Received 18/01/12

OPTIMAL HARMONIC INTERPOLATIONS BETWEEN SEIFFERT MEANS

ALFRED WITKOWSKI

ABSTRACT. We provide set of optimal estimates of the form

$$\frac{1-\mu}{\mathcal{A}(x,y)}+\frac{\mu}{\mathcal{M}(x,y)}\le\frac{1}{\mathcal{B}(x,y)}\le\frac{1-\nu}{\mathcal{A}(x,y)}+\frac{\nu}{\mathcal{M}(x,y)}$$

where $\mathcal{A}<\mathcal{B}$ are two of the Seifert-like means L,P,M,T, while $\mathcal{M}$ is another mean greater than the two.

1. Introduction

The two means introduced by Seiffert in [13]

$$P(x,y)=\begin{cases}\dfrac{x-y}{2\arcsin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

and in [15]

$$T(x,y)=\begin{cases}\dfrac{x-y}{2\arctan\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

are being currently investigated by many mathematicians. Interesting inequalities between P, T, arithmetic, geometric, logarithmic, identric and power means were obtained by many authors (see the bibliography) using analytic approach or properties of the Schwab-Borchardt algorithm.

It is worth noting two other Seiffert-like means involving inverse hyperbolic function: introduced in [9]

$$M(x,y)=\begin{cases}\dfrac{x-y}{2\operatorname{arsinh}\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

and well known logarithmic mean

$$L(x,y)=\begin{cases}\dfrac{x-y}{2\operatorname{artanh}\frac{x-y}{x+y}}=\dfrac{x-y}{\log x-\log y} & x\neq y\\ x & x=y\end{cases}.$$

2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Seiffert means, logarithmic mean.

The four means satisfy inequalities $L < P < M < T$ if $x \neq y$. The goal of this paper is to provide optimal bounds of the form

$$\frac{1-\mu}{\mathcal{A}(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{\mathcal{B}(x,y)} \le \frac{1-\nu}{\mathcal{A}(x,y)} + \frac{\nu}{\mathcal{M}(x,y)} \tag{1.1}$$

where $\mathcal{A} < \mathcal{B}$ are two of the means L, P, M, T, while $\mathcal{M}$ is another mean greater than the two.

In [19] we developed a geometric method to obtain interpolations for means of the form $SB_{M,N}(x,y) = \frac{\sqrt{N^2(x,y)-M^2(x,y)}}{\arccos(M(x,y)/N(x,y))}$. This method proves to be very efficient in reaching our goal.

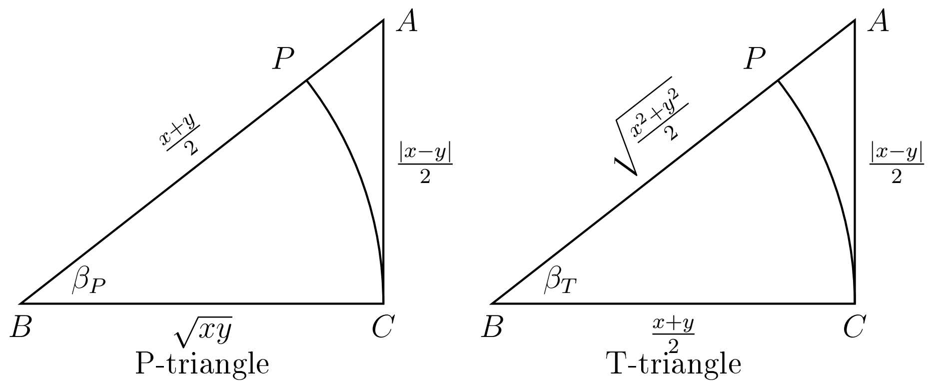
2. Notation and definitions

We shall be using the following notation: x, y are always positive. We shall be considering the geometric, arithmetic, root-mean square and contraharmonic means denoted respectively by

$$G(x,y) = \sqrt{xy},\ A(x,y) = \frac{x+y}{2},\ R(x,y) = \sqrt{\frac{x^2+y^2}{2}},\ C(x,y) = \frac{x^2+y^2}{x+y}$$

In most cases we shall omit the arguments to simplify notation.
We use the symbol $a \cong b$ to indicate that a and b are of the same sign.

We shall be using two variables. The first one: β_P denotes the radial measure of the angle $\angle ABC$ in the P-triangle in the picture below, while β_T denotes the same angle in the T-triangle.



