\langle \sum_{j=0}^{n} |a_j|\,\left|N^j\right|^2 x, x\right\rangle^{\alpha}\left\langle \sum_{j=0}^{n} |a_j|\,\left|N^j\right|^2 y, y\right\rangle^{1-\alpha} \\ &= \left\langle \sum_{j=0}^{n} |a_j|\,|N|^{2j} x, x\right\rangle^{\alpha}\left\langle \sum_{j=0}^{n} |a_j|\,|N|^{2j} y, y\right\rangle^{1-\alpha}\end{aligned} \tag{3.15}$$

for any $n \in \mathbb{N}$ and $x, y \in H$.

By making use of (3.14) and (3.15) we get

$$\begin{aligned}&\left|\left\langle \sum_{j=0}^{n} a_j z^j N^j x, y\right\rangle\right|^2 \\ &\le \sum_{j=0}^{n} |a_j|\,|z|^{2j}\left\langle \sum_{j=0}^{n} |a_j|\,|N|^{2j} x, x\right\rangle^{\alpha}\left\langle \sum_{j=0}^{n} |a_j|\,|N|^{2j} y, y\right\rangle^{1-\alpha}\end{aligned} \tag{3.16}$$

for any $n \in \mathbb{N}$ and $x, y \in H$.

Since the series $\sum_{j=0}^{\infty} |a_j| |N|^{2j}$ is absolutely convergent, taking the limit over $n \to \infty$ in (3.16) produces the desired result (3.12). □

Remark 5. *Assume that f, R, z, N and α are as in Theorem 4. If we take the supremum in (3.12) over $y \in H, \|y\| = 1$, then we get*

$$\|f(zN)x\|^2 \leq f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x, x\right\rangle^{\alpha}\left\|f_A\left(|N|^2\right)\right\|^{1-\alpha} \tag{3.17}$$

for any $x \in H$, which produces the operator norm inequality

$$\|f(zN)\|^2 \leq f_A\left(|z|^2\right)\left\|f_A\left(|N|^2\right)\right\|. \tag{3.18}$$

If we take $y = x$ in (3.12), then we get

$$|\langle f(zN)x, x\rangle|^2 \leq f_A\left(|z|^2\right)\left\langle f_A\left(|N|^2\right)x, x\right\rangle \tag{3.19}$$

for any $x \in H$.

From (3.12) we get the vector inequalities

$$|\langle \exp(zN)x, y\rangle|^2 \leq \exp\left(|z|^2\right)\left\langle \exp\left(|N|^2\right)x, x\right\rangle^{\alpha}\left\langle \exp\left(|N|^2\right)y, y\right\rangle^{1-\alpha},$$

$$|\langle \sin(zN)x, y\rangle|^2 \leq \sinh\left(|z|^2\right)\left\langle \sinh\left(|N|^2\right)x, x\right\rangle^{\alpha}\left\langle \sinh\left(|N|^2\right)y, y\right\rangle^{1-\alpha},$$

and

$$\begin{aligned}&|\langle \cos(zN)x, y\rangle|^2 \\ &\leq \cosh\left(|z|^2\right)\left\langle \cosh\left(|N|^2\right)x, x\right\rangle^{\alpha}\left\langle \cosh\left(|N|^2\right)y, y\right\rangle^{1-\alpha},\end{aligned}$$

for any normal operator N, any complex number z and any $x, y \in H$.

We have, for instance, from (3.18) the following norm inequalities as well:

$$\|\exp(zN)\|^2 \leq \exp\left(|z|^2\right)\left\|\exp\left(|N|^2\right)\right\|$$

and

$$\|\sin(zN)\|^2 \leq \sinh\left(|z|^2\right)\left\|\sinh\left(|N|^2\right)\right\|$$

for any normal operator N and any complex number z.

Similar results can be stated for other functions, however the details are omitted.

