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Inequality chains for Wilker, Huygens and Lazarević type inequalities

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Abstract

We offer various refinements of inequalities related to the Wilker, Huygens, or Lazarević inequalities.

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1 Introduction

Wilker in [17] proposed two open problems:

(a) Prove that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2\ . \tag{1.1}$$

(b) Find the largest constant c such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

for $0 < x < \pi/2$.

In [16], Wilker inequality (1.1) was proved, and the following inequality

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x \quad \text{for } 0 < x < \frac{\pi}{2}\ , \tag{1.2}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ are best possible, was also established.

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Wilker type inequalities (1.1) and (1.2) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs and various generalizations and improvements (cf. [5, 9, 10, 11, 12, 14, 16, 18, 19, 21, 23, 24, 25, 28, 29, 30] and the references cited therein).

Another inequality which is of interest to us is Huygens inequality [6], which asserts that

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}\,. \tag{1.3}$$

Wu and Srivastava [22, Lemma 3] established another inequality

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{1.4}$$

The following inequality

$$(\cos x)^{1/3}<\frac{\sin x}{x}, \quad 0<|x|<\frac{\pi}{2} \tag{1.5}$$

was established by Adamović and Mitrinović (see, e.g., [8, p. 238]). Inequality (1.5) can be re-written as

$$\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}>1 \quad \left(\text{or} \quad \sqrt[3]{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}}>1\right) \quad \text{for all} \quad 0<|x|<\frac{\pi}{2}\,. \tag{1.6}$$

Baricz and Sándor [1] pointed out that inequalities (1.1) and (1.3) are simple consequences of the arithmetic-geometric mean inequality together with inequality (1.6), and inequality (1.4) is in fact an immediate consequence of inequality (1.1). Here we point out that inequality (1.4) is sharper than inequality (1.1), this fact follows from the following result:

$$\frac{(\sin x/x)^2+\tan x/x}{(x/\sin x)^2+x/\tan x}=\frac{(\sin x/x)^3}{\cos x}>1\Longrightarrow\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2 \tag{1.7}$$

for $0<|x|<\frac{\pi}{2}$. Neuman and Sándor [12] have pointed out that (1.3) implies (1.1). This fact is easily seen from the following result:

$$\left[\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2\right]-\left[2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}-3\right]=\left(1-\frac{\sin x}{x}\right)^2>0 \tag{1.8}$$

for $0<|x|<\pi/2$.

Chen and Cheung [3] pointed out that inequalities (1.1), (1.3), (1.6) and (1.4) can be grouped into the following inequality chain:

$$\begin{aligned}\frac{(\sin x/x)^2+\tan x/x}{2}>\frac{2\,(\sin x/x)+\tan x/x}{3}&>\sqrt[3]{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}}>1\\&>\frac{2}{1/\,(\sin x/x)^2+1/\,(\tan x/x)}\end{aligned} \tag{1.9}$$

for $0<|x|<\pi/2$, in terms of the arithmetic, geometric and harmonic means.

The first aim of this paper is to prove Theorem 1.1 below, which shows that the following inequality chain holds:

$$\frac{(\sin x/x)^2+\tan x/x}{2} > \left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right) > \frac{2\,(\sin x/x)+\tan x/x}{3}$$
$$> \left(\frac{\sin x}{x}\right)^{2/3}\left(\frac{\tan x}{x}\right)^{1/3} > \frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right] > 1 \qquad (1.10)$$

for $0<|x|<\pi/2$.

Now it is easy to see that among inequalities (1.1), (1.3), (1.4) and (1.6), inequality (1.4) is the sharpest.

Theorem 1.1. (i) *For* $0<|x|<\pi/2$,

$$\frac{(\sin x/x)^2+\tan x/x}{2} > \left(\frac{\sin x}{x}\right)^{2p}\left(\frac{\tan x}{x}\right)^{p} > \frac{2\,(\sin x/x)+\tan x/x}{3} \qquad (1.11)$$

with the best possible constant $p=1$.

(ii) *For* $0<|x|<\pi/2$,

$$\left(\frac{\sin x}{x}\right)^{2r}\left(\frac{\tan x}{x}\right)^{r} > \frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right] > 1 \qquad (1.12)$$

with the best possible constant $r=1/3$.

