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THE HADAMARD INEQUALITY FOR CONVEX FUNCTION VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish several inequalities for some differentiable mappings that are connected with the Riemann-Liouville fractional integrals. The analysis used in the proofs is fairly elementary.

1. INTRODUCTION

One of the most famous inequalities for convex functions is Hadamard's inequality. This double inequality is stated as follows(see for example [7] and [12]): Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For several recent results concerning the inequality (1.1) we refer the interested reader to ([1], [7]-[11]).

Definition 1. *The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds:*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

In [10], Özdemir and Kırmacı proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I, $a, r \in I$. Furthermore, if $f' \in L[a, r]$ $(a < r)$, then*

$$\frac{f(a)+f(r)}{2} - \frac{1}{r-a}\int_a^r f(x)dx = \frac{r-a}{2}\int_0^1 (1-2t)f'(r+(a-r)t)dt.$$

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I, $a, r \in I$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(r)}{2} - \frac{1}{r-a}\int_a^r f(x)dx\right| \leq \frac{(r-a)\left(|f'(a)| + |f'(r)|\right)}{8}. \tag{1.2}$$

In [8], Dragomir and Agarwal established the following result.

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Theorem 2. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on l°, $a, b \in I^{\circ}$ with $a < b$, and let $p > 1$. If the new mapping $|f'|^{p/(p-1)}$ convex on $[a, b]$, then the £ollowing inequality holds:*

(1.3)
$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.$$

In [11], Pearce and Pečarić established the following result which holds for convex functions.

Theorem 3. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on l°, $a, b \in I^{\circ}$ with $a < b$, and let $q \geq 1$. If the mapping $|f'|^q$ convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

In [13], Sarıkaya *et. al.* proved the following result for fractional integrals.

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(b)\right] \\ &= \frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)dt. \end{aligned} \tag{1.5}$$

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(b)\right]\right| \\ &\leq \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right]. \end{aligned} \tag{1.6}$$

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 2. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x<b$$

respectively where $\Gamma(\alpha)=\int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here is $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see ([2]-[6]).

Motivated by the above results we consider new inequalites that related of right hand Hadamard integral inequalities for some convex functions by using Hölder inequality, properties of modulus, power mean inequality and elementary inequality.

The aim of this paper is to establish some Hadamard's inequality for convex functions via Riemann-Liouville fractional integral. In order to obtain our results, we modified Lemma 2 as following.

2. HERMITE-HADAMARD'S INEQUALITIES FOR FRACTIONAL INTEGRALS

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I with $a < r$, $a, r \in I$. If $f' \in L[a, r]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r)\right] \\ = & \frac{r-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(r+(a-r)t)dt. \end{aligned} \tag{2.1}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(r+(a-r)t)dt \\ &= \left[\int_0^1 (1-t)^{\alpha} f'(r+(a-r)t)dt\right] + \left[-\int_0^1 t^{\alpha} f'(r+(a-r)t)dt\right] \\ &= I_1 + I_2. \end{aligned}$$

Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha} f'(r+(a-r)t)dt \\ &= (1-t)^{\alpha}\frac{f(r+(a-r)t)}{a-r}\Big|_0^1 + \int_0^1 \alpha(1-t)^{\alpha-1}\frac{f(r+(a-r)t)}{a-r}dt \end{aligned} \tag{2.2}$$

On using the change of the variable $x = r + (a-r)t$, $t \in [0, 1]$ equality (2.2) can written as

$$\begin{aligned} I_1 &= -\frac{f(r)}{a-r} + \frac{\alpha}{a-r}\int_r^a \left(\frac{a-x}{a-r}\right)^{\alpha-1}\frac{f(x)}{a-r}dx \\ &= -\frac{f(r)}{a-r} - \frac{\Gamma(\alpha+1)}{(r-a)^{\alpha+1}} J_{r^-}^{\alpha} f(a) \end{aligned}$$

and similarly we get

$$\begin{aligned} I_2 &= -\int_0^1 t^{\alpha} f'(r+(a-r)t)dt \\ &= -t^{\alpha}\frac{f(r+(a-r)t)}{a-r}\Big|_0^1 + \int_0^1 \alpha t^{\alpha-1}\frac{f(r+(a-r)t)}{a-r}dt \\ &= -\frac{f(a)}{a-r} - \frac{\Gamma(\alpha+1)}{(r-a)^{\alpha+1}} J_{a^+}^{\alpha} f(r). \end{aligned} \tag{2.3}$$

