Received 14/01/12

SOME GRÜSS TYPE INEQUALITIES FOR N-TUPLES OF VECTORS

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Abstract. Some Grüss type inequalities in inner product modules over C^*-algebras, proper H^*-algebras and unital Banach $*$-algebras for n-Tuples of Vectors are established.
Mathematics Subject Classification (2000). Primary 46L08, 46H25; Secondary 26D99, 46C99.
Keywords: Inner product C^*-modules, inner product $*$-modules, Grüss inequality, C^*-seminorms.

1. Introduction

We start by recalling some of the most important Grüss type discrete inequalities for inner product spaces that are available in [3].

Theorem 1. *Let $(H;\langle,\rangle)$ be an inner product space over $\mathbb{K}$; $(\mathbb{K}=\mathbb{C},\mathbb{R})$, $x_i,\ y_i \in H$, $p_i \geq 0$ $(i=1,...,n)$ $(n\geq 2)$ with $\sum_{i-1}^{n} p_i = 1$. If $x,X,y,Y \in H$ are such that*

$$\operatorname{Re}\langle X-x_i, x_i-x\rangle \geq 0 \quad and \quad \operatorname{Re}\langle Y-y_i, y_i-y\rangle \geq 0$$

for all $i\in\{1,...,n\}$, or, equivalently,

$$\left\|x_i-\frac{x+X}{2}\right\| \leq \frac{1}{2}\|X-x\| \quad and \quad \left\|y_i-\frac{y+Y}{2}\right\| \leq \frac{1}{2}\|Y-y\|$$

for all $i\in\{1,...,n\}$, then we have the inequality

$$\left|\sum_{i=1}^{n} p_i\langle x_i,y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right\rangle\right| \leq \frac{1}{4}\|X-x\|\|Y-y\|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Theorem 2. *Let $(H;\langle,\rangle)$ and $\mathbb{K}$ be as above and $\overline{x}=(x_1,...,x_n)\in H^n$, $\overline{\alpha}=(\alpha_1,...,\alpha_n)\in\mathbb{K}^n$ and $\overline{p}=(p_1,...,p_n)$ a probability vector. If $x,X\in H$ are such that*

$$\operatorname{Re}\langle X-x_i, x_i-x\rangle \geq 0 \ for\ all\ i\in\{1,...,n\},$$

or, equivalently,

$$\left\|x_i-\frac{x+X}{2}\right\| \leq \frac{1}{2}\|X-x\| \ for\ all\ i\in\{1,...,n\},$$

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holds, then we have the inequality

$$\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2} \|X - x\| \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{j=1}^{n} p_j \alpha_j \right|$$

$$\leq \frac{1}{2} \|X - x\| \left[\sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \right]^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ *in the first and second inequalities is best possible.*

The main aim of this paper is to present some extensions of the above results holding in the general setting of n-tuples of vectors in an inner product modules over C^*-algebras, proper H^*-algebras and unital Banach $*$-algebras.

2. Preliminaries

A proper H^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|.\|)$ where the underlying Banach space is a Hilbert space with respect to the inner product $\langle ., . \rangle$ satisfying the properties $\langle ab, c \rangle = \langle b, a^*c \rangle$ and $\langle ba, c \rangle = \langle b, ca^* \rangle$ for all $a, b, c \in \mathcal{A}$. A C^*-algebra is a complex Banach $*$-algebra $(\mathcal{A}, \|.\|)$ such that $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{A}$. If $\mathcal{A}$ is a proper H^*-algebra or a C^*-algebra and $a \in \mathcal{A}$ is such that $\mathcal{A}a = 0$ or $a\mathcal{A} = 0$ then $a = 0$.

For a proper H^*-algebra $\mathcal{A}$, the trace class associated with $\mathcal{A}$ is $\tau(\mathcal{A}) = \{ab : a, b \in \mathcal{A}\}$. For every positive $a \in \tau(\mathcal{A})$ there exists the square root of a, that is, a unique positive $a^{\frac{1}{2}} \in \mathcal{A}$ such that $(a^{\frac{1}{2}})^2 = a$, the square root of a^*a is denoted by $|a|$. There are a positive linear functional tr on $\tau(\mathcal{A})$ and a norm τ on $\tau(\mathcal{A})$, related to the norm of A by the equality $tr(a^*a) = \tau(a^*a) = \|a\|^2$ for every $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a proper H^*-algebra or a C^*-algebra. A semi-inner product module over $\mathcal{A}$ is a right module X over $\mathcal{A}$ together with a generalized semi-inner product, that is with a mapping $\langle ., . \rangle$ on $X \times X$, which is $\tau(\mathcal{A})$-valued if $\mathcal{A}$ is a proper H^*-algebra, or $\mathcal{A}$-valued if $\mathcal{A}$ is a C^*-algebra, having the following properties:

(i) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in X, a \in \mathcal{A}$,
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in X$,
(iv) $\langle x, x \rangle \geq 0$ for $x \in X$.