Definition of β_P and β_T

Note that as x, y vary, the angle β_P assumes all values between $(0, \pi/2)$, while β_T does not exceed $\pi/4$. This follows from the fact that in the T-triangle $|AC| < |BC|$ and makes an essential difference between the two pictures.
Thanks to this geometric interpretation we can express the means to be

considered as functions of variable β_P and β_T respectively. In the P-triangle we have

$$\begin{gathered} P=\frac{|AC|}{\beta_P},\ G=\frac{|AC|}{\tan\beta_P},\ A=\frac{|AC|}{\sin\beta_P},\\ L=\frac{|AC|}{\operatorname{artanh}(\sin\beta_P)},\ M=\frac{|AC|}{\operatorname{arsinh}(\sin\beta_P)}. \end{gathered} \tag{2.1}$$

while in the T-triangle

$$\begin{gathered} P=\frac{|AC|}{\beta_T},\ A=\frac{|AC|}{\tan\beta_T},\ R=\frac{|AC|}{\sin\beta_T},\\ L=\frac{|AC|}{\operatorname{artanh}(\tan\beta_T)},\ M=\frac{|AC|}{\operatorname{arsinh}(\tan\beta_T)}. \end{gathered} \tag{2.2}$$

In many cases the bounds obtained in (1.1) are absolute (i.e. valid for all arguments (x, y)), while some bounds will be trivial. For example, if $\mathcal{A} = L$, the only possible left-hand side bound is $\mu = 1$. This is a consequence of the fact that $\lim_{x\to 0} L(x, 1) = 0$, while $\lim_{x\to 0} P(x, 1) = 1/\pi$. In such a case we shall provide additional bounds assuming (x, y) vary over a restricted area.

Definition 2.1. For $0 < \alpha < \pi/2$ we say that (x, y) satisfy P_α *condition* if

$$\frac{1-\sin\alpha}{1+\sin\alpha} \le \frac{x}{y} \le \frac{1+\sin\alpha}{1-\sin\alpha} \tag{2.3}$$

Definition 2.2. For $0 < \alpha < \pi/2$ we say that (x, y) satisfy T_α *condition* if

$$\frac{1-\tan\alpha}{1+\tan\alpha} \le \frac{x}{y} \le \frac{1+\tan\alpha}{1-\tan\alpha} \tag{2.4}$$

Geometrically the P_α condition is equivalent to $\frac{|x-y|}{x+y} \le \sin\alpha$, which means the angle β in the P-traingle varies over the interval $[0, \alpha]$ only. T_α condition means the same in the T-triangle.

For the convenience of the reader we provide Appendix 1 with functions corresponding to reciprocals of used means, and their respective second derivatives.

3. Main proving tool

If $\mathcal{A} < \mathcal{B} < \mathcal{M}$, and τ varies from 0 to 1, then the expression

$$\frac{1}{\mathcal{B}(x,y)} - \frac{1-\tau}{\mathcal{A}(x,y)} - \frac{\tau}{\mathcal{M}(x,y)} \tag{3.1}$$

is negative at $\tau = 0$ and strictly increases, to become positive at the other end. We shall be seeking for those values of parameter τ, where (3.1) is negative while the ange β_P or β_T varies over its maximal range, (or over

$(0, \alpha)$ if we consider P_α/T_α condition), and for values where it is always positive. In most cases we face a situation described in following lemma.

Lemma 3.1. *Suppose* $u_\tau : [0, a] \to \mathbb{R},\ \tau \in [0, 1]$ *is a family of functions satisfying the following assumptions:*

- f_τ *increases with* τ*,*
- $f_\tau(0) = f'_\tau(0) = 0$ *for every* τ*,*
- *there exists* τ_0 *such that* $u_\tau(x)$ *are strictly concave in* x *for every* $\tau \le \tau_0$*,*
- *if* $\tau > \tau_0$*, then* $u_\tau(x)$ *are strictly convex for small* x *and* u''_τ *changes sign at most once.*

Let $0 < \alpha \le a$*. Then*

- $u_\tau(x) \le 0$ *holds for all* $x \in [0, \alpha]$ *if and only if* $\tau \le \tau_0$*,*
- $u_\tau(x) \ge 0$ *holds for all* $x \in [0, \alpha]$ *if and only if* $u_\tau(\alpha) \ge 0$*.*

In particular, if $u_{\tau(\alpha)}(\alpha) = 0$*, then* u_τ *is nonnegative for all* $\tau \ge \tau(\alpha)$*.*

Proof. If $\tau \le \tau_0$ the functions u_τ are concave, thus negative. Otherwise they are convex, thus positive for small arguments, so we are done with the first part.

In case $\tau > \tau_0$ the functions u_τ are initially convex and positive, then they may reach a local maximum and become decreasing, which yields the second part.

□

4. Harmonic interpolations with P and L

In this section we deal with approximations of the form

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{P(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{4.1}$$

where $\mathcal{M}$ is a mean bounding P from above.

For the arithmetic mean we have

Theorem 4.1. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{A(x,y)} \le \frac{1}{P(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{A(x,y)}$$

hold if and only if $\nu \le 1/2$ *and* $\mu = 1$*. If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{A(x,y)} \le \frac{1}{P(x,y)}$$

holds if and only $\mu \ge \dfrac{\operatorname{artanh}(\sin\alpha) - \alpha}{\operatorname{artanh}(\sin\alpha) - \sin\alpha}$.