Hence, from (2.2) and (2.3), we obtain equality (2.1). □

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I, $a, r \in I$ and $a < r$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r-}^{\alpha} f(a) + J_{a+}^{\alpha} f(r)\right]\right| \leq \frac{r-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|+|f'(r)|}{2}\right]. \tag{2.4}$$

Proof. Using Lemma 2 and the convexity of $|f'|$, it follows that

$$\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r-}^{\alpha} f(a) + J_{a+}^{\alpha} f(r)\right]\right| \leq \frac{r-a}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| \left|f'(r+(a-r)t)\right| dt,$$

using the convexity of $|f'|$, we obtain inequality

$$\left|f'(r+(a-r)t)\right| = \left|f'(ta+(1-t)r)\right| \leq t\left|f'(a)\right| + (1-t)\left|f'(r)\right|, \; t \in (0,1).$$

Hence,

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r-}^{\alpha} f(a) + J_{a+}^{\alpha} f(r)\right]\right| \\
\leq\; & \frac{r-a}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right| \left[t\left|f'(a)\right| + (1-t)\left|f'(r)\right|\right] dt \\
=\; & \frac{r-a}{2}\left\{\int_0^{\frac{1}{2}} \left[(1-t)^{\alpha} - t^{\alpha}\right]\left[t\left|f'(a)\right| + (1-t)\left|f'(r)\right|\right] dt \right. \\
& \left. + \int_{\frac{1}{2}}^1 \left[t^{\alpha} - (1-t)^{\alpha}\right]\left[t\left|f'(a)\right| + (1-t)\left|f'(r)\right|\right] dt\right\} \\
=\; & \frac{r-a}{2}\left\{\left|f'(a)\right|\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]\right. \\
& \left. + \left|f'(r)\right|\left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]\right\} \\
& + \frac{r-a}{2}\left\{\left|f'(a)\right|\left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]\right. \\
& \left. + \left|f'(r)\right|\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]\right\} \\
=\; & \frac{r-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|+|f'(r)|}{2}\right].
\end{aligned}$$

□

Remark 1. *If in Theorem 5, we choose $r = b$, then, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha} f(a) + J_{a+}^{\alpha} f(b)\right]\right| \leq \frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|+|f'(b)|}{2}\right]. \tag{2.5}$$

which gives the inequality (1.6).

Remark 2. *If in Remark 1, we choose $\alpha = 1$, then the inequalities (2.5) become the inequalities (1.2) of Theorem 1.*

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a dfferentiable mapping on (a, b) with $a < r$ such that $f' \in L_1[a, r]$. If $|f'|^q$ is convex on $[a, b]$, and $p > 1$, then the following inequality for fractional integrals hold:*

$$\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J^{\alpha}_{r^-}f(a) + J^{\alpha}_{a^+}f(r)\right]\right| \leq \frac{r-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q + |f'(r)|^q}{2}\right)^{\frac{1}{q}} \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0, 1]$.

Proof. From Lemma 2 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J^{\alpha}_{r^-}f(a) + J^{\alpha}_{a^+}f(r)\right]\right| \\
\leq\ & \frac{r-a}{2}\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\,|f'(r+(a-r)t)|\,dt \\
\leq\ & \frac{r-a}{2}\left(\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(r+(a-r)t)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

We know that for $\alpha \in [0, 1]$ and $\forall t_1, t_2 \in [0, 1]$,

$$|t_1^{\alpha} - t_2^{\alpha}| \leq |t_1 - t_2|^{\alpha},$$

therefore

$$\begin{aligned}
\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p\,dt &\leq \int_0^1 |1-2t|^{\alpha p}\,dt \\
&= \int_0^{\frac{1}{2}} [1-2t]^{\alpha p}\,dt + \int_{\frac{1}{2}}^1 [2t-1]^{\alpha p}\,dt \\
&= \frac{1}{\alpha p+1}.
\end{aligned}$$

Since $|f'|^q$ is convex on $[a, b]$,we get

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J^{\alpha}_{r^-}f(a) + J^{\alpha}_{a^+}f(r)\right]\right| \\
\leq\ & \frac{r-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q + |f'(r)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

Corollary 1. *If in Theorem 6, we choose $r = b$, we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(b)\right]\right| \leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.7}$$

Remark 3. *In Corollary 1, if we choose $\alpha = 1$, then we obtain the inequality (1.3).*

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a dfferentiable mapping on (a, b) with $a < r$ such that $f' \in L_1[a, r]$. If $|f'|^q$ is concave on $[a, b]$, and $p > 1$, then the following inequality for fractional integrals hold:*

$$\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r)\right]\right| \leq \frac{r-a}{2(\alpha p+1)^{\frac{1}{p}}}\left|f'\left(\frac{a+r}{2}\right)\right| \tag{2.8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0, 1]$.