We shall say that X is a semi-inner product H^*-module if $\mathcal{A}$ is a proper H^*-algebra and that X is a semi-inner product C^*-module if $\mathcal{A}$ is a C^*-algebra.
If, in addition,

(v) $\langle x, x \rangle = 0$ implies $x = 0$,

then X is called an inner product module over $\mathcal{A}$. The absolute value of $x \in X$ is defined as the square root of $\langle x, x \rangle$ and it is denoted by $|x|$.

Let $\mathcal{A}$ be a $*$-algebra. A seminorm γ on $\mathcal{A}$ is a real-valued function on $\mathcal{A}$ such that for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: $\gamma(a) \geq 0, \quad \gamma(\lambda a) = |\lambda| \gamma(a), \quad \gamma(a + b) \leq \gamma(a) + \gamma(b)$. A seminorm γ on $\mathcal{A}$ is called a C^*-seminorm if it satisfies the C^*-condition: $\gamma(a^*a) = (\gamma(a))^2 \quad (a \in \mathcal{A})$. By Sebestyen's theorem [2, Theorem 38.1] every C^*-seminorm γ on a $*$-algebra $\mathcal{A}$ is submultiplicative, i.e., $\gamma(ab) \leq \gamma(a)\gamma(b) \quad (a, b \in \mathcal{A})$, and by [1, Section 39, Lemma 2 (i)] $\gamma(a) = \gamma(a^*)$. For every $a \in \mathcal{A}$, the spectral radius of a is defined to be $r(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}$.

The Pták function ρ on $*$-algebra $\mathcal{A}$ is defined to be $\rho : \mathcal{A} \to [0, \infty)$, where $\rho(a) = (r(a^*a))^{1/2}$. This function has important roles in Banach $*$-algebras, for example, on C^*-algebras, ρ is equal to the norm and on hermitian Banach $*$-algebras ρ is the greatest C^*-seminorm. By utilizing properties of the spectral radius and the Pták function, V. Pták [6] showed in 1970 that an elegant theory for Banach $*$-algebras arises from the inequality $r(a) \le \rho(a)$.
This inequality characterizes hermitian (and symmetric) Banach $*$-algebras, and further characterizations of C^*-algebras follow as a result of Pták theory.

Let $\mathcal{A}$ be a $*$-algebra. We define $\mathcal{A}^+$ by

$$\mathcal{A}^+ = \left\{ \sum_{k=1}^{n} a_k^* a_k : n \in \mathbb{N}, a_k \in \mathcal{A} \text{ for } k = 1, 2, ..., n \right\},$$

and call the elements of $\mathcal{A}^+$ positive.
The set $\mathcal{A}^+$ of positive elements is obviously a convex cone (i.e., it is closed under convex combinations and multiplication by positive constants). Hence we call $\mathcal{A}^+$ the positive cone. By definition, zero belongs to $\mathcal{A}^+$. It is also clear that each positive element is hermitian.

We recall that a Banach $*$-algebra $(\mathcal{A}, \|.\|)$ is said to be an A^*-algebra provided there exists on $\mathcal{A}$ a second norm $|.|$, not necessarily complete, which is a C^*-norm. The second norm will be called an auxiliary norm.

Definition 1. Let $\mathcal{A}$ be a $*$-algebra. A semi-inner product $\mathcal{A}$-module (or semi-inner product $*$-module) is a complex vector space which is also a right $\mathcal{A}$-module X with a sesquilinear semi-inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathcal{A}$, fulfilling

$$\langle x, ya \rangle = \langle x, y \rangle a \qquad \text{(right linearity)}$$
$$\langle x, x \rangle \in \mathcal{A}^+ \qquad \text{(positivity)}$$

for $x, y \in X,\ a \in \mathcal{A}$. Furthermore, if X satisfies the strict positivity condition

$$x = 0 \quad \text{if } \langle x, x \rangle = 0, \quad \text{(strict positivity)}$$

then X is called an inner product $\mathcal{A}$-module (or inner product $*$-module).