Lazarević [7] (see, e.g., [8, p. 238]) proved the following inequality:

$$(\cosh x)^{1/3} < \frac{\sinh x}{x}, \quad x\neq 0, \qquad (1.13)$$

which can be re-written as

$$\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x} > 1 \quad \left(\text{or} \quad \sqrt[3]{\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}} > 1\right) \quad \text{for} \quad x\neq 0. \qquad (1.14)$$

Zhu [26] established hyperbolic versions of inequality (1.1):

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x} > 2\,, \qquad x\neq 0. \qquad (1.15)$$

Baricz and Sándor [1] have pointed out that (1.14) implies (1.15) and the following inequality

$$2\left(\frac{\sinh x}{x}\right)+\frac{\tanh x}{x} > 3\,, \qquad x\neq 0, \qquad (1.16)$$

by the arithmetic-geometric mean inequality.

Neuman and Sándor [12, Theorem 2.4] proved hyperbolic version of inequality (1.4):

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x} > 2, \qquad x\neq 0. \qquad (1.17)$$

In [12] not only hyperbolic versions of (1.1) and (1.3) are studied, but many other facts, e.g. Cusa-Huygens, Huygens, Wilker type inequalities and their connections to each others in the trigonometric and also the hyperbolic case. Wilker-type inequalities for hyperbolic functions are studied by Wu and Debnath [20]. In [2, 3] inverse trigonometric and inverse hyperbolic versions of inequalities (1.1)–(1.3) were established. Zhu [27] established some new inequalities of the Huygens type for trigonometric and hyperbolic functions. As meantime there have been published other papers in the field, see the works by (for example) Sándor [15] and Neuman and Sándor [13]. Chen and Cheung [3] pointed out that hyperbolic version of inequality chain (1.9) holds true:

$$\frac{(\sinh x/x)^2+\tanh x/x}{2} > \frac{2\,(\sinh x/x)+\tanh x/x}{3} > \sqrt[3]{\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}} > 1$$
$$> \frac{2}{1/\left(\sinh x/x\right)^2+1/\left(\tanh x/x\right)} \tag{1.18}$$

for $x \neq 0$.

The second aim of this paper is to prove inequality chain involving the inequalities (1.14) to (1.17). Theorem 1.2 shows that among inequalities (1.14) to (1.17), inequality (1.17) is the sharpest.

Theorem 1.2. *For $x \neq 0$,*

$$\frac{(\sinh x/x)^2+\tanh x/x}{2} > \left(\frac{\sinh x}{x}\right)^2\left(\frac{\tanh x}{x}\right) > \frac{2\,(\sinh x/x)+\tanh x/x}{3}$$
$$> \frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}\right] > 1. \tag{1.19}$$

Remark 1.1. *For $x \neq 0$, there is no strict comparison between the representatives $\sqrt[3]{\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}}$ and $\frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}\right]$.*

The third aim of this paper is to present sharp version of Huygens' inequality (1.3).

Theorem 1.3. *For $0 < |x| < \frac{\pi}{2}$,*

$$3+\frac{3}{20}x^3\tan x < 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x} < 3+\left(\frac{2}{\pi}\right)^4 x^3\tan x. \tag{1.20}$$

The constants $\frac{3}{20}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

The inequality (1.20) is an interesting analogue of the inequality (1.2). It is easy to see that the inequalities (1.2) and (1.20) can be grouped into the following inequality chain:

$$\frac{8}{45}x^3\tan x > \left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2 > \left(\frac{2}{\pi}\right)^4 x^3\tan x$$
$$> 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}-3 > \frac{3}{20}x^3\tan x \tag{1.21}$$

for $0 < |x| < \pi/2$, and the constants $\frac{8}{45}$, $\left(\frac{2}{\pi}\right)^4$ and $\frac{3}{20}$ are the best possible.

Inequality (1.21) shows again that (1.3) implies (1.1).

2 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1. Elementary calculations show that

$$\frac{(\sin x/x)^2+\tan x/x}{2}-\left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right)=\frac{\tan x}{x^3}U(x),$$

where

$$U(x)=x^2+\frac{1}{2}x\sin(2x)-1+\cos(2x).$$

Differentiation yields

$$\begin{aligned}U'(x)&=-\frac{3}{2}\sin(2x)+2x+x\cos(2x),\\ U''(x)&=2\sin(2x)(\tan x-x)>0\quad\text{for}\quad 0<x<\frac{\pi}{2}.\end{aligned}$$

Hence, we have for $0<x<\pi/2$,

$$U'(x)>U'(0)=0\Longrightarrow U(x)>U(0)=0.$$

Therefore, the first inequality in (1.11) holds for $p=1$. The first inequality in (1.11) can be re-written as

$$p<\frac{\ln\left(\frac{(\sin x/x)^2+\tan x/x}{2}\right)}{\ln\left(\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}\right)},\qquad 0<|x|<\frac{\pi}{2}. \tag{2.22}$$

In (2.22) let x tend to $\pi/2$, we imply that the ratio on the right-hand side of (2.22) tends to 1. This means that the first inequality in (1.11) holds for $0<|x|<\pi/2$ with the best possible constant $p=1$.