Proof. From Lemma 2 and using Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r)\right]\right| \\
\leq\ & \frac{r-a}{2}\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|\,|f'(r+(a-r)t)|\,dt \\
\leq\ & \frac{r-a}{2}\left(\int_0^1 |(1-t)^{\alpha} - t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(r+(a-r)t)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a, b]$, we can use the integral Jensen's inequality to obtain

$$\begin{aligned}
\int_0^1 |f'(r+(a-r)t)|^q\,dt &= \int_0^1 t^0\,|f'(r+(a-r)t)|^q\,dt \\
&\leq \left(\int_0^1 t^0 dt\right)\left|f'\left(\frac{\int_0^1 [r+(a-r)t]\,dt}{\int_0^1 t^0 dt}\right)\right|^q \\
&= \left|f'\left(\frac{a+r}{2}\right)\right|^q.
\end{aligned}$$

Therefore

$$\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha} f(a) + J_{a^+}^{\alpha} f(r)\right]\right| \leq \frac{r-a}{2(\alpha p+1)^{\frac{1}{p}}}\left|f'\left(\frac{a+r}{2}\right)\right|$$

which completes the proof. □

Corollary 2. *If in Theorem 7, we choose $r=b$, we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(b)\right]\right| \leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{2.9}$$

Theorem 8. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I with $a<r$ and $q\geq 1$. If $|f'|^q$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(r)\right]\right| \leq \frac{r-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|^q+|f'(r)|^q}{2}\right]^{\frac{1}{q}}. \tag{2.10}$$

Proof. From Lemma 2 and using the well known power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(r)\right]\right| \\
\leq\ & \frac{r-a}{2}\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'(r+(a-r)t)|\,dt \\
\leq\ & \frac{r-a}{2}\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'(r+(a-r)t)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,dt &= \int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right).
\end{aligned}$$

Since $|f'|^q$ is convex, we obtain

$$|f'(r+(a-r)t)|^q=|f'(ta+(1-t)r)|^q\leq t\,|f'(a)|^q+(1-t)\,|f'(r)|^q\,,\ \ t\in(0,1)$$

and

$$\begin{aligned}
\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'(r+(a-r)t)|^q\,dt &\leq \int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\left[t\,|f'(a)|^q+(1-t)\,|f'(r)|^q\right]dt \\
&= \int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[t\,|f'(a)|^q+(1-t)\,|f'(r)|^q\right]dt \\
&\quad +\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left[t\,|f'(a)|^q+(1-t)\,|f'(r)|^q\right]dt \\
&= \frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|^q+|f'(r)|^q\right].
\end{aligned}$$

Therefore, we have

$$\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r-}^{\alpha}f(a)+J_{a+}^{\alpha}f(r)\right]\right| \leq \frac{r-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|^{q}+|f'(r)|^{q}}{2}\right]^{\frac{1}{q}}$$

which is required. □

Corollary 3. *If we choose $r=b$ in the inequality (2.10) , we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right| \leq \frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}. \tag{2.11}$$

Remark 4. *If in Corollary 3, we choose $\alpha=1$, then the inequalities (2.11) become the inequalities (1.4) of Theorem 3.*

References

[1] M. Alomari and M. Darus, On the Hadamard's inequality for log-convex functions on the coordinateds, Journal of Inequalities and Applications, vol. 2009, Article ID 283147, 13 pages, 2009.
[2] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.
[3] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Science, 9(4) (2010), 493-497.
[4] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. 1(1) (2010), 51-58.
[5] Z. Dahmani, L. Tabharit, S. Taf, Some fractional integral inequalities, Nonl. Sci. Lett. A., 1(2) (2010), 155-160.
[6] Z. Dahmani, L. Tabharit, S. Taf, New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.
[7] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[8] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91-95.
[9] U.S. Kırmacı, M.K. Bakula, M.E. Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions, Appl. Math. and Comp., 193(2007), 26-35.
[10] M.E. Özdemir and U.S. Kırmacı, Two new theorem on mappings uniformly continuous and convex with applications to quadrature rules and means. Appl. Math. and Comp., 143(2003), 269-274.
[11] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula. Appl. Math. Lett., 13 (2000), 51-55.
[12] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992.
[13] M.Z. Sarıkaya, E. Set, H. Yaldız and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Submitted.

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