Let γ be a seminorm or a positive linear functional on $\mathcal{A}$ and $\Gamma(x) = (\gamma(\langle x, x \rangle))^{1/2}$ $(x \in X)$. If Γ is a seminorm on a semi-inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a semi-Hilbert $\mathcal{A}$-module.

If Γ is a norm on an inner product $\mathcal{A}$-module X, then (X, Γ) is said to be a pre-Hilbert $\mathcal{A}$-module.

A pre-Hilbert $\mathcal{A}$-module which is complete with respect to its norm is called a Hilbert $\mathcal{A}$-module.

Since $\langle x + y, x + y \rangle$ and $\langle x + iy, x + iy \rangle$ are self adjoint, therefore we get the following Corollary.

Corollary 1. *If X is a semi-inner product $*$-module then the following symmetry condition holds:*

$$\langle x, y \rangle^* = \langle y, x \rangle \quad \text{for } x, y \in X. \qquad \text{(symmetry)}$$

Example 1.

(a) Let $\mathcal{A}$ be a $*$-algebra and γ a positive linear functional or a C^*-seminorm on $\mathcal{A}$. It is known that $(\mathcal{A}, \gamma)$ is a semi-Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a, b \rangle := a^* b$, in this case $\Gamma = \gamma$.

(b) Let $\mathcal{A}$ be a hermitian Banach $*$-algebra and ρ be the Pták function on $\mathcal{A}$. If X is a semi-inner product $\mathcal{A}$-module and $P(x) = (\rho(\langle x, x\rangle))^{1/2}(x \in X)$, then (X, P) is a semi-Hilbert $\mathcal{A}$-module.

(c) Let $\mathcal{A}$ be a A^*-algebra and $|\cdot|$ be the auxiliary norm on $\mathcal{A}$. If X is an inner product $\mathcal{A}$-module and $|x| = |\langle x, x\rangle|^{1/2}(x \in X)$, then $(X, |\cdot|)$ is a pre-Hilbert $\mathcal{A}$-module.

(d) Let $\mathcal{A}$ be a H^*-algebra and X (a semi-inner product) an inner product $\mathcal{A}$-module. Since tr is a positive linear functional on $\tau(\mathcal{A})$ and for every $x \in X$ we have $tr(\langle x, x\rangle) = \| \, |x| \, \|^2$ therefore $(X, \| \, |.| \, \|)$ is a (semi-Hilbert) pre-Hilbert $\mathcal{A}$-module.

If X is a semi-inner product C^*-module, then the following Schwarz inequality holds:

$$\|\langle x, y\rangle\|^2 \leq \|\langle x, x\rangle\| \|\langle y, y\rangle\| \quad (x, y \in X). \tag{2.1}$$

(e.g. [8, Lemma 15.1.3]).

If X is a semi-inner product H^*-module, then there are two forms of the Schwarz inequality: for every $x, y \in X$

$$tr(\langle x, y\rangle)^2 \leq tr(\langle x, x\rangle) tr(\langle y, y\rangle) \qquad \text{(the weak Schwarz inequality)}, \tag{2.2}$$

$$\tau(\langle x, y\rangle)^2 \leq tr(\langle x, x\rangle) tr(\langle y, y\rangle) \qquad \text{(the strong Schwarz inequality)}. \tag{2.3}$$

First Saworotnow in [7] proved the strong Schwarz inequality, but the direct proof of that for a semi-inner product H^*-module can be found in [5].

Now let $\mathcal{A}$ be a $*$-algebra, φ a positive linear functional on $\mathcal{A}$ and X be a semi-inner $\mathcal{A}$-module. We can define a sesquilinear form on $X \times X$ by $\sigma(x, y) = \varphi(\langle x, y\rangle)$; the Schwarz inequality for σ implies that

$$|\varphi(\langle x, y\rangle)|^2 \leq \varphi(\langle x, x\rangle) \varphi(\langle y, y\rangle). \tag{2.4}$$

In [4, Proposition 1, Remark 1] the authors present two other forms of the Schwarz inequality in semi-inner $\mathcal{A}$-module X, one for positive linear functional φ on $\mathcal{A}$:

$$\varphi(\langle x, y\rangle \langle x, y\rangle) \leq \varphi(\langle x, x\rangle) r(\langle y, y\rangle), \tag{2.5}$$

another one for C^*-seminorm γ on $\mathcal{A}$:

$$\gamma(\langle x, y\rangle)^2 \leq \gamma(\langle x, x\rangle) \gamma(\langle y, y\rangle). \tag{2.6}$$