Proof. Using (2.1) we can write

$$\frac{1}{P} - \frac{1-\tau}{L} - \frac{\tau}{A} \cong \beta_P - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\sin\beta_P.$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\sin x) - \tau\sin x$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$u_\tau''(x) = -\sin x\left(\frac{1-\tau}{\cos^2 x} - \tau\right).$$

The expression in brackest increases from $1-2\tau$ to infinity, so Lemma 3.1 applies with $\tau_0 = 1/2$. Since for $\tau < 1$ we have $\lim_{x\to\pi/2} u_\tau(x) = -\infty$, our functions cannot be globally positive. Thus the only global left bound is $\mu = 1$. Solving in τ inequality $u_\tau(\alpha) \ge 0$ leads us to optimal μ in case variable satisfy P_α condition.

□

Note: the concluding statement in the above proof applies to all theorems in all sections where the logarithmic mean is involved.

For the M mean holds

Theorem 4.2. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{M(x,y)} \le \frac{1}{P(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{M(x,y)}$$

hold if and only if $\nu \le 1/3$ *and* $\mu = 1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{M(x,y)} \le \frac{1}{P(x,y)}$$

holds if and only $\mu \ge \dfrac{\operatorname{artanh}(\sin\alpha) - \alpha}{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}$.

Proof. We have

$$\frac{1}{P} - \frac{1-\tau}{L} - \frac{\tau}{M} \cong \beta_P - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\operatorname{arsinh}(\sin\beta_P).$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\sin x) - \tau\operatorname{arsinh}(\sin x)$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$u_\tau''(x) = -\frac{\sin x}{\cos^2 x}\left(1 - \tau - 2\tau\frac{\cos^2 x}{(2-\cos^2 x)^{3/2}}\right).$$

The expression in brackest increases from $1-3\tau$ to $1-\tau$, so Lemma 3.1 applies with $\tau_0 = 1/3$. □

Theorem 4.3. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{T(x,y)}\le\frac{1}{P(x,y)}\le\frac{1-\nu}{L(x,y)}+\frac{\nu}{T(x,y)}$$

hold if and only if $\nu\le 1/4$ *and* $\mu=1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{T(x,y)}\le\frac{1}{P(x,y)}$$

holds if and only $\mu\ge\dfrac{\operatorname{artanh}(\sin\alpha)-\alpha}{\operatorname{artanh}(\sin\alpha)-\arctan(\sin\alpha)}$.

Proof. We have

$$\frac{1}{P}-\frac{1-\tau}{L}-\frac{\tau}{T}\cong\beta_P-(1-\tau)\operatorname{artanh}(\sin\beta_P)-\tau\arctan(\sin\beta_P).$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\sin x)-\tau\arctan(\sin x)$ satisfy $u_\tau(0)=u'_\tau(0)=0$ and

$$u''_\tau(x)=-\sin x\frac{4(1-\tau)-2(2-\tau)\cos^2 x+(1-2\tau)\cos^4 x}{\cos^2 x(1+\sin^2 x)^2}. \tag{4.2}$$

The critical point of the function $p(z)=4(1-\tau)-2(2-\tau)z+(1-2\tau)z^2$ lies outside the interval $(0,1)$ thus we conclude the numerator in (4.2) increases from $1-4\tau$ to $4-4\tau$. Again Lemma 3.1 applies with $\tau_0=1/4$. □

The root-mean square mean can be written in the language of the P-triangle as

$$R=\sqrt{2A^2-G^2}=\sqrt{|AB|^2+|AC|^2}=|AC|\frac{\sqrt{1+\sin^2\beta_P}}{\sin\beta_P}.$$

Theorem 4.4. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{P(x,y)}\le\frac{1-\nu}{L(x,y)}+\frac{\nu}{R(x,y)}$$

hold if and only if $\nu\le 1/5$ *and* $\mu=1$. *If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{P(x,y)}$$

holds if and only $\mu\ge\dfrac{\operatorname{artanh}(\sin\alpha)-\alpha}{\operatorname{artanh}(\sin\alpha)-\frac{\sin\alpha}{\sqrt{1+\sin^2\alpha}}}$.

Proof. We have

$$\frac{1}{P}-\frac{1-\tau}{L}-\frac{\tau}{R}\cong\beta_P-(1-\tau)\operatorname{artanh}(\sin\beta_P)-\tau\frac{\sin\beta_p}{\sqrt{1+\sin^2\beta_P}}.$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\sin x) - \tau\frac{\sin x}{\sqrt{1+\sin^2 x}}$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$u''_\tau(x) = -\frac{\sin x}{\cos^2 x}\left(1 - \tau - 2\tau\frac{\cos^2 x(\cos^2 x + 1)}{(1+\sin^2 x)^{5/2}}\right).$$

The expression in brackets strictly increases from $1-5\tau$ to $1-\tau$ and reasoning seen before leads us to the end of the proof. □

The contraharmonic mean can be written as

$$C = 2A - \frac{G^2}{A} = |AC|\frac{1+\sin^2\beta_P}{\sin\beta_P}.$$

Theorem 4.5. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{P(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 1/8$ *and* $\mu = 1$*. If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{P(x,y)}$$

holds if and only $\mu \ge \dfrac{\operatorname{artanh}(\sin\alpha) - \alpha}{\operatorname{artanh}(\sin\alpha) - \frac{\sin\alpha}{1+\sin^2\alpha}}$.