4. Applications for the Euclidian Norm

In [21], the author has introduced the following norm on the Cartesian product $\mathcal{B}^{(n)}(H) := \mathcal{B}(H) \times \cdots \times \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1, \ldots, T_n)\|_e := \sup_{(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \tag{4.1}$$

where $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$ and $\mathbb{B}_n := \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \left| \sum_{j=1}^{n} |\lambda_j|^2 \leq 1\right.\right\}$ is the Euclidean closed ball in $\mathbb{C}^n$.

It is clear that $\|\cdot\|_e$ is a norm on $B^{(n)}(H)$ and for any $(T_1, \ldots, T_n) \in B^{(n)}(H)$ we have

$$\|(T_1, \ldots, T_n)\|_e = \|(T_1^*, \ldots, T_n^*)\|_e,$$

where T_j^* is the adjoint operator of T_j, $j \in \{1, \dots, n\}$. We call this the *Euclidian norm* of an n-tuple of operators $(T_1, \dots, T_n) \in B^{(n)}(H)$.

It has been shown in [21] that the following basic inequality for the Euclidian norm holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^{n} \left| T_j^* \right|^2 \right\|^{\frac{1}{2}} \leq \|(T_1, \dots, T_n)\|_e \leq \left\| \sum_{j=1}^{n} \left| T_j^* \right|^2 \right\|^{\frac{1}{2}} \tag{4.2}$$

for any n-tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [21] the author has introduced the *Euclidean operator radius* of an n-tuple of operators $(T_1, \dots, T_n)$ by

$$w_e(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \tag{4.3}$$

and proved that $w_e(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2} \|(T_1, \dots, T_n)\|_e \leq w_e(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_e \tag{4.4}$$

for each n-tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$.

As pointed out in [21], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^{n} \left| T_j^* \right|^2 \right\|^{\frac{1}{2}} \leq w_e(T_1, \dots, T_n) \leq \left\| \sum_{j=1}^{n} \left| T_j^* \right|^2 \right\|^{\frac{1}{2}} \tag{4.5}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

In [1], by utilizing the concept of *hypo-Euclidean norm* on H^n we obtained the following representation for the Euclidian norm:

Proposition 1. *For any $(T_1, \dots, T_n) \in B^{(n)}(H)$ we have*

$$\|(T_1, \dots, T_n)\|_e = \sup_{\|y\|=1, \|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j y, x \rangle|^2 \right)^{\frac{1}{2}}. \tag{4.6}$$

The following different lower bound for the Euclidean operator norm $\|\cdot\|_e$ was also obtained in [1]:

Proposition 2. *For any $(T_1, \dots, T_n) \in B^{(n)}(H)$, we have*

$$\|(T_1, \dots, T_n)\|_e \geq \frac{1}{\sqrt{n}} \|T_1 + \dots + T_n\|. \tag{4.7}$$

Utilizing some techniques based on the Boas-Bellman and Bombieri type inequalities we obtained in [1] the following upper bounds:

Proposition 3. *For any* $(T_1,\ldots,T_n)\in B^{(n)}(H)$, *we have the inequalities:*

$$\|(T_1,\ldots,T_n)\|_e^2 \le \begin{cases} \max\limits_{1\le j\le n}\left\{\|T_j\|^2\right\} + \left[\sum\limits_{1\le j\ne k\le n} w^2\left(T_k^*T_j\right)\right]^{\frac{1}{2}}; \\ \max\limits_{1\le j\le n}\left\{\|T_j\|^2\right\} + (n-1)\max\limits_{1\le j\ne k\le n}\left\{w\left(T_k^*T_j\right)\right\}; \\ \left[\max\limits_{1\le j\le n}\left\{\|T_j\|^2\right\}\left\|\sum_{j=1}^n |T_j|^2\right\|^2\right. \\ \qquad\left. + \max\limits_{1\le j\ne k\le n}\left\{\|T_j\|\|T_k\|\right\}\sum\limits_{1\le j\ne k\le n} w\left(T_kT_j^*\right)\right]^{\frac{1}{2}} \end{cases} \tag{4.8}$$