3. The version for inner product modules over C^*-algebras

Before stating the main results, let us fix the rest of our notation. We assume, unless stated otherwise, throughout this section that $\mathcal{A}$ is a C^*-algebra and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector i.e. $p_i \geq 0 \quad (i = 1, ..., n)$ and $\sum_{i=1}^n p_i = 1$. If X is a semi-inner product $\mathcal{A}$-module and $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ we put

$$G_{\overline{p}}(\overline{x}, \overline{y}) = \sum_{i=1}^n p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle$$

we use $G_{\overline{p}}(\overline{x})$ instead of $G_{\overline{p}}(\overline{x}, \overline{x})$.

In the following Theorem we give a generalization of Theorem 1 for $\mathcal{A}$-modules over C^*-algebras, which is a refinement of it.

Theorem 3. *Let X be a semi-inner product C^*-module, $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a, b \in X, r \geq 0, s \geq 0$ such that*

$$\|x_i - a\| \leq r, \quad \|y_i - b\| \leq s, \text{ for all } i \in \{1, ..., n\}, \tag{3.1}$$

then we have the inequality

$$\begin{aligned} \|G_{\overline{p}}(\overline{x}, \overline{y})\| &\leq rs - \sqrt{r^2 - \|G_{\overline{p}}(\overline{x})\|}\sqrt{s^2 - \|G_{\overline{p}}(\overline{y})\|} \\ &\leq rs - \sqrt{r^2 - \sum_{i=1}^n p_i \|x_i - a\|^2}\sqrt{s^2 - \sum_{i=1}^n p_i \|y_i - b\|^2} \leq rs. \end{aligned} \tag{3.2}$$

The constant 1 coefficient of rs in the inequalities (3.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. A simple calculation shows that

$$\sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, y_i - y_j \rangle, \tag{3.3}$$

therefore

$$G_{\overline{p}}(\overline{x}) = \frac{1}{2} \sum_{i,j=1}^n p_i p_j \langle x_i - x_j, x_i - x_j \rangle \geq 0. \tag{3.4}$$

It is easy to show that $G_{\overline{p}}(.,.)$ is an $\mathcal{A}$-value semi-inner product on X^n, so Schwarz inequality holds i.e.,

$$\|G_{\overline{p}}(\overline{x}, \overline{y})\|^2 \leq \|G_{\overline{p}}(\overline{x})\| \|G_{\overline{p}}(\overline{y})\|. \tag{3.5}$$

Also a simple calculation shows that for every $a, b \in X$

$$\begin{aligned} &\sum_{i=1}^n p_i \langle x_i - a, y_i - b \rangle - \left\langle \sum_{i=1}^n p_i (x_i - a), \sum_{i=1}^n p_i (y_i - b) \right\rangle \\ &\qquad = \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, \sum_{i=1}^n p_i y_i \right\rangle = G_{\overline{p}}(\overline{x}, \overline{y}). \end{aligned} \tag{3.6}$$

particular

$$\begin{aligned} G_{\overline{p}}(\overline{x}) &= \sum_{i=1}^n p_i \langle x_i - a, x_i - a \rangle - \left\langle \sum_{i=1}^n p_i (x_i - a), \sum_{i=1}^n p_i (x_i - a) \right\rangle \\ &\leq \sum_{i=1}^n p_i \langle x_i - a, x_i - a \rangle. \end{aligned} \tag{3.7}$$

This implies that

$$\|G_{\overline{p}}(\overline{x})\| \leq \sum_{i=1}^n p_i \|x_i - a\|^2 \leq r^2, \tag{3.8}$$

similarly

$$\|G_{\overline{p}}(\overline{y})\| \leq \sum_{i=1}^n p_i \|y_i - b\|^2 \leq s^2. \tag{3.9}$$

Now using the elementary inequality for real numbers

$$(m^2-n^2)(p^2-q^2)\le (mp-nq)^2 \tag{3.10}$$

on

$$m=r,\qquad n^2=r^2-\sum_{i=1}^n p_i\|x_i-a\|^2$$
$$p=s,\qquad q^2=s^2-\sum_{i=1}^n p_i\|y_i-b\|^2$$

we get the inequality (3.2). To prove the sharpness of the constant 1 in the inequalities in (3.2), let us assume that, under the assumptions of the theorem, the inequalities hold with a constant $c>0$, i.e.,

$$\begin{aligned}\|G_{\overline{p}}(\overline{x},\overline{y})\| &\le crs-\sqrt{r^2-\|G_{\overline{p}}(\overline{x})\|}\sqrt{s^2-\|G_{\overline{p}}(\overline{y})\|}\\ &\le crs-\sqrt{r^2-\sum_{i=1}^n p_i\|x_i-a\|^2}\sqrt{s^2-\sum_{i=1}^n p_i\|y_i-b\|^2}\le crs.\end{aligned} \tag{3.11}$$