Proof. We have

$$\frac{1}{P} - \frac{1-\tau}{L} - \frac{\tau}{C} \cong \beta_P - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\frac{\sin\beta_p}{1+\sin^2\beta_P}.$$

The functions $u_\tau(x) = x - (1-\tau)\operatorname{artanh}(\sin x) - \tau\frac{\sin x}{1+\sin^2 x}$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$u''_\tau(x) = -\frac{\sin x}{\cos^2 x}\left(1 - \tau - \tau\frac{\cos^4 x(\cos^2 x + 6)}{(1+\sin^2 x)^3}\right).$$

The expression in brackets strictly increases (numerator decreases, denominator increases) from $1-8\tau$ to $1-\tau$ etc. □

5. Harmonic interpolations with M and L

In this section we deal with approximations of the form

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{5.1}$$

where $\mathcal{M}$ is a mean bounding M from above. Let us begin with the T mean.

Theorem 5.1. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{T(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{T(x,y)}$$

hold if and only if $\nu \le 3/4$ *and* $\mu = 1$*. If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{T(x,y)} \le \frac{1}{M(x,y)}$$

holds if and only $\mu \ge \dfrac{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}{\operatorname{artanh}(\sin\alpha) - \arctan(\sin\alpha)}$.

Proof. We have

$$\begin{aligned}\frac{1}{M} - \frac{1-\tau}{L} - \frac{\tau}{T} &\cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\operatorname{artanh}(\sin\beta_P)\\ &\quad - \tau\arctan(\sin\beta_P) \triangleq u_\tau(\sin\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned}u''_\tau(x) &= x\left(\frac{2(\tau-1)}{(1-x^2)^2} + \frac{2\tau}{(1+x^2)^2} - \frac{1}{(1+x^2)^{3/2}}\right)\\ &\cong 2(\tau-1)(1+x^2)^2 + 2\tau(1-x^2)^2 - (1-x^2)^2\sqrt{1+x^2} \triangleq p(x^2). \qquad (5.2)\end{aligned}$$

We shall show that p strictly decreases. Indeed, $p'(z) = (8\tau-4)z - 4 + \frac{3+2z-5z^2}{2\sqrt{z+1}} \le 4z - 4 + \frac{3+2z-5z^2}{2\sqrt{z+1}} = h(z)$. The function h is concave, attains its maximum at $z_0 \approx 1.631$ and $h(1) = 0$ which means it is negative for $z < 1$. Therefore p decreases and so does $p(x^2)$ (or (5.2)). Since $p(0) = 4\tau - 3$ we are able to apply Lemma 3.1 with $\tau_0 = 3/4$. □

Now it is turn for the R mean

Theorem 5.2. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{R(x,y)}$$

hold if and only if $\nu \le 3/5$ *and* $\mu = 1$*. If the* P_α *condition is satisfied, then*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{M(x,y)}$$

holds if and only $\mu \ge \dfrac{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}{\operatorname{artanh}(\sin\alpha) - \frac{\sin\alpha}{\sqrt{1+\sin^2\alpha}}}$.

Proof. We have

$$\begin{aligned}\frac{1}{M} - \frac{1-\tau}{L} - \frac{\tau}{R} &\cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \frac{\tau\sin\beta_p}{\sqrt{1+\sin^2\beta_P}}\\ &\triangleq u_\tau(\sin\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$u_\tau''(x) = x\left(\frac{2(\tau-1)}{(1-x^2)^2} - \frac{1}{(1+x^2)^{3/2}} + \frac{3\tau}{(1+x^2)^{5/2}}\right)$$
$$\cong 2(\tau-1)(1+x^2)^{5/2} - (1-x^2)^2(1+x^2) + 3\tau(1-x^2)^2 \triangleq p(x^2).$$

We shall show that p decreases in $(0,1)$. We have $-(3z^2-2z-1) \le 4/3$ and $5(z+1)^{3/2} \ge 5$ and

$$p'(z) = \tau(5(z+1)^{3/2} + 6z - 6) - (3z^2 - 2z - 1) - 5(z+1)^{3/2}.$$

If $5(z+1)^{3/2}+6z-6 < 0$, then $p'(z) < -(3z^2-2z-1)-5(z+1)^{3/2} < -11/5 < 0$. Otherwise $p'(z) \le 5(z+1)^{3/2} + 6z - 6 - (3z^2-2z-1) - 5(z+1)^{3/2} = -3z^2 + 8z - 5 < 0$ in $(0,1)$. Since $p(0) = 5\tau - 3$ Lemma 3.1 applies with $\tau_0 = 3/5$.