and

$$\|(T_1,\ldots,T_n)\|_e^2 \le \begin{cases} \max\limits_{1\le j\le n}\left\{\sum\limits_{k=1}^n w\left(T_k^*T_j\right)\right\}; \\ \left[\sum\limits_{j,k=1}^n w^2\left(T_k^*T_j\right)\right]^{\frac{1}{2}}; \\ n\max\limits_{1\le j\le n}\left[\sum\limits_{k=1}^n w^2\left(T_k^*T_j\right)\right]^{\frac{1}{2}}; \\ n\left[\sum\limits_{j=1}^n \max\limits_{1\le k\le n}\left\{w^2\left(T_k^*T_j\right)\right\}\right]^{\frac{1}{2}}. \end{cases} \tag{4.9}$$

Now we can provide now a different upper bound for the Euclidian norm:

Proposition 4. *Let* $(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$ *be an n-tuple of bounded linear operators on the Hilbert space* $(H;\langle\cdot,\cdot\rangle)$. *Then we have*

$$\|(T_1,\ldots,T_n)\|_e^2 \le \left\|\sum_{j=1}^n |T_j|^2\right\|^{\alpha}\left\|\sum_{j=1}^n \left|T_j^*\right|^2\right\|^{1-\alpha} \tag{4.10}$$

and

$$\begin{aligned} w_e^2(T_1,\ldots,T_n) &\le \sup_{\|x\|=1}\left[\left\langle\sum_{j=1}^n |T_j|^2 x,x\right\rangle^{\alpha}\left\langle\sum_{j=1}^n \left|T_j^*\right|^2 x,x\right\rangle^{1-\alpha}\right] \\ &\le \begin{cases} \left[\left\|\sum_{j=1}^n |T_j|^2\right\|\right]^{\alpha}\left[\left\|\sum_{j=1}^n \left|T_j^*\right|^2\right\|\right]^{1-\alpha}, \\ \left\|\sum_{j=1}^n\left[\alpha|T_j|^2+(1-\alpha)\left|T_j^*\right|^2\right]\right\|, \end{cases} \end{aligned} \tag{4.11}$$

for any $\alpha\in[0,1]$.

Proof. Utilizing the vector inequality (2.1) and taking the supremum over $\|y\| = 1, \|x\| = 1$ we have

$$\|(T_1,\ldots,T_n)\|_e^2 \le \left[\sup_{\|x\|=1}\left\langle\sum_{j=1}^n |T_j|^2 x,x\right\rangle\right]^{\alpha}\left[\sup_{\|y\|=1}\left\langle\sum_{j=1}^n \left|T_j^*\right|^2 y,y\right\rangle\right]^{1-\alpha} \tag{4.12}$$

for any $\alpha \in [0,1]$ and since

$$\sup_{\|x\|=1} \left\langle \sum_{j=1}^{n} |T_j|^2 x, x \right\rangle = \left\| \sum_{j=1}^{n} |T_j|^2 \right\|$$

and

$$\sup_{\|y\|=1} \left\langle \sum_{j=1}^{n} |T_j^*|^2 y, y \right\rangle = \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|$$

we get from (4.12) the desired result (4.10).

Now from the first inequality in (2.4) we have

$$\begin{aligned} w_e^2(T_1,\dots,T_n) &\le \sup_{\|x\|=1} \left[\left\langle \sum_{j=1}^{n} |T_j|^2 x, x \right\rangle^{\alpha} \left\langle \sum_{j=1}^{n} |T_j^*|^2 x, x \right\rangle^{1-\alpha} \right] \\ &\le \left[\sup_{\|x\|=1} \left\langle \sum_{j=1}^{n} |T_j|^2 x, x \right\rangle \right]^{\alpha} \left[\sup_{\|x\|=1} \left\langle \sum_{j=1}^{n} |T_j^*|^2 x, x \right\rangle \right]^{1-\alpha} \\ &= \left[\left\| \sum_{j=1}^{n} |T_j|^2 \right\| \right]^{\alpha} \left[\left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right]^{1-\alpha} \end{aligned} \tag{4.13}$$