Assume that $n=2, p_1=p_2=\frac{1}{2}$ and e is an element of X such that $\|\langle e,e\rangle\|=1$. We put

$$x_1=a+re,\ \ y_1=b+se$$
$$x_2=a-re,\ \ y_2=b-se,$$

then, obviously,

$$\|x_i-a\|\le r,\quad \|y_i-b\|\le s,\quad (i=1,2),$$

which shows that the condition (3.1) holds. If we replace $n, p_1, p_2, x_1, x_2, y_1, y_2$ in (3.11), we obtain

$$\|G_{\overline{p}}(\overline{x},\overline{y})\|=rs\le crs.$$

from where we deduce that $c\ge 1$, which proves the sharpness of the constant 1. □

The following Remark 1(ii) is a generalization of Theorem 2 for $\mathcal{A}$-modules over C^*-algebras.

Remark 1.

(i) Let $\mathcal{A}$ be a C^*-algebra, it is known that $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a,b\rangle:=a^*b$. In this case (3.2) implies that

$$\left\|\sum_{i=1}^n p_ia_i^*b_i-\sum_{i=1}^n p_ia_i^*.\sum_{i=1}^n p_ib_i\right\|\le rs-\sqrt{r^2-\sum_{i=1}^n p_i\|a_i-a\|^2}\sqrt{s^2-\sum_{i=1}^n p_i\|b_i-b\|^2}$$
$$\le rs.$$

(ii) Let X be a semi-inner product C^*-module, $\overline{x}=(x_1,...,x_n), \overline{\alpha}=(\alpha_1,...,\alpha_n)\in\mathbb{K}^n$ and $\overline{p}=(p_1,...,p_n)\in\mathbb{R}^n$ a probability vector. If $a\in X, r\ge 0$, such that

$$\|x_i-a\|\le r,\ \text{ for all i }\in\{1,...,n\},$$

holds, since

$$\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i(\alpha_i - \sum_{j=1}^{n} p_j\alpha_j)(x_i - a) \tag{3.12}$$

then we have the inequalities

$$\left\| \sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i \right\| \leq r \sum_{i=1}^{n} p_i \left| \alpha_i - \sum_{i=1}^{n} p_j\alpha_j \right| \tag{3.13}$$
$$\leq r \left[\sum_{i=1}^{n} p_i|\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i\alpha_i \right|^2 \right]^{\frac{1}{2}}.$$

The constant 1 in the first and second inequalities in (3.13) is best possible.

4. The version for inner product modules over proper H^*-algebras

We assume, unless stated otherwise, throughout this section that $\mathcal{A}$ is a proper H^*-algebra. The following Theorem is a version of Theorem 3 for $\mathcal{A}$-modules over proper H^*-algebras.

Theorem 4. *Let X be a semi-inner product $\mathcal{A}$-module, $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a, b \in X, r \geq 0, s \geq 0$ such that*

$$|||x_i - a||| \leq r, \quad |||y_i - b||| \leq s, \text{ for all } i \in \{1, ..., n\}, \tag{4.1}$$

then we have the inequality

$$\tau(G_{\overline{p}}(\overline{x}, \overline{y})) \leq rs - \sqrt{r^2 - tr(G_{\overline{p}}(\overline{x}))}\sqrt{s^2 - tr(G_{\overline{p}}(\overline{y}))} \tag{4.2}$$
$$\leq rs - \sqrt{r^2 - \sum_{i=1}^{n} p_i|||x_i - a|||^2}\sqrt{s^2 - \sum_{i=1}^{n} p_i|||y_i - b|||^2} \leq rs.$$

The constant 1 coefficient of rs in the inequalities (4.2) is sharp.