□

In case of contraharmonic mean we obtain

Theorem 5.3. *The inequalities*

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{M(x,y)} \le \frac{1-\nu}{L(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 3/8$ and $\mu = 1$. If the P_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{M(x,y)}$$

holds if and only $\mu \ge \dfrac{\operatorname{artanh}(\sin\alpha) - \operatorname{arsinh}(\sin\alpha)}{\operatorname{artanh}(\sin\alpha) - \frac{\sin\alpha}{1+\sin^2\alpha}}$.

Proof. We have

$$\frac{1}{M} - \frac{1-\tau}{L} - \frac{\tau}{C} \cong \operatorname{arsinh}(\sin\beta_P) - (1-\tau)\operatorname{artanh}(\sin\beta_P) - \tau\frac{\sin\beta_p}{1+\sin^2\beta_P}$$
$$\triangleq u_\tau(\sin\beta_P).$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$u_\tau''(x) = x\left(\frac{2(\tau-1)}{(1-x^2)^2} - \frac{1}{(1+x^2)^{3/2}} - \frac{2\tau(x^2-3)}{(1+x^2)^3}\right)$$
$$\cong 2(\tau-1)(1+x^2)^3 - (1-x^2)^2(1+x^2)^{3/2} - 2\tau(1-x^2)^2(x^2-3)$$
$$\triangleq p(x^2).$$

We shall show that p decreases in $(0,1)$. We have $-(7z^2-6z-1)\sqrt{z+1} \le 25\sqrt{2}/7$ and $12(z+1)^2 \ge 5$ and

$$2p'(z) = 16\tau(4z-1) - \sqrt{z+1}(7z^2-6z-1) - 12(z+1)^2.$$

If $4z-1<0$, then $p'(z)<25\sqrt{2}/7-12<0$. Otherwise $p'(z)\le 48z-16+25\sqrt{2}/7-12(z+1)^2=-12(z-1)^2-16+25\sqrt{2}/7<0$. Since $p(0)=8\tau-3$ Lemma 3.1 applies with $\tau_0=3/8$.

□

6. Harmonic interpolations with T and L

Theorem 6.1. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{L(x,y)}+\frac{\nu}{R(x,y)}$$

hold if and only if $\nu\le 4/5$ and $\mu=1$. If the T_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{T(x,y)}$$

holds if and only $\mu\ge\dfrac{\operatorname{artanh}(\tan\alpha)-\alpha}{\operatorname{artanh}(\tan\alpha)-\sin\alpha}$.

Proof. We have

$$\frac{1}{T}-\frac{1-\tau}{L}-\frac{\tau}{R}\cong\beta_T-(1-\tau)\operatorname{artanh}(\tan\beta_T)-\tau\sin\beta_T.$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\tan x)-\tau\sin x$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$u_\tau''(x)=-\frac{\sin x}{\cos^2 x}p(\cos x),$$

where $p(z)=4(1-\tau)z-\tau(2z^2-1)^2$. On the interval $(\sqrt{2}/2,1)$ the function p is concave, $p(\sqrt{2}/2)=2\sqrt{2}(1-\tau)\ge 0,\ p(1)=4-5\tau$, so it is positive for $\tau\le 4/5$ and changes sign once otherwise, thus known argument applies.

□

In the T-triangle the contraharmonic mean is represented as

$$C=\frac{R^2}{A}=|AC|\frac{2}{\sin 2x}.$$

Theorem 6.2. *The inequalities*

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{C(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{L(x,y)}+\frac{\nu}{C(x,y)}$$

hold if and only if $\nu\le 1/2$ and $\mu=1$. If the T_α condition is satisfied, then

$$\frac{1-\mu}{L(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{T(x,y)}$$

holds if and only $\mu\ge\dfrac{\operatorname{artanh}(\tan\alpha)-\alpha}{\operatorname{artanh}(\tan\alpha)-\frac{1}{2}\sin 2\alpha}$.

Proof. We have

$$\frac{1}{T}-\frac{1-\tau}{L}-\frac{\tau}{C}\cong \beta_T-(1-\tau)\operatorname{artanh}(\tan\beta_T)-\frac{\tau}{2}\sin 2\beta_T.$$

The functions $u_\tau(x)=x-(1-\tau)\operatorname{artanh}(\tan x)-\frac{\tau}{2}\sin 2x$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$u_\tau''(x)=-2\sin 2x\left(\frac{1-\tau}{\cos^2 2x}-\tau\right).$$

The expression in brackets increases from $1-2\tau$ to infinity, which allows us to complete the proof as usual.

□

7. Harmonic interpolations with M and P

In this section we deal with approximations of the form

(7.1) $$\frac{1-\mu}{P(x,y)}+\frac{\mu}{\mathcal{M}(x,y)}\le\frac{1}{M(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{\mathcal{M}(x,y)},$$

where $\mathcal{M}$ is a mean bounding M from above. We go back to the P-triangle and begin with the T mean. From now on we obtain absolute bounds.

Theorem 7.1. *The inequalities*

$$\frac{1-\mu}{P(x,y)}+\frac{\mu}{T(x,y)}\le\frac{1}{M(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{T(x,y)}$$

hold if and only if $\nu\le 2/3$ *and* $\mu>\dfrac{2\pi-4\operatorname{arsinh}1}{\pi}$.

Proof. We have

$$\begin{aligned}\frac{1}{M}-\frac{1-\tau}{P}-\frac{\tau}{T}&\cong \operatorname{arsinh}(\sin\beta_P)-(1-\tau)\beta_P-\tau\arctan(\sin\beta_P)\\&\triangleq u_\tau(\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$\begin{aligned}u_\tau''(x)&=-\sin x\left(\frac{2}{(1+\sin^2 x)^{3/2}}-\tau\frac{3-\sin^2 x}{(1+\sin^2 x)^2}\right)\\&\cong -\sin x\left(\frac{2(1+\sin^2 x)^{1/2}}{3-\sin^2 x}-\tau\right).\end{aligned}$$

The expression in brackets increases from $2/3-\tau$ thus is positive for $\tau\le 2/3$. Applying Lemma 3.1 we see that u_τ are positive on $(0,\pi/2)$ if and only if $u_\tau(\pi/2)\ge 0$, which completes the proof. □

Now it is turn for the R mean

Theorem 7.2. *The inequalities*

$$\frac{1-\mu}{P(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{M(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{R(x,y)}$$

hold if and only if $\nu\le 1/2$ *and* $\mu>\dfrac{\pi-2\operatorname{arsinh}1}{\pi-\sqrt{2}}$.

Proof. We have

$$\begin{aligned}\frac{1}{M}-\frac{1-\tau}{P}-\frac{\tau}{R}&\cong\operatorname{arsinh}(\sin\beta_P)-(1-\tau)\beta_P-\tau\frac{\sin\beta_P}{\sqrt{1+\sin^2\beta_P}}\\&\triangleq u_\tau(\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u'_\tau(0)=0$ and

$$\begin{aligned}u''_\tau(x)&=-2\sin x\left(\frac{1}{(1+\sin^2 x)^{3/2}}-\tau\frac{2-\sin^2 x}{(1+\sin^2 x)^{5/2}}\right)\\&\cong-\sin x\left(\frac{1+\sin^2 x}{2-\sin^2 x}-\tau\right).\end{aligned}$$

The expression in brackets increases from $1/2-\tau$ application of Lemma 3.1 completes the proof. □

In case of contraharmonic mean we obtain

Theorem 7.3. *The inequalities*

$$\frac{1-\mu}{P(x,y)}+\frac{\mu}{C(x,y)}\le\frac{1}{M(x,y)}\le\frac{1-\nu}{P(x,y)}+\frac{\nu}{C(x,y)}$$

hold if and only if $\nu\le 2/7$ *and* $\mu>\dfrac{\pi-2\operatorname{arsinh}1}{\pi-1}$.

Proof. We have

$$\begin{aligned}\frac{1}{M}-\frac{1-\tau}{P}-\frac{\tau}{C}&\cong\operatorname{arsinh}(\sin\beta_P)-(1-\tau)\beta_P-\tau\frac{\sin\beta_P}{1+\sin^2\beta_P}\\&\triangleq u_\tau(\beta_P).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u'_\tau(0)=0$ and

$$\begin{aligned}u''_\tau(x)&=-\sin x\left(\frac{2}{(1+\sin^2 x)^{3/2}}-\tau\frac{\cos^4 x+6\cos^2 x}{(1+\sin^2 x)^2}\right)\\&\cong-\sin x\left(\frac{2(1+\sin^2 x)^{1/2}}{\cos^4 x+6\cos^2 x}-\tau\right).\end{aligned}$$

The expression in brackets increases from $2/7-\tau$ and again application of Lemma 3.1 completes the proof. □

8. Harmonic interpolations with T and M

In this section we deal with approximations of the form

$$\frac{1-\mu}{M(x,y)}+\frac{\mu}{\mathcal{M}(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{M(x,y)}+\frac{\nu}{\mathcal{M}(x,y)},\tag{8.1}$$

where $\mathcal{M}$ is a mean bounding T from above. We switch back to the T-triangle.

The first upper bound for T is the root-mean square mean.

Theorem 8.1. *The inequalities*

$$\frac{1-\mu}{M(x,y)}+\frac{\mu}{R(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{M(x,y)}+\frac{\nu}{R(x,y)}$$

hold if and only if $\nu\le 1/2$ *and* $\mu>\dfrac{\pi}{2(\pi-\sqrt{2})}$.

Proof. We have

$$\begin{aligned}\frac{1}{T}-\frac{1-\tau}{M}-\frac{\tau}{R}&\cong\beta_T-(1-\tau)\operatorname{arsinh}(\tan\beta_T)-\tau\sin\beta_T\\&\triangleq u_\tau(\beta_T).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$\begin{aligned}u_\tau''(x)&=-(1-\tau)\frac{\sin x}{\cos^2 x}+\tau\sin x\\&\cong-(1-\tau)\sin x\left(\frac{1}{\cos^2 x}-\frac{\tau}{1-\tau}\right).\end{aligned}$$

The expression in brackets increases from $1-\frac{\tau}{1-\tau}$ and once more application of Lemma 3.1 completes the proof. □

For the contraharmonic mean we obtain

Theorem 8.2. *The inequalities*

$$\frac{1-\mu}{M(x,y)}+\frac{\mu}{C(x,y)}\le\frac{1}{T(x,y)}\le\frac{1-\nu}{M(x,y)}+\frac{\nu}{C(x,y)}$$

hold if and only if $\nu\le 1/5$ *and* $\mu>\dfrac{\pi}{2(\pi-1)}$.

Proof. We have

$$\begin{aligned}\frac{1}{T}-\frac{1-\tau}{M}-\frac{\tau}{C}&\cong\beta_T-(1-\tau)\operatorname{arsinh}(\tan\beta_T)-\tau\frac{\sin 2\beta_T}{2}\\&\triangleq u_\tau(\beta_T).\end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0)=u_\tau'(0)=0$ and

$$\begin{aligned} u_\tau''(x) &= -(1-\tau)\frac{\sin x}{\cos^2 x} + 4\tau \sin x \cos x \\ &= -4(1-\tau)\sin x \cos x \left(\frac{1}{4\cos^3 x} - \frac{\tau}{1-\tau}\right). \end{aligned}$$

The expression in brackets increases from $\frac{1}{4} - \frac{\tau}{1-\tau}$ and once more application of Lemma 3.1 completes the proof. □

9. Harmonic interpolations with T and P

In this section we deal with approximations of the form

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{\mathcal{M}(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{\mathcal{M}(x,y)}, \tag{9.1}$$

where $\mathcal{M}$ is a mean bounding T from above. We switch back to the P-triangle.

The first upper bound for T is the root-mean square mean.

Theorem 9.1. *The inequalities*

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{R(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{R(x,y)}$$

hold if and only if $\nu \le 3/4$ *and* $\mu > \dfrac{\pi}{2(\pi - \sqrt{2})}$.

Proof. We have

$$\begin{aligned} \frac{1}{T} - \frac{1-\tau}{P} - \frac{\tau}{R} &\cong \arctan(\sin \beta_P) - (1-\tau)\beta_P - \tau \frac{\sin \beta_P}{(1+\sin^2 \beta_P)^{1/2}} \\ &\triangleq u_\tau(\beta_P). \end{aligned}$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u_\tau'(0) = 0$ and

$$\begin{aligned} u_\tau''(x) &= -\frac{\sin x(\cos^2 x + 2)}{(1+\sin^2 x)^2} + \tau \frac{2\sin x(\cos^2 x + 1)}{(1+\sin^2 x)^{5/2}} \\ &= -\frac{2\sin x(\cos^2 x + 1)}{(1+\sin^2 x)^{5/2}} \left(\frac{(\cos^2 x + 2)(1+\sin^2 x)^{1/2}}{2(\cos^2 x + 1)} - \tau\right). \end{aligned}$$

The expression in brackets increases from $3/4 - \tau$ (because both $1 + \frac{1}{1+\cos^2 x}$ and $\sqrt{1+\sin^2 x}$ increase) and once more application of Lemma 3.1 completes the proof. □

For the contraharmonic mean we obtain

Theorem 9.2. *The inequalities*

$$\frac{1-\mu}{P(x,y)} + \frac{\mu}{C(x,y)} \le \frac{1}{T(x,y)} \le \frac{1-\nu}{P(x,y)} + \frac{\nu}{C(x,y)}$$

hold if and only if $\nu \le 3/7$ *and* $\mu > \dfrac{\pi}{2(\pi-1)}$.

Proof. We have

$$\frac{1}{T} - \frac{1-\tau}{P} - \frac{\tau}{C} \cong \arctan(\sin\beta_P) - (1-\tau)\beta_P - \tau\frac{\sin\beta_P}{1+\sin^2\beta_P}$$
$$\triangleq u_\tau(\beta_P).$$

The functions $u_\tau(x)$ satisfy $u_\tau(0) = u'_\tau(0) = 0$ and

$$\begin{aligned} u''_\tau(x) &= -\frac{\sin x(\cos^2 x + 2)}{(1+\sin^2 x)^2} + \tau\frac{\sin x(\cos^4 x + 6\cos^2 x)}{(1+\sin^2 x)^3} \\ &= -\frac{\sin x(\cos^4 x + 6\cos^2 x)}{(1+\sin^2 x)^3}\left(\frac{(4-\cos^4 x)}{\cos^4 x + 6\cos^2 x} - \tau\right). \end{aligned}$$

The expression in brackets increases from $3/7-\tau$ etc. □

10. Appendix A

P-triangle

Mean	f	f''
$\frac{1}{L}$	$\operatorname{artanh}(\sin x)$	$\dfrac{\sin x}{\cos^2 x}$
$\frac{1}{P}$	x	0
$\frac{1}{A}$	$\sin x$	$-\sin x$
$\frac{1}{M}$	$\operatorname{arsinh}(\sin x)$	$-\dfrac{2\sin x}{(1+\sin^2 x)^{3/2}}$
$\frac{1}{T}$	$\arctan(\sin x)$	$-\dfrac{\sin x(\cos^2 x+2)}{(1+\sin^2 x)^2}$
$\frac{1}{R}$	$\dfrac{\sin x}{(1+\sin^2 x)^{1/2}}$	$-\dfrac{2\sin x(\cos^2 x+1)}{(1+\sin^2 x)^{5/2}}$
$\frac{1}{C}$	$\dfrac{\sin x}{1+\sin^2 x}$	$-\dfrac{\sin x(\cos^4 x+6\cos^2 x)}{(1+\sin^2 x)^3}$

T-triangle

Mean	f	f''
$\frac{1}{L}$	$\operatorname{artanh}(\tan x)$	$\dfrac{2\sin 2x}{\cos^2 2x}$
$\frac{1}{A}$	$\tan x$	$\dfrac{2\sin x}{\cos^3 x}$
$\frac{1}{M}$	$\operatorname{arsinh}(\tan x)$	$\dfrac{\sin x}{\cos^2 x}$
$\frac{1}{T}$	x	0
$\frac{1}{R}$	$\sin x$	$-\sin x$
$\frac{1}{C}$	$\dfrac{\sin 2x}{2}$	$-2\sin 2x$

References

[1] B.C. Carlsson, *Algorithms involving arithmetic and geometic means*, Amer. Math. Monthly **78** (1971), 496-505.

[2] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang, and G.-D. Wang, *The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean*, Journal of Inequalities and Applications, Article ID 436457, 7 pages, 2010.

[3] P.A. Hästö *A Monotonicity Property of Ratios of Symmetric Homogeneous Means*, J. Ineq. Pure and Appl. Math., **3**(5) (2002), Article 71. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=223`].

[4] P.A. Hästö *Optimal inequalities between Seiffert's mean and power means*, Math. Ineq. Appl. **7** (1) (2004) 47-53.

[5] A.A. Jager, *Solution of Problem 887* Nieuv Arch. Wisk. (**4**) 12 (1994) 230-231.

[6] Neuman E., Sándor J. *On the Schwab-Borchardt Mean* Math. Pannonica **14**/2 (2003) 253-260

[7] H. Liu and X-J Meng, *The Optimal Convex Combination Bounds for Seiffert's Mean*, Journal of Inequalities and Applications Volume 2011, Article ID 686834, 9 pages.

[8] D.S. Mitrinovi'c, *Elementary inequalities*, P. Noordhoff Ltd., Groningen, 1964.

[9] E. Neuman and J. Sándor, *On the Schwab-Borchardt mean*, Math. Pannonica. **14**,2 (2003) 253-266,

[10] J. Sándor, *On certain inequalities for means, III*, Arch. Math. **76** (2001) 34-40, [ONLINE: `http:/rgmia.vu.edu.au/v2n3.html`].

[11] J. Sándor, *Über zwei Mittel von Seiffert*, Die Wurzel **5**(2002) 104-107.

[12] J. Sándor, E. Neuman, *On certain menas of two arguments and their extensions*, IJMMS, (**16**) (2003) 981-993

[13] H.-J. Seiffert, *Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen*, Elem. Math. 42 (1987), 105-107.

[14] H.-J. Seiffert, *Problem 887*, Nieuv Arch. Wisk. (Ser. 4), **11** (1993), 196.

[15] H.-J. Seiffert, *Aufgabe β16*, Die Wurzel, **29** (1995), 221-222.

[16] H.-J. Seiffert, *Ungleichungen für einen bestimmten Mittelwert*, Nieuv Arch. Wisk. (Ser. 4), **13** (1995), 195-198.

[17] S. Wang, Y. Chu, *The Best Bounds of the Combination of Arithmetic and Harmonic Means for the Seiffert's Mean* Int. Journal of Math. Analysis, Vol. **4**, 2010, no. 22, 1079–1084.

[18] J.B. Wilker *Problem 3306*, Amer. Math. Monthly **96** (1898), 55.

[19] A. Witkowski, *Interpolations of Schwab-Borchartdt mean"*, in preparation.

[20] C. Zong, Y. Chu, *An Inequality Among Identric, Geometric and Seiffert's Means*, International Mathematical Forum, **5**, 2010, no. 26, 1297–1302,

Institute of Mathematics and Physics, University of Technology and Life Sciences, al. prof. Kaliskiego 7, 85-796 Bydgoszcz, Poland

E-mail address: `alfred.witkowski@utp.edu.pl`