and from the second inequality in (2.4) we also have

$$\begin{aligned} w_e^2(T_1,\dots,T_n) &\le \sup_{\|x\|=1} \left\langle \sum_{j=1}^{n} \left[\alpha |T_j|^2 + (1-\alpha) |T_j^*|^2 \right] x, x \right\rangle \\ &= \left\| \sum_{j=1}^{n} \alpha |T_j|^2 + (1-\alpha) |T_j^*|^2 \right\| \end{aligned} \tag{4.14}$$

for any $\alpha \in [0,1]$.

Utilizing (4.13) and (4.14) we get (4.11). □

Remark 6. *The case when $\alpha = 1/2$ provides the inequalities*

$$\|(T_1,\dots,T_n)\|_e^2 \le \left\| \sum_{j=1}^{n} |T_j|^2 \right\|^{1/2} \left\| \sum_{j=1}^{n} |T_j^*|^2 \right\|^{1/2} \tag{4.15}$$

and

$$\begin{aligned} w_e^2(T_1,\dots,T_n) &\le \sup_{\|x\|=1} \left[\left\langle \sum_{j=1}^{n} |T_j|^2 x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} |T_j^*|^2 x, x \right\rangle^{1/2} \right] \\ &\le \begin{cases} \left[\left\| \sum_{j=1}^{n} |T_j|^2 \right\| \right]^{1/2} \left[\left\| \sum_{j=1}^{n} |T_j^*|^2 \right\| \right]^{1/2}, \\ \\ \left\| \sum_{j=1}^{n} \left[\frac{|T_j|^2 + |T_j^*|^2}{2} \right] \right\|. \end{cases} \end{aligned} \tag{4.16}$$

5. Applications for s-1-Norm and s-1-Numerical Radius

We can introduce the *s-p-norm* of the n-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$ by

$$\|(T_1,\dots,T_n)\|_{s,p}:=\sup_{\|y\|=1,\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_jy,x\rangle|^p\right)^{\frac{1}{p}}\right]. \tag{5.1}$$

Indeed this is a norm, since by the Minkowski inequality we have

$$\begin{aligned}
&\|(T_1,\dots,T_n)+(V_1,\dots,V_n)\|_{s,p} \\
&=\sup_{\|y\|=1,\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_jy,x\rangle+\langle V_jy,x\rangle|^p\right)^{\frac{1}{p}}\right] \\
&\le\sup_{\|y\|=1,\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_jy,x\rangle|^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}|\langle V_jy,x\rangle|^p\right)^{\frac{1}{p}}\right] \\
&\le\sup_{\|y\|=1,\|x\|=1}\left(\sum_{j=1}^{n}|\langle T_jy,x\rangle|^p\right)^{\frac{1}{p}}+\sup_{\|y\|=1,\|x\|=1}\left(\sum_{j=1}^{n}|\langle V_jy,x\rangle|^p\right)^{\frac{1}{p}} \\
&=\|(T_1,\dots,T_n)\|_{s,p}+\|(V_1,\dots,V_n)\|_{s,p},
\end{aligned} \tag{5.2}$$

which proves the triangle inequality. The other properties of the norm are obvious.