Proof. By strong Schwarz inequality (2.3) we have

$$\tau(G_{\overline{p}}(\overline{x}, \overline{y}))^2 \leq tr(G_{\overline{p}}(\overline{x}))tr(G_{\overline{p}}(\overline{y})). \tag{4.3}$$

From (3.7) we obtain

$$tr(G_{\overline{p}}(\overline{x})) \leq \sum_{i=1}^{n} p_i|||x_i - a|||^2 \leq r^2, \tag{4.4}$$

similarly

$$tr(G_{\overline{p}}(\overline{y})) \leq \sum_{i=1}^{n} p_i|||y_i - b|||^2 \leq s^2. \tag{4.5}$$

Now (4.3), (4.4), (4.5) and (3.10) imply (4.2).

The fact that the constant 1 is sharp may be proven in a similar manner to the one embodied in the proof of Theorem 3. We omit the details. □

The following companion of the Grüss inequality for $\mathcal{A}$-modules over proper H^*-algebras holds:

Theorem 5. *Let X be a semi-inner product $\mathcal{A}$-module, $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a, b \in X, r \geq 0, s \geq 0$ such that*

$$|||x_i - a||| \leq r, \ \ |||y_i - b||| \leq s \text{ for all } i \in \{1, ..., n\} \tag{4.6}$$

then we have the inequality

$$|\tau(G_{\overline{p}}(\overline{x}, \overline{y}))| \leq rs - \left|\left|\left|\sum_{i=1}^{n} p_i(x_i - a)\right|\right|\right| \left|\left|\left|\sum_{i=1}^{n} p_i(y_i - b)\right|\right|\right| \leq rs. \tag{4.7}$$

Proof. From the inequality (3.7) we get

$$tr(G_{\overline{p}}(\overline{x})) = \sum_{i=1}^{n} p_i |||x_i - a|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(x_i - a)\right|\right|\right|^2. \tag{4.8}$$

Similarly

$$tr(G_{\overline{p}}(\overline{y})) = \sum_{i=1}^{n} p_i |||y_i - b|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(y_i - b)\right|\right|\right|^2. \tag{4.9}$$

By strong Schwarz inequality (2.3) we have

$$\begin{aligned} |\tau(G_{\overline{p}}(\overline{x}, \overline{y}))| &\leq \left[\sum_{i=1}^{n} p_i |||x_i - a|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(x_i - a)\right|\right|\right|^2\right]^{\frac{1}{2}} \\ &\quad \left[\sum_{i=1}^{n} p_i |||y_i - b|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(y_i - b)\right|\right|\right|^2\right]^{\frac{1}{2}} \\ &\leq \left[r^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(x_i - a)\right|\right|\right|^2\right]^{\frac{1}{2}} \left[s^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(y_i - b)\right|\right|\right|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

By using (3.10) we obtain the inequality (4.7). □

Corollary 2. *Let X be a semi-inner product $\mathcal{A}$-module, $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a \in X, r \geq 0$ such that*

$$|||x_i - a||| \leq r \text{ for all } i \in \{1, ..., n\} \tag{4.11}$$

then we have the inequality

$$|\tau(G_{\overline{p}}(\overline{x}, \overline{y}))| \leq r \left[\sum_{i=1}^{n} p_i |||y_i|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i y_i\right|\right|\right|^2\right]^{\frac{1}{2}}. \tag{4.12}$$

Proof. From (4.8) we have

$$\sum_{i=1}^{n} p_i |||x_i - a|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(x_i - a)\right|\right|\right|^2 \leq \sum_{i=1}^{n} p_i r^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(x_i - a)\right|\right|\right|^2 \leq r^2.$$

The first inequality in (4.10) implies that

$$|\tau(G_{\overline{p}}(\overline{x},\overline{y}))| \leq r\left[\sum_{i=1}^{n} p_i|||y_i-b|||^2 - \left|\left|\left|\sum_{i=1}^{n} p_i(y_i-b)\right|\right|\right|^2\right]^{\frac{1}{2}}, \tag{4.13}$$

and for $b=0$ we get the inequality (4.12). □

The following Remark is a version of Remark 1 for $\mathcal{A}$-modules over H^*-algebras.

Remark 2.