Elementary calculations show that

$$\left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right)-\frac{2\left(\sin x/x\right)+\tan x/x}{3}=\frac{\tan x}{x}Q(x),$$

where

$$Q(x)=\left(\frac{\sin x}{x}\right)^2-\frac{1+2\cos x}{3}. \tag{2.23}$$

Differentiation yields

$$\frac{3x^3}{2\sin x}Q'(x)=W(x),$$

where

$$W(x)=x^3-3\sin x+3x\cos x.$$

Since

$$W'(x)=3x(x-\sin x)>0\quad\text{for}\quad 0<x<\pi/2,$$

we have for $0 < x < \pi/2$,

$$W(x) > W(0) = 0 \Longrightarrow Q'(x) > 0 \Longrightarrow Q(x) > Q(0) = 0.$$

Therefore, the second inequality in (1.11) holds for $p = 1$. The second inequality in (1.11) can be re-written as

$$\frac{\ln\left(\frac{2(\sin x/x)+\tan x/x}{3}\right)}{\ln\left(\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}\right)} < p, \qquad 0 < |x| < \frac{\pi}{2}. \tag{2.24}$$

In (2.24) let x tend to $\pi/2$, we imply that the ratio on the left-hand side of (2.24) tends to 1. This means that the second inequality in (1.11) holds for $0 < |x| < \pi/2$ with the best possible constant $p = 1$.

Consider the function $R(x)$ defined by

$$R(x) = \frac{\sqrt[3]{\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}}}{\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right]} = \frac{2\sin^3 x}{x^2(x+\sin x\cos x)(\cos x)^{1/3}}, \quad 0 < x < \frac{\pi}{2}.$$

Differentiation yields

$$R'(x) = \frac{\sin^2 x S(x)}{3x^3\Big(x^2 + x\sin(2x) + \frac{1}{4}\sin^2(2x)\Big)\big(\cos x\big)^{4/3}}$$

where

$$\begin{aligned} S(x) &= \big(8x^2 + x\sin(2x)\big)\big(2\cos^2 x\big) - 5x\sin(2x) - 3\sin^2(2x) + 2x^2 \\ &= \big(8x^2 + x\sin(2x)\big)\big(1+\cos(2x)\big) - 5x\sin(2x) - \frac{3\big(1-\cos(4x)\big)}{2} + 2x^2 \\ &= -4x\sin(2x) + \frac{1}{2}x\sin(4x) + 8x^2\cos(2x) + \frac{3}{2}\cos(4x) - \frac{3}{2} + 10x^2. \end{aligned}$$

By using the power series expansions of $\sin x$ and $\cos x$, we have

$$2S(x) = \frac{128}{315}x^8 - \frac{256}{4725}x^{10} - \frac{128}{31185}x^{12} + \sum_{n=7}^{\infty}(-1)^{n-1}v_n(x), \tag{2.25}$$

where

$$v_n(x) = \frac{32n(n-1)\cdot 4^n + (n-6)\cdot 4^{2n}}{2\cdot(2n)!}x^{2n}.$$

Elementary calculations show that for $0 < x < \pi/2$ and $n \geq 7$,

$$\begin{aligned} \frac{u_{n+1}(x)}{u_n(x)} &= \frac{x^2\Big(128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}\Big)}{2(2n+1)(n+1)\Big(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\Big)} \\ &< \frac{\left(\frac{\pi}{2}\right)^2\Big(128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}\Big)}{2(2n+1)(n+1)\Big(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\Big)} \\ &< \frac{128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}}{8(n+1)\Big(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\Big)} \end{aligned}$$

and

$$8(n+1)\Big(32n(n-1)\cdot 2^{2n}+(n-6)\cdot 2^{4n}\Big)-\Big(128n(n+1)\cdot 2^{2n}+(16n-80)\cdot 2^{4n}\Big)$$
$$=128n(n+1)(2n-3)\cdot 2^{2n}+\Big(8n(n-7)+32\Big)\cdot 2^{4n}>0.$$