For $p=2$ we get

$$\|(T_1,\dots,T_n)\|_{s,2}=\|(T_1,\dots,T_n)\|_{e}.$$

We are interested in this section in the case $p=1$, namely on the s-1-norm defined by

$$\|(T_1,\dots,T_n)\|_{s,1}:=\sup_{\|y\|=1,\|x\|=1}\sum_{j=1}^{n}|\langle T_jy,x\rangle|.$$

Since for any $x,y\in H$ we have $\sum_{j=1}^{n}|\langle T_jy,x\rangle|\ge\left|\left\langle\sum_{j=1}^{n}T_jy,x\right\rangle\right|$, then by the properties of the supremum we get the basic inequality

$$\left\|\sum_{j=1}^{n}T_j\right\|\le\|(T_1,\dots,T_n)\|_{s,1}\le\sum_{j=1}^{n}\|T_j\|. \tag{5.3}$$

Similarly, we can also introduce the *s-p-numerical radius* of the n-tuple of operators $(T_1,\dots,T_n)\in B^{(n)}(H)$ by

$$w_{s,p}(T_1,\dots,T_n):=\sup_{\|x\|=1}\left[\left(\sum_{j=1}^{n}|\langle T_jx,x\rangle|^p\right)^{\frac{1}{p}}\right], \tag{5.4}$$

which for $p=2$ reduces to the Euclidean operator radius introduced previously. We observe that the s-p-numerical radius is also a norm on $B^{(n)}(H)$ for $p\ge1$ and for $p=1$ it satisfies the basic inequality

$$w\left(\sum_{j=1}^{n}T_j\right)\leq w_{s,1}\left(T_1,\dots,T_n\right)\leq\sum_{j=1}^{n}w\left(T_j\right). \tag{5.5}$$

Proposition 5. *Let $(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Then we have*

$$\left\|(T_1,\dots,T_n)\right\|_{s,1}\leq\left\|\sum_{j=1}^{n}\left|T_j\right|^{2\alpha}\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^{2(1-\alpha)}\right\|^{1/2} \tag{5.6}$$

for any $\alpha\in[0,1]$, and in particular, the following refinement of the triangle inequality for operator norm:

$$\begin{aligned}\left\|\sum_{j=1}^{n}T_j\right\| &\leq\left\|(T_1,\dots,T_n)\right\|_{s,1} \\ &\leq\left\|\sum_{j=1}^{n}\left|T_j\right|\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|T_j^*\right|\right\|^{1/2} \\ &\leq\frac{1}{2}\left[\left\|\sum_{j=1}^{n}\left|T_j\right|\right\|+\left\|\sum_{j=1}^{n}\left|T_j^*\right|\right\|\right]\leq\sum_{j=1}^{n}\left\|T_j\right\|.\end{aligned} \tag{5.7}$$

Proof. Utilizing the vector inequality (2.7) and taking the supremum over $\|y\|=1,\|x\|=1$ we have

$$\begin{aligned}&\left\|(T_1,\dots,T_n)\right\|_{s,1} \\ &\leq\left\{\sup_{\|x\|=1}\left\langle\sum_{j=1}^{n}\left|T_j\right|^{2\alpha}x,x\right\rangle\right\}^{1/2}\left\{\sup_{\|y\|=1}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle\right\}^{1/2}\end{aligned} \tag{5.8}$$

and since

$$\sup_{\|x\|=1}\left\langle\sum_{j=1}^{n}\left|T_j\right|^{2\alpha}x,x\right\rangle=\left\|\sum_{j=1}^{n}\left|T_j\right|^{2\alpha}\right\|$$

and

$$\sup_{\|y\|=1}\left\langle\sum_{j=1}^{n}\left|T_j^*\right|^{2(1-\alpha)}y,y\right\rangle=\left\|\sum_{j=1}^{n}\left|T_j^*\right|^{2(1-\alpha)}\right\|$$

then we get from (5.8) the desired inequality (5.6).