(i) Let $\mathcal{A}$ be a C^*-algebra, it is known that $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a,b\rangle := a^*b$. In this case (4.2) implies that

$$\tau\left(\sum_{i=1}^{n} p_i a_i^* b_i - \sum_{i=1}^{n} p_i a_i^* . \sum_{i=1}^{n} p_i b_i\right) \leq rs - \sqrt{r^2 - \sum_{i=1}^{n} p_i \tau(a_i-a)^2}\sqrt{s^2 - \sum_{i=1}^{n} p_i\tau(b_i-b)^2}$$
$$\leq rs.$$

(ii) Let $\mathcal{A}$ be a proper H^*-module and X be a semi-inner product $\mathcal{A}$-module, $\overline{x}=(x_1,...,x_n), \overline{\alpha}=(\alpha_1,...,\alpha_n)\in\mathbb{K}^n$ and $\overline{p}=(p_1,...,p_n)\in\mathbb{R}^n$ a probability vector. If $a\in X, r\geq 0$, such that

$$|||x_i-a|||\leq r, \text{ for all i } \in\{1,...,n\},$$

holds, by (3.12) we have the inequalities

$$\left|\left|\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right|\right|\right| \leq r\sum_{i=1}^{n} p_i\left|\alpha_i - \sum_{i=1}^{n} p_j\alpha_j\right| \tag{4.14}$$
$$\leq r\left[\sum_{i=1}^{n} p_i|\alpha_i|^2 - \left|\sum_{i=1}^{n} p_i\alpha_i\right|^2\right]^{\frac{1}{2}}.$$

The constant 1 in the first and second inequalities in (4.14) is best possible.

5. The version for inner product modules over unital Banach ∗-algebras

We assume, unless stated otherwise, throughout this section that $\mathcal{A}$ is a unital Banach $*$-algebra. Also if X is a semi-inner product $\mathcal{A}$-module and γ is a C^*-seminorm or a positive linear functional on $\mathcal{A}$ we put $\Gamma(x)=(\gamma(\langle x,x\rangle))^{1/2}(x\in X)$. The following Theorem is a version of Theorem 3 for $\mathcal{A}$-modules over unital Banach $*$-algebras.

Theorem 6. *Let X be a semi-inner product $\mathcal{A}$-module, $\overline{x}=(x_1,...,x_n), \overline{y}=(y_1,...,y_n)\in X^n$ and $\overline{p}=(p_1,...,p_n)\in\mathbb{R}^n$ a probability vector. If $a,b\in X, r\geq 0, s\geq 0$ such that*

$$\Gamma(x_i-a)\leq r, \quad \Gamma(y_i-b)\leq s, \text{ for all } i \in\{1,...,n\}, \tag{5.1}$$

then we have the inequality

$$|\gamma(G_{\overline{p}}(\overline{x},\overline{y}))| \leq rs - \sqrt{r^2-\gamma(G_{\overline{p}}(\overline{x}))}\sqrt{s^2-\gamma(G_{\overline{p}}(\overline{y}))} \tag{5.2}$$
$$\leq rs - \sqrt{r^2-\sum_{i=1}^{n}p_i\Gamma(x_i-a)^2}\sqrt{s^2-\sum_{i=1}^{n}p_i\Gamma(y_i-b)^2} \leq rs.$$

Proof. By Schwarz inequalities (2.4) and (2.6) we have

$$\gamma(G_{\overline{p}}(\overline{x},\overline{y}))^2 \leq \gamma(G_{\overline{p}}(\overline{x}))\gamma(G_{\overline{p}}(\overline{y})). \tag{5.3}$$

From (3.7) we obtain

$$\gamma(G_{\overline{p}}(\overline{x})) \leq \sum_{i=1}^{n}p_i\Gamma(x_i-a)^2 \leq r^2, \tag{5.4}$$

similarly

$$\gamma(G_{\overline{p}}(\overline{y})) \leq \sum_{i=1}^{n}p_i\Gamma(y_i-b)^2 \leq s^2. \tag{5.5}$$

Now (5.3), (5.4), (5.5) and (3.10) imply (5.2).

The fact that the constant 1 is sharp may be proven in a similar manner to the one embodied in the proof of Theorem 3. We omit the details. □

A version of Theorem 5 for $\mathcal{A}$-modules over unital Banach $*$-algebras holds.

Theorem 7. *Let X be a semi-inner product $\mathcal{A}$-module, $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a, b \in X, r \geq 0, s \geq 0$ and φ a positive linear functional on $\mathcal{A}$, such that*

$$\Phi(x_i-a) \leq r,\ \ \Phi(y_i-b) \leq s \text{ for all } i \in \{1, ..., n\} \tag{5.6}$$

then we have the inequality

$$|\varphi(G_{\overline{p}}(\overline{x},\overline{y}))| \leq rs - \Phi\left(\sum_{i=1}^{n}p_i(x_i-a)\right)\Phi\left(\sum_{i=1}^{n}p_i(y_i-b)\right) \leq rs. \tag{5.7}$$

Proof. from the inequality (3.7) we get

$$\varphi(G_{\overline{p}}(\overline{x})) = \sum_{i=1}^{n}p_i\Phi(x_i-a)^2 - \Phi\left(\sum_{i=1}^{n}p_i(x_i-a)\right)^2. \tag{5.8}$$

Similarly

$$\varphi(G_{\overline{p}}(\overline{y})) = \sum_{i=1}^{n}p_i\Phi(y_i-b)^2 - \Phi\left(\sum_{i=1}^{n}p_i(y_i-b)\right)^2. \tag{5.9}$$

Therefore

$$
\begin{aligned}
(5.10)\quad |\varphi(G_{\overline{p}}(\overline{x},\overline{y}))| &\leq \left[\sum_{i=1}^{n} p_i \Phi(x_i-a)^2 - \Phi\left(\sum_{i=1}^{n} p_i(x_i-a)\right)^2\right]^{\frac{1}{2}} \\
&\quad \left[\sum_{i=1}^{n} p_i \Phi(y_i-b)^2 - \Phi\left(\sum_{i=1}^{n} p_i(y_i-b)\right)^2\right]^{\frac{1}{2}} \\
&\leq \left[r^2 - \Phi\left(\sum_{i=1}^{n} p_i(x_i-a)\right)^2\right]^{\frac{1}{2}} \left[s^2 - \Phi\left(\sum_{i=1}^{n} p_i(y_i-b)\right)^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

By using (3.10) we obtain the inequality (5.7). □

Corollary 3. *Let X be a semi-inner product $\mathcal{A}$-module, $\overline{x} = (x_1, ..., x_n), \overline{y} = (y_1, ..., y_n) \in X^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a \in X, r \geq 0$ and φ a positive linear functional on $\mathcal{A}$, such that*

$$(5.11)\qquad \Phi(x_i - a) \leq r \text{ for all } i \in \{1, ..., n\}$$

then we have the inequality

$$(5.12)\qquad |\varphi(G_{\overline{p}}(\overline{x},\overline{y}))| \leq r\left[\sum_{i=1}^{n} p_i \Phi(y_i)^2 - \Phi\left(\sum_{i=1}^{n} p_i y_i\right)^2\right]^{\frac{1}{2}}.$$

Proof. The first inequality in (5.10) implies that

$$(5.13)\qquad |\varphi(G_{\overline{p}}(\overline{x},\overline{y}))| \leq r\left[\sum_{i=1}^{n} p_i \Phi(y_i-b)^2 - \Phi\left(\sum_{i=1}^{n} p_i(y_i-b)\right)^2\right]^{\frac{1}{2}},$$

and for $b = 0$ we get the inequality (5.12). □

The following Remark is a version of Remark 1 for $\mathcal{A}$-modules over unital Banach $*$-algebras.

Remark 3.

(i) Let $\mathcal{A}$ be a C^*-algebra, it is known that $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module over itself with the inner product defined by $\langle a, b\rangle := a^*b$. In this case (5.2) implies that

$$
\begin{aligned}
\gamma\left(\sum_{i=1}^{n} p_i a_i^* b_i - \sum_{i=1}^{n} p_i a_i^* . \sum_{i=1}^{n} p_i b_i\right) &\leq rs - \sqrt{r^2 - \sum_{i=1}^{n} p_i \gamma(a_i-a)^2}\sqrt{s^2 - \sum_{i=1}^{n} p_i \gamma(b_i-b)^2} \\
&\leq rs.
\end{aligned}
$$

(ii) Let $\mathcal{A}$ be a Banach $*$-module, γ a C^*-seminorm or a positive linear functional on $\mathcal{A}$ and (X, Γ) be a semi-inner product $\mathcal{A}$-module, $\overline{x} = (x_1, ..., x_n), \overline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$ and $\overline{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ a probability vector. If $a \in X, r \geq 0$, such that

$$\Gamma(x_i - a) \leq r, \text{ for all i } \in \{1, ..., n\},$$

holds, by (3.12) we have the inequalities

$$\Gamma\left(\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right) \leq r\sum_{i=1}^{n} p_i\left|\alpha_i - \sum_{i=1}^{n} p_j\alpha_j\right| \leq r\left[\sum_{i=1}^{n} p_i|\alpha_i|^2 - \left|\sum_{i=1}^{n} p_i\alpha_i\right|^2\right]^{\frac{1}{2}}. \tag{5.14}$$

The constant 1 in the first and second inequalities in (5.14) is best possible.

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