Hence, for every $x\in(0,\pi/2)$, the sequence $n\longmapsto v_n(x)$ is strictly decreasing for $n\geq 7$. Therefore, we obtain from (2.25) that

$$2S(x)>x^8\left(\frac{128}{315}-\frac{256}{4725}x^2-\frac{128}{31185}x^4\right)>0,\quad 0<x<\frac{\pi}{2},$$

which implies $R'(x)>0$ for $0<x<\pi/2$. Hence, the function $R(x)$ is strictly increasing for $(0,\pi/2)$, and we have

$$R(x)>\lim_{x\to 0^+}R(x)=1,\quad 0<x<\frac{\pi}{2}.$$

Therefore, the first inequality in (1.12) holds for $r=1/3$. The first inequality in (1.12) can be re-written as

$$\frac{\ln\left(\frac{(x/\sin x)^2+x/\tan x}{2}\right)}{\ln\left(\left(\frac{\sin x}{x}\right)^2\frac{\tan x}{x}\right)}<r,\qquad 0<|x|<\frac{\pi}{2}. \tag{2.26}$$

In (2.26) let x tend to 0, we imply that the ratio on the left-hand side of (2.26) tends to $1/3$. This means that the first inequality in (1.12) holds for $0<|x|<\pi/2$ with the best possible constant $r=1/3$.

The second inequality in (1.12) has been proved by Wu and Srivastava [22, Lemma 3]. This completes the proof of Theorem 1.1. □

Remark 2.1. *The first inequality in* (1.5) *can be separated. Indeed, we have*

$$(\cos x)^{1/3}<\left(\frac{1+2\cos x}{3}\right)^{1/2}<\frac{\sin x}{x},\quad 0<|x|<\frac{\pi}{2}. \tag{2.27}$$

In fact, the first inequality in (2.27) *may be written, after some elementary transformations, as* $(1-\cos x)^2(1+8\cos x)>0$ *for* $0<|x|<\pi/2$, *while the second inequality in* (2.27) *follows from the inequality* $Q(x)>0$ *of* (2.23).

The first inequality in (1.13) *can be separated. Indeed, we have*

$$(\cosh x)^{1/3}<\left(\frac{1+2\cosh x}{3}\right)^{1/2}<\frac{\sinh x}{x},\quad x\neq 0. \tag{2.28}$$

In fact, the first inequality in (2.28) *may be written, after some elementary transformations, as* $(1-\cosh x)^2(1+8\cosh x)>0$ *for* $x\neq 0$, *while the second inequality in* (2.28) *follows from the following result:*

$$\left(\frac{\sinh x}{x}\right)^2-\frac{1+2\cosh x}{3}=\frac{-x^2-2x^2\cosh x-3+3\cosh^2 x}{3x^2}$$
$$=\sum_{n=3}^{\infty}\frac{3\cdot 2^{2n-1}-4n(2n-1)}{3\cdot(2n)!}x^{2n-2}>0,\qquad x\neq 0.$$

Proof of Theorem 1.2. Elementary calculations show that

$$\frac{(\sinh x/x)^2+\tanh x/x}{2}-\left(\frac{\sinh x}{x}\right)^2\left(\frac{\tanh x}{x}\right)=\frac{\tanh x}{2x^3}V(x),$$

where

$$V(x)=x^2+\frac{1}{2}x\sinh(2x)+1-\cosh(2x).$$

Differentiation yields

$$\begin{aligned} V'(x)&=-\frac{3}{2}\sinh(2x)+2x+x\cosh(2x),\\ V''(x)&=2\sinh(2x)(x-\tan x)>0 \quad \text{for} \quad x>0. \end{aligned}$$

Hence, we have for $x>0$,

$$V'(x)>V'(0)=0\Longrightarrow V(x)>V(0)=0.$$

Therefore, the first inequality in (1.19) holds.

Elementary calculations show that

$$\left(\frac{\sinh x}{x}\right)^2\left(\frac{\tanh x}{x}\right)-\frac{2\,(\sinh x/x)+\tanh x/x}{3}=\frac{\tanh x}{3x^3}T(x),$$

with

$$T(x)=-2x^2\cosh x-x^2+\frac{3}{2}\cosh(2x)-\frac{3}{2}=\sum_{n=3}^{\infty}\frac{3\cdot 4^n-8n(2n-1)}{2\cdot(2n)!}x^{2n}>0, \quad x\neq 0.$$

Therefore, the second inequality in (1.19) holds.

Elementary calculations show that

$$\frac{2\,(\sinh x/x)+\tanh x/x}{3}-\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]=\frac{x^2J(x)}{6\cosh x\sinh^2 x},$$

where

$$J(x)=4\left(\frac{\sinh x}{x}\right)^3\cosh x-3\left(\frac{\sinh x}{x}\right)\cosh^2 x+2\left(\frac{\sinh x}{x}\right)^3-3\cosh x.$$

By (1.13), we have for $x\neq 0$,

$$\begin{aligned} J(x)&>4\left(\frac{\sinh x}{x}\right)^3\cosh x-3\left(\frac{2+\cosh x}{3}\right)\cosh^2 x-\cosh x\\ &=4\cosh x\left[\left(\frac{\sinh x}{x}\right)^3-\left(\frac{1+\cosh x}{2}\right)^2\right]\\ &=4\cosh x\left[\left(\frac{2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)}{x}\right)^3-\cosh^4\left(\frac{x}{2}\right)\right]\\ &=4\cosh x\cosh^3\left(\frac{x}{2}\right)\left[\left(\frac{\sinh\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^3-\cosh\left(\frac{x}{2}\right)\right]\\ &>0. \end{aligned}$$

Therefore, the third inequality in (1.19) holds.

The last inequality in (1.19) has been proved by Neuman and Sándor [12, Theorem 2.4]. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. Consider the function $F(x)$ defined by

$$F(x) = \frac{2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} - 3}{x^3 \tan x}, \quad 0 < x < \frac{\pi}{2}.$$

Elementary calculations show that

$$\begin{aligned}
2x^5 \sin^2 x F'(x) &= 9x\sin(2x) - 4x\sin^3 x - 16\cos x \sin^2 x - 8\sin^2 x + 6x^2 \\
&= 9x\sin(2x) + x\sin(3x) - 3x\sin x + 4\cos(3x) - 4\cos x + 4\cos(2x) + 6x^2 - 4 \\
&= \sum_{n=4}^{\infty} (-1)^{n-1} \frac{(27n-12)\cdot 4^n + 2(n-6)9^n - 18n + 12}{3\cdot(2n)!} x^{2n} \\
&= \frac{1}{70}x^8 + \frac{1}{1400}x^{10} - \frac{79}{184800}x^{12} \\
&\quad + \sum_{n=7}^{\infty} (-1)^{n-1} \frac{(27n-12)\cdot 4^n + 2(n-6)9^n - 18n + 12}{3\cdot(2n)!} x^{2n} \\
&= \frac{1}{70}x^8 + \frac{1}{1400}x^{10} - \frac{79}{184800}x^{12} + \sum_{n=7}^{\infty} (-1)^{n-1} u_n(x), \qquad (2.29)
\end{aligned}$$

where

$$u_n(x) = \frac{(27n-12)\cdot 4^n + 2(n-6)9^n - 18n + 12}{3\cdot(2n)!} x^{2n}.$$

Elementary calculations show that for $0 < x < \pi/2$ and $n \geq 7$,

$$\begin{aligned}
\frac{u_{n+1}(x)}{u_n(x)} &= \frac{3x^2}{2n+1} \frac{(18n+10)4^n + (3n-15)9^n - 3n - 1}{(n+1)\Big((27n-12)4^n + (2n-12)9^n - 18n + 12\Big)} \\
&< \frac{(18n+10)4^n + (3n-15)9^n - 3n - 1}{(n+1)\Big((27n-12)4^n + (2n-12)9^n - 18n + 12\Big)}
\end{aligned}$$

and

$$\begin{aligned}
&(n+1)\Big((27n-12)4^n + (2n-12)9^n - 18n + 12\Big) - \Big((18n+10)4^n + (3n-15)9^n - 3n - 1\Big) \\
&= (27n^2 - 3n - 22)4^4 + (2n^2 - 13n + 3)9^n - (18n^2 + 3n - 13) > 0.
\end{aligned}$$

Hence, for every $x \in (0, \pi/2)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing for $n \geq 7$. Therefore, we obtain from (2.29) that

$$2x^5 \sin^2 x F'(x) > x^8 \left(\frac{1}{70} + \frac{1}{1400}x^2 - \frac{79}{184800}x^4\right) > 0, \quad 0 < x < \frac{\pi}{2}.$$

Hence, the function $F(x)$ is strictly increasing for $(0, \pi/2)$, and we have

$$\frac{3}{20} = \lim_{x\to 0^+} F(x) < F(x) = \frac{2\left(\dfrac{\sin x}{x}\right) + \dfrac{\tan x}{x} - 3}{x^3 \tan x} < \lim_{x\to(\pi/2)^-} F(x) = \left(\frac{2}{\pi}\right)^4, \quad 0 < x < \frac{\pi}{2}.$$

This completes the proof of Theorem 1.3. □

Remark 2.2. *Inequality* (1.20) *first appeared in [4]. Here an elementary proof has been provided.*

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