The inequality (5.7) follows from (5.6). □

The case of normal operators provides a simpler bound:

Corollary 1. *Let $(N_1,...,N_n)\in\mathcal{B}^{(n)}(H)$ be an n-tuple of normal operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Then we have*

$$\left\|(N_1,\dots,N_n)\right\|_{s,1}\leq\left\|\sum_{j=1}^{n}\left|N_j\right|^{2\alpha}\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|N_j\right|^{2(1-\alpha)}\right\|^{1/2} \tag{5.9}$$

for any $\alpha \in [0,1]$, *and in particular,*

$$\left\| \sum_{j=1}^{n} N_j \right\| \le \|(N_1, \ldots, N_n)\|_{s,1} \le \left\| \sum_{j=1}^{n} |N_j| \right\| \le \sum_{j=1}^{n} \|N_j\| . \tag{5.10}$$

The above results provide an interesting criterion of convergence in the Banach algebra $\mathcal{B}(H)$ for the series of operators $\sum_{j=0}^{\infty} T_j$.

Criterion 1. *Let* $\{T_j\}_{j\in\mathbb{N}}$ *be a sequence of operators in* $\mathcal{B}(H)$. *If there exists an* $\alpha \in (0,1)$ *such that the series* $\sum_{j=0}^{\infty} |T_j|^{2\alpha}$ *and* $\sum_{j=0}^{\infty} \left|T_j^*\right|^{2(1-\alpha)}$ *are convergent in the Banach algebra* $\mathcal{B}(H)$, *then* $\sum_{j=0}^{\infty} T_j$ *is convergent in* $\mathcal{B}(H)$ *and*

$$\left\| \sum_{j=0}^{\infty} T_j \right\| \le \left\| \sum_{j=0}^{\infty} |T_j|^{2\alpha} \right\|^{1/2} \left\| \sum_{j=0}^{\infty} \left|T_j^*\right|^{2(1-\alpha)} \right\|^{1/2} .$$

In particular, the convergence of the series $\sum_{j=0}^{\infty} |T_j|$ *and* $\sum_{j=0}^{\infty} \left|T_j^*\right|$ *imply the convergence of* $\sum_{j=0}^{\infty} T_j$ *in* $\mathcal{B}(H)$ *with the estimate for the sums as follows:*

$$\left\| \sum_{j=0}^{\infty} T_j \right\| \le \left\| \sum_{j=0}^{\infty} |T_j| \right\|^{1/2} \left\| \sum_{j=0}^{\infty} \left|T_j^*\right| \right\|^{1/2} .$$

The following result for the s-1-numerical radius may be stated as well:

Proposition 6. *Let* $(T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$ *be an* n*-tuple of bounded linear operators on the Hilbert space* $(H; \langle\cdot,\cdot\rangle)$. *Then we have*

$$\begin{aligned} & w_{s,1}(T_1, \ldots, T_n) \\ & \le \sup \left\{ \left\langle \sum_{j=1}^{n} p_j |T_j|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j \left|T_j^*\right|^{2(1-\alpha)} x, x \right\rangle^{1/2} \right\} \\ & \le \begin{cases} \left\| \sum_{j=1}^{n} |T_j|^{2\alpha} \right\|^{1/2} \left\| \sum_{j=1}^{n} \left|T_j^*\right|^{2(1-\alpha)} \right\|^{1/2} ; \\ \left\| \sum_{j=1}^{n} \frac{|T_j|^{2\alpha} + \left|T_j^*\right|^{2(1-\alpha)}}{2} \right\| \end{cases} \end{aligned} \tag{5.11}$$

for any $\alpha \in [0,1]$ *, and, in particular,*

$$w\left(\sum_{j=1}^{n} T_j\right) \le w_{s,1}(T_1,\dots,T_n) \le \begin{cases} \left\|\sum_{j=1}^{n} |T_j|\right\|^{1/2} \left\|\sum_{j=1}^{n} |T_j^*|\right\|^{1/2}; \\ \left\|\sum_{j=1}^{n} \frac{|T_j|+|T_j^*|}{2}\right\|. \end{cases} \tag{5.12}$$

Remark 7. *We observe that due to the inequality*

$$\frac{1}{2}\left\|\sum_{j=1}^{n} T_j\right\| \le w\left(\sum_{j=1}^{n} T_j\right) \le \left\|\sum_{j=1}^{n} \frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2}\right\|, \tag{5.13}$$

the convergence of the series $\sum_{k=0}^{\infty}\left[|T_k|^{2\alpha} + |T_k^*|^{2(1-\alpha)}\right]$ *in the Banach algebra* $\mathcal{B}(H)$ *for some* $\alpha \in (0,1)$ *sufficies for the convergence of* $\sum_{k=0}^{\infty} T_k$*, which is a slight improvement of the result from Criterion 1.*

The case $\alpha = \frac{1}{2}$ *produces the simpler inequality of interest for the numerical radius of a sum:*

$$\frac{1}{2}\left\|\sum_{j=1}^{n} T_j\right\| \le w\left(\sum_{j=1}^{n} T_j\right) \le \frac{1}{2}\left\|\sum_{j=1}^{n} \left[|T_j| + |T_j^*|\right]\right\|. \tag{5.14}$$

References

[1] S.S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 52, 22 pp.

[2] M. Fujii, C.-S. Lin and R. Nakamoto, Alternative extensions of Heinz-Kato-Furuta inequality. *Sci. Math.* **2** (1999), no. 2, 215–221.

[3] M. Fujii and T. Furuta, Löwner-Heinz, Cordes and Heinz-Kato inequalities. *Math. Japon.* **38** (1993), no. 1, 73–78.

[4] M. Fujii, E. Kamei, C. Kotari and H. Yamada, Furuta's determinant type generalizations of Heinz-Kato inequality. *Math. Japon.* **40** (1994), no. 2, 259–267

[5] M. Fujii, Y.O. Kim, and Y. Seo, Further extensions of Wielandt type Heinz-Kato-Furuta inequalities via Furuta inequality. *Arch. Inequal. Appl.* **1** (2003), no. 2, 275–283

[6] M. Fujii, Y.O. Kim and M. Tominaga, Extensions of the Heinz-Kato-Furuta inequality by using operator monotone functions. *Far East J. Math. Sci.* (FJMS) **6** (2002), no. 3, 225–238

[7] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **128** (2000), no. 1, 223–228.

[8] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. II. *J. Inequal. Appl.* **3** (1999), no. 3, 293–302,

[9] T. Furuta, Equivalence relations among Reid, Löwner-Heinz and Heinz-Kato inequalities, and extensions of these inequalities. *Integral Equations Operator Theory* **29** (1997), no. 1, 1–9.

[10] T. Furuta, Determinant type generalizations of Heinz-Kato theorem via Furuta inequality. *Proc. Amer. Math. Soc.* **120** (1994), no. 1, 223–231.

[11] T. Furuta, An extension of the Heinz-Kato theorem. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 785–787.

[12] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.
[13] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208-212.
[14] F. Kittaneh, Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.
[15] F. Kittaneh, Norm inequalities for fractional powers of positive operators. *Lett. Math. Phys.* **27** (1993), no. 4, 279–285.
[16] C.-S. Lin, On Heinz-Kato-Furuta inequality with best bounds. J. *Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **15** (2008), no. 1, 93–101.
[17] C.-S. Lin, On chaotic order and generalized Heinz-Kato-Furuta-type inequality. *Int. Math. Forum* **2** (2007), no. 37-40, 1849–1858,
[18] C.-S. Lin, On inequalities of Heinz and Kato, and Furuta for linear operators. *Math. Japon.* **50** (1999), no. 3, 463–468.
[19] C.-S. Lin, On Heinz-Kato type characterizations of the Furuta inequality. II. *Math. Inequal. Appl.* **2** (1999), no. 2, 283–287.
[20] C.A. McCarthy, c_p. *Israel J. Math.*, **5**(1967), 249-271.
[21] G. Popescu, Unitary invariants in multivariable operator theory. *Mem. Amer. Math. Soc.* **200** (2009), no. 941, vi+91 pp.
[22] M. Uchiyama, Further extension of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **127** (1999), no. 10, 2899–2904.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa