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GENERALIZATIONS OF FURUTA'S INEQUALITY

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ABSTRACT. We obtain amongst other the following result

$$\left|\left\langle Tf\left(|T|^{\alpha+\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \left\langle f_A\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle f_A\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle$$

for any $x,y\in H$, where $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is a function defined by power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$, $f_A(z)=\sum_{n=0}^{\infty}|a_n|z^n$, $T\in\mathcal{B}(H)$, $\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$ and $\|T\|^{2\alpha},\|T\|^{2\beta}<R$. For constant functions this produces Furuta's inequality which in its turn is a generalization of Kato's inequality that is obtained for $\alpha\in[0,1]$ and $\beta=1-\alpha$.

1. Introduction

In the following we denote by $\mathcal{B}(H)$ the *Banach algebra* of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$.

In 1952, Kato [11] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator T on H:

$$|\langle Tx,y\rangle|^2\leq\langle(T^*T)^{\alpha}x,x\rangle\left\langle(TT^*)^{1-\alpha}y,y\right\rangle, \tag{K}$$

for any $x,y\in H$ and $\alpha\in[0,1]$. Utilizing the modulus notation, we can write (K) as follows

$$|\langle Tx,y\rangle|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2(1-\alpha)}y,y\right\rangle \tag{1.1}$$

for any $x,y\in H$ and $\alpha\in[0,1]$.

In order to generalize this result, in 1994 Furuta [10] obtained the following result:

$$\left|\left\langle T|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2\beta}y,y\right\rangle \tag{F}$$

for any $x,y\in H$ and $\alpha,\beta\in[0,1]$ with $\alpha+\beta\geq 1$.

From the proof in [10], one realizes that the condition $\alpha,\beta\in[0,1]$ is taken only to fit with the result from the *Heinz-Kato inequality*

$$|\langle Tx,y\rangle|\leq\|A^{\alpha}x\|\left\|B^{1-\alpha}y\right\|, \tag{HK}$$

for any $x,y\in H$ and $\alpha\in[0,1]$, where A and B are positive operators such that $\|Tx\|\leq\|Ax\|$ and $\|T^*y\|\leq\|By\|$ for all $x,y\in H$.

Therefore, one can state the more general result:

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Theorem 1 (Furuta Inequality, 1994, [10])**.** *Let $T \in \mathcal{B}(H)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$. Then for any $x, y \in H$ we have the inequality (F).*

We observe that this fact allows for some particular instances of interest that were not possible in the case when $\alpha, \beta \in [0,1]$.

If we take $\beta = \alpha$ in (F) then we get

$$\left|\left\langle T|T|^{2\alpha-1}x, y\right\rangle\right|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T^*|^{2\alpha}y, y\right\rangle \tag{1.2}$$

for any $x, y \in H$ and $\alpha \geq \frac{1}{2}$. In particular, for $\alpha = 1$ we get

$$|\langle T|T|x, y\rangle|^2 \leq \left\langle |T|^2 x, x\right\rangle \left\langle |T^*|^2 y, y\right\rangle \tag{1.3}$$

for any $x, y \in H$.

If we take $T = N$ a *normal operator*, i.e., we recall that $NN^* = N^*N$, then we get from (F) the following inequality for normal operators

$$\left|\left\langle N|N|^{\alpha+\beta-1}x, y\right\rangle\right|^2 \leq \left\langle |N|^{2\alpha}x, x\right\rangle \left\langle |N|^{2\beta}y, y\right\rangle \tag{1.4}$$

for any $x, y \in H$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$.

This implies the inequalities

$$\left|\left\langle N|N|^{2\alpha-1}x, y\right\rangle\right|^2 \leq \left\langle |N|^{2\alpha}x, x\right\rangle \left\langle |N|^{2\alpha}y, y\right\rangle \tag{1.5}$$

for any $x, y \in H$ and $\alpha \geq \frac{1}{2}$ and, in particular,

$$|\langle N|N|x, y\rangle|^2 \leq \left\langle |N|^2 x, x\right\rangle \left\langle |N|^2 y, y\right\rangle \tag{1.6}$$

for any $x, y \in H$.

Making $y = x$ in (1.5) produces

$$\left|\left\langle N|N|^{2\alpha-1}x, x\right\rangle\right| \leq \left\langle |N|^{2\alpha}x, x\right\rangle$$

for any $x \in H$ and $\alpha \geq \frac{1}{2}$ and, in particular,

$$|\langle N|N|x, x\rangle| \leq \left\langle |N|^2 x, x\right\rangle$$

for any $x \in H$.

For various interesting generalizations and extensions of Kato and Furuta inequalities, see the papers [1]-[10], and [12]-[18].

In this paper we obtain some functional generalizations of Furuta and Kato inequalities. Particular cases of interest are presented as well.

2. Functional Generalizations

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which has as coefficients the absolute values of the coefficients of the original series, namely, $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series has the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_A = f$.

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Let $T \in \mathcal{B}(H)$, $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and such that*

$$\|T\|^{2\alpha}, \|T\|^{2\beta} < R, \tag{2.1}$$

then we have the inequality

$$\left|\left\langle Tf\left(|T|^{\alpha+\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \left\langle f_A\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle f_A\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle \tag{2.2}$$

for any $x,y\in H$.

Proof. Since $\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$, then $n\alpha+n\beta\geq 1$ for any $n\geq 1$.

From Furuta's inequality (F) we have the power inequality

$$\left|\left\langle T|T|^{n\alpha+n\beta-1}x,y\right\rangle\right|\leq\left\langle |T|^{2n\alpha}x,x\right\rangle^{1/2}\left\langle |T^*|^{2n\beta}y,y\right\rangle^{1/2}, \tag{2.3}$$

for any natural numbers $n\geq 1$ and $x,y\in H$.

If we multiply this inequality with the positive quantities $|a_{n-1}|$, use the triangle inequality and the Cauchy-Bunyakowsky-Schwarz discrete inequality we have successively

$$\begin{aligned}
&\left|\left\langle\sum_{n=1}^{k}a_{n-1}T|T|^{n(\alpha+\beta)-1}x,y\right\rangle\right| \\
&\leq\sum_{n=1}^{k}|a_{n-1}|\left|\left\langle T|T|^{n(\alpha+\beta)-1}x,y\right\rangle\right| \\
&\leq\sum_{n=1}^{k}|a_{n-1}|\left\langle |T|^{2n\alpha}x,x\right\rangle^{1/2}\left\langle |T^*|^{2n\beta}y,y\right\rangle^{1/2} \\
&\leq\left\langle\sum_{n=1}^{k}|a_{n-1}||T|^{2n\alpha}x,x\right\rangle^{1/2}\left\langle\sum_{n=1}^{k}|a_{n-1}||T^*|^{2n\beta}y,y\right\rangle^{1/2}
\end{aligned} \tag{2.4}$$

for any $x,y\in H$ and $k\geq 1$.

Observe also that

$$\sum_{n=1}^{k}a_{n-1}T|T|^{n(\alpha+\beta)-1}=T\left(\sum_{n=1}^{k}a_{n-1}|T|^{(n-1)(\alpha+\beta)}\right)|T|^{(\alpha+\beta)-1},$$

$$\sum_{n=1}^{k}|a_{n-1}||T|^{2n\alpha}=\left(\sum_{n=1}^{k}|a_{n-1}||T|^{2(n-1)\alpha}\right)|T|^{2\alpha}$$

and

$$\sum_{n=1}^{k}|a_{n-1}||T^*|^{2n\beta}=\left(\sum_{n=1}^{k}|a_{n-1}||T^*|^{2(n-1)\beta}\right)|T^*|^{2\beta}$$

for any $k\geq 1$.

Therefore, by (2.4) we have the inequality

$$\left|\left\langle T\left(\sum_{n=1}^{k} a_{n-1}\left|T\right|^{(n-1)(\alpha+\beta)}\right)\left|T\right|^{(\alpha+\beta)-1}x,y\right\rangle\right|^{2} \tag{2.5}$$
$$\leq\left\langle\left(\sum_{n=1}^{k}\left|a_{n-1}\right|\left|T\right|^{2(n-1)\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle$$
$$\times\left\langle\left(\sum_{n=1}^{k}\left|a_{n-1}\right|\left|T^{*}\right|^{2(n-1)\beta}\right)\left|T^{*}\right|^{2\beta}y,y\right\rangle$$

for any $x,y\in H$ and $k\geq 1$.

From (2.1) we have that the series $\sum_{n=0}^{\infty}a_n\left(\left|T\right|^{\alpha+\beta}\right)^n$, $\sum_{n=0}^{\infty}\left|a_n\right|\left(\left|T\right|^{2\alpha}\right)^n$ and $\sum_{n=0}^{\infty}\left|a_n\right|\left(\left|T^{*}\right|^{2\beta}\right)^n$ are convergent in $\mathcal{B}(H)$ and taking the limit over $k\to\infty$ in (2.5) we deduce the desired result from (2.2). □

Corollary 1. *With the assumptions of Theorem 2 we have the norm inequality*

$$\left\|Tf\left(\left|T\right|^{\alpha+\beta}\right)\left|T\right|^{\alpha+\beta-1}\right\|^{2}\leq\left\|f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}\right\|\left\|f_A\left(\left|T^{*}\right|^{2\beta}\right)\left|T^{*}\right|^{2\beta}\right\| \tag{2.6}$$

and the numerical radius inequality

$$w\left(Tf\left(\left|T\right|^{\alpha+\beta}\right)\left|T\right|^{\alpha+\beta-1}\right)\leq\frac{1}{2}\left\|f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}+f_A\left(\left|T^{*}\right|^{2\beta}\right)\left|T^{*}\right|^{2\beta}\right\|. \tag{2.7}$$

Proof. The proof of (2.6) follows by (2.2) on taking the supremum over $x,y\in H$ with $\|x\|=\|y\|=1$.

By the inequality (2.2) we also have

$$\left|\left\langle Tf\left(\left|T\right|^{\alpha+\beta}\right)\left|T\right|^{\alpha+\beta-1}x,x\right\rangle\right|$$
$$\leq\left\langle f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle^{1/2}\left\langle f_A\left(\left|T^{*}\right|^{2\beta}\right)\left|T^{*}\right|^{2\beta}x,x\right\rangle^{1/2}$$
$$\leq\frac{1}{2}\left\langle\left[f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}+f_A\left(\left|T^{*}\right|^{2\beta}\right)\left|T^{*}\right|^{2\beta}\right]x,x\right\rangle$$

for any $x\in H$.

Taking the supremum over $\|x\|=1$ we deduce the desired inequality (2.7). □

Remark 1. *If we take $f(z)=1$, then we get from (2.2) the Furuta's inequality (F).*

If we take $\beta=\alpha$ in (2.2), then we get

$$\left|\left\langle Tf\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha-1}x,y\right\rangle\right|^{2} \tag{2.8}$$
$$\leq\left\langle f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle\left\langle f_A\left(\left|T^{*}\right|^{2\alpha}\right)\left|T^{*}\right|^{2\alpha}y,y\right\rangle$$

provided $\alpha\geq\frac{1}{2}$ and $\left\|T\right\|^{2\alpha}<R$.

In particular, we have

$$\left|\left\langle Tf\left(\left|T\right|\right)x,y\right\rangle\right|^{2}\leq\left\langle f_A\left(\left|T\right|\right)\left|T\right|x,x\right\rangle\left\langle f_A\left(\left|T^{*}\right|\right)\left|T^{*}\right|y,y\right\rangle \tag{2.9}$$

for any $T\in\mathcal{B}(H)$ with $\|T\|<R$.

Remark 2. *If we take $\beta = 1-\alpha$ with $\alpha \in [0,1]$ in (2.2) then we get the following generalization of Kato's inequality (1.1)*

$$\begin{aligned} &|\langle Tf(|T|)x,y\rangle|^2 \\ &\le \left\langle f_A\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle \left\langle f_A\left(|T^*|^{2(1-\alpha)}\right)|T^*|^{2(1-\alpha)}y,y\right\rangle \end{aligned} \tag{2.10}$$

for any $x,y\in H$ and $T\in\mathcal{B}(H)$ with $\|T\|^{2\alpha}, \|T\|^{2(1-\alpha)} < R$.

The following result concerning two functions also holds:

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and be $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions defined by power series with real coefficients and both of them convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Let $T\in\mathcal{B}(H)$, $\alpha,\beta\ge 0$ with $\alpha+\beta\ge 1$ and $z,u\in\mathbb{C}$ such that*

$$|z|^2, |u|^2, \|T\|^{2\alpha}, \|T\|^{2\beta} < R, \tag{2.11}$$

then we have the inequality

$$\begin{aligned} &\left|\left\langle Tf(z|T|^{\alpha})\, g\left(u|T|^{\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \\ &\le f_A\left(|z|^2\right) g_A\left(|u|^2\right)\left\langle f_A\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle \left\langle g_A\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle \end{aligned} \tag{2.12}$$

for any $x,y\in H$.

Proof. Since $\alpha,\beta\ge 0$ with $\alpha+\beta\ge 1$, then for any $n,m\ge 1$ natural numbers we also have that $n\alpha+m\beta\ge 1$.

From Furuta's inequality (F) written for $n\alpha+m\beta\ge 1$ we have for any natural numbers $n\ge 1$ and $m\ge 1$ the following power inequality

$$\left|\left\langle T|T|^{n\alpha+m\beta-1}x,y\right\rangle\right| \le \left\langle |T|^{2n\alpha}x,x\right\rangle^{1/2}\left\langle |T^*|^{2m\beta}y,y\right\rangle^{1/2}, \tag{2.13}$$

where $x,y\in H$.

If we multiply this inequality with the positive quantities $|a_{n-1}||z|^{n-1}$ and $|b_{m-1}||u|^{m-1}$, use the triangle inequality and the Cauchy-Bunyakowsky-Schwarz discrete inequality we have successively:

$$\begin{aligned} &\left|\sum_{n=1}^{k}\sum_{m=1}^{l} a_{n-1}z^{n-1}b_{m-1}u^{m-1}\left\langle T|T|^{n\alpha+m\beta-1}x,y\right\rangle\right| \\ &\le \sum_{n=1}^{k}\sum_{m=1}^{l}|a_{n-1}||z|^{n-1}|b_{m-1}||u|^{m-1}\left|\left\langle T|T|^{n\alpha+m\beta-1}x,y\right\rangle\right| \\ &\le \sum_{n=1}^{k}|a_{n-1}||z|^{n-1}\left\langle |T|^{2n\alpha}x,x\right\rangle^{1/2}\sum_{m=1}^{l}|b_{m-1}||u|^{m-1}\left\langle |T^*|^{2m\beta}y,y\right\rangle^{1/2} \\ &\le \left(\sum_{n=1}^{k}|a_{n-1}||z|^{2(n-1)}\right)^{1/2}\left\langle \sum_{n=1}^{k}|a_{n-1}||T|^{2n\alpha}x,x\right\rangle^{1/2} \\ &\times\left(\sum_{m=1}^{l}|b_{m-1}||u|^{2(m-1)}\right)^{1/2}\left\langle \sum_{m=1}^{l}|b_{m-1}||T^*|^{2m\beta}y,y\right\rangle^{1/2} \end{aligned} \tag{2.14}$$

for any $x,y\in H$ and $k\ge 1, l\ge 1$.

Observe also that

$$\sum_{n=1}^{k}\sum_{m=1}^{l}a_{n-1}z^{n-1}b_{m-1}u^{m-1}\left\langle T\left|T\right|^{n\alpha+m\beta-1}x,y\right\rangle = \left\langle T\left(\sum_{n=1}^{k}a_{n-1}z^{n-1}\left|T\right|^{(n-1)\alpha}\right)\left(\sum_{m=1}^{l}b_{m-1}u^{m-1}\left|T\right|^{(m-1)\beta}\right)\left|T\right|^{\alpha+\beta-1}x,y\right\rangle \tag{2.15}$$

for any $x,y\in H$ and $k\geq 1, l\geq 1$.

Making use of (2.14) and (2.15) we get

$$\begin{aligned}&\left|\left\langle T\left(\sum_{n=1}^{k}a_{n-1}z^{n-1}\left|T\right|^{(n-1)\alpha}\right)\left(\sum_{m=1}^{l}b_{m-1}u^{m-1}\left|T\right|^{(m-1)\beta}\right)\left|T\right|^{\alpha+\beta-1}x,y\right\rangle\right|\\&\leq\left(\sum_{n=1}^{k}\left|a_{n-1}\right|\left|z\right|^{2(n-1)}\right)^{1/2}\left\langle\left(\sum_{n=1}^{k}\left|a_{n-1}\right|\left|T\right|^{2(n-1)\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle^{1/2}\\&\times\left(\sum_{m=1}^{l}\left|b_{m-1}\right|\left|u\right|^{2(m-1)}\right)^{1/2}\left\langle\left(\sum_{m=1}^{l}\left|b_{m-1}\right|\left|T^{*}\right|^{2(m-1)\beta}\right)\left|T^{*}\right|^{2\beta}y,y\right\rangle^{1/2}\end{aligned} \tag{2.16}$$

for any $x,y\in H$ and $k\geq 1, l\geq 1$.

From (2.11) we have that the series

$$\sum_{n=0}^{\infty}a_nz^n\left|T\right|^{n\alpha},\sum_{m=0}^{\infty}b_mu^m\left|T\right|^{m\beta},\sum_{n=0}^{\infty}\left|a_n\right|\left|T\right|^{2n\alpha}$$

and

$$\sum_{m=0}^{\infty}\left|b_m\right|\left|T^{*}\right|^{2m\beta}$$

are convergent in $\mathcal{B}(H)$ and the series $\sum_{n=0}^{\infty}\left|a_n\right|\left|z\right|^{2n}$ and $\sum_{m=0}^{\infty}\left|b_m\right|\left|u\right|^{2m}$ are convergent in $\mathbb{R}$ and then, by taking the limit over $k\to\infty$ and $l\to\infty$ in (2.16) we deduce desired result (2.12). □

Remark 3. *The above inequality (2.12) can provide various particular instances of interest.*

For instance, if we take $g=f$ and $z=u$ in Theorem 3 then we get

$$\begin{aligned}&\left|\left\langle Tf\left(z\left|T\right|^{\alpha}\right)f\left(z\left|T\right|^{\beta}\right)\left|T\right|^{\alpha+\beta-1}x,y\right\rangle\right|^{2}\\&\leq f_A^2\left(\left|z\right|^2\right)\left\langle f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle\left\langle f_A\left(\left|T^{*}\right|^{2\beta}\right)\left|T^{*}\right|^{2\beta}y,y\right\rangle\end{aligned} \tag{2.17}$$

for any $x,y\in H$.

Also if we take $f(z)=1$ in (2.17), then we get Furuta's inequality (F).

Corollary 2. *With the assumptions of Theorem 3 we have the norm inequality*

$$\begin{aligned}&\left\|Tf\left(z\left|T\right|^{\alpha}\right)g\left(u\left|T\right|^{\beta}\right)\left|T\right|^{\alpha+\beta-1}\right\|^{2}\\&\leq f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\left\|f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}\right\|\left\|g_A\left(\left|T^{*}\right|^{2\beta}\right)\left|T^{*}\right|^{2\beta}\right\|\end{aligned} \tag{2.18}$$

and the numerical radius inequality

$$w\left(Tf\left(z\left|T\right|^{\alpha}\right)g\left(u\left|T\right|^{\beta}\right)\left|T\right|^{\alpha+\beta-1}\right) \le \frac{1}{2}\left[f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\right]^{1/2}\left\|f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}+g_A\left(\left|T^*\right|^{2\beta}\right)\left|T^*\right|^{2\beta}\right\|. \tag{2.19}$$

Proof. The inequality (2.18) follows from (2.12) by taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, from (2.12) we also have

$$\begin{aligned}&\left|\left\langle Tf\left(z\left|T\right|^{\alpha}\right)g\left(u\left|T\right|^{\beta}\right)\left|T\right|^{\alpha+\beta-1}x,x\right\rangle\right|\\ &\le\left[f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\right]^{1/2}\left\langle f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle^{1/2}\left\langle g_A\left(\left|T^*\right|^{2\beta}\right)\left|T^*\right|^{2\beta}x,x\right\rangle^{1/2}\\ &\le\frac{1}{2}\left[f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\right]^{1/2}\left\langle\left[f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}+g_A\left(\left|T^*\right|^{2\beta}\right)\left|T^*\right|^{2\beta}\right]x,x\right\rangle\end{aligned}$$

for any $x \in H$.

Taking the supremum over $x \in H$ with $\|x\| = 1$ we get the desired result (2.19). □

Remark 4. *If we take $\beta = \alpha$ in (2.12) then we get the inequality*

$$\left|\left\langle Tf\left(z\left|T\right|^{\alpha}\right)g\left(u\left|T\right|^{\alpha}\right)\left|T\right|^{2\alpha-1}x,y\right\rangle\right|^2 \le f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\left\langle f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle\left\langle g_A\left(\left|T^*\right|^{2\alpha}\right)\left|T^*\right|^{2\alpha}y,y\right\rangle \tag{2.20}$$

provided $\alpha \ge \frac{1}{2}$ and

$$\left|z\right|^2, \left|u\right|^2, \left\|T\right\|^{2\alpha} < R.$$

In particular we have

$$\left|\left\langle Tf\left(z\left|T\right|\right)g\left(u\left|T\right|\right)x,y\right\rangle\right|^2 \le f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\left\langle f_A\left(\left|T\right|\right)\left|T\right|x,x\right\rangle\left\langle g_A\left(\left|T^*\right|\right)\left|T^*\right|y,y\right\rangle \tag{2.21}$$

provided

$$\left|z\right|^2, \left|u\right|^2, \left\|T\right\| < R.$$

Remark 5. *If we take $\beta = 1-\alpha$ with $\alpha \in [0,1]$ in (2.12) then we have the inequality*

$$\left|\left\langle Tf\left(z\left|T\right|^{\alpha}\right)g\left(u\left|T\right|^{1-\alpha}\right)x,y\right\rangle\right|^2 \le f_A\left(\left|z\right|^2\right)g_A\left(\left|u\right|^2\right)\left\langle f_A\left(\left|T\right|^{2\alpha}\right)\left|T\right|^{2\alpha}x,x\right\rangle\left\langle g_A\left(\left|T^*\right|^{2(1-\alpha)}\right)\left|T^*\right|^{2(1-\alpha)}y,y\right\rangle \tag{2.22}$$

for any $x, y \in H$, where $z, u \in \mathbb{C}$ and $T \in \mathcal{B}(H)$ such that $\left|z\right|^2, \left|u\right|^2, \left\|T\right\|^{2\alpha}, \left\|T\right\|^{2(1-\alpha)} < R$.

3. Some Examples

As some natural examples that are useful for applications, we can point out that, if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n=\ln\frac{1}{1+z},\ \ z\in D(0,1);\tag{3.1}$$
$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z,\ \ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z,\ \ z\in\mathbb{C};$$
$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z},\ \ z\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(z)=\sum_{n=1}^{\infty}\frac{1}{n!}z^n=\ln\frac{1}{1-z},\ \ z\in D(0,1);\tag{3.2}$$
$$g_A(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ \ z\in\mathbb{C};$$
$$h_A(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ \ z\in\mathbb{C};$$
$$l_A(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ \ z\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n\qquad z\in\mathbb{C},\tag{3.3}$$
$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D(0,1);$$
$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\qquad z\in D(0,1);$$
$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\qquad z\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is the *Gamma function*.

Utilising the inequality (2.2) and the power series representations from (3.1) and (3.2) we have

$$\left|\left\langle T\left(1_H \pm |T|^{\alpha+\beta}\right)^{-1}|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \left\langle\left(1_H - |T|^{2\alpha}\right)^{-1}|T|^{2\alpha}x,x\right\rangle\left\langle\left(1_H - |T^*|^{2\beta}\right)^{-1}|T^*|^{2\beta}y,y\right\rangle \tag{3.4}$$

and

$$\begin{aligned}\left|\left\langle T\ln\left[\left(1_H \pm |T|^{\alpha+\beta}\right)^{-1}\right]|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 &\leq \left\langle \ln\left[\left(1_H - |T|^{2\alpha}\right)^{-1}\right]|T|^{2\alpha}x,x\right\rangle \\ &\times\left\langle \ln\left[\left(1_H - |T^*|^{2\beta}\right)^{-1}\right]|T^*|^{2\beta}y,y\right\rangle\end{aligned} \tag{3.5}$$

for $T \in \mathcal{B}(H)$, $\alpha, \beta \geq 0$ with $\alpha+\beta \geq 1$ and such that $\|T\| < 1$ and for any $x, y \in H$.

From (2.2) we also have the inequalities

$$\left|\left\langle T\sin\left(|T|^{\alpha+\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \left\langle \sinh\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle \sinh\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle, \tag{3.6}$$

$$\left|\left\langle T\cos\left(|T|^{\alpha+\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \left\langle \cosh\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle \cosh\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle, \tag{3.7}$$

and

$$\left|\left\langle T\exp\left(|T|^{\alpha+\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \left\langle \exp\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle \exp\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle \tag{3.8}$$

for $T \in \mathcal{B}(H)$, $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and for any $x, y \in H$.

Utilizing the inequality (2.12) we have

$$\left|\left\langle T\exp\left(z|T|^{\alpha} + u|T|^{\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \leq \exp\left(|z|^2 + |u|^2\right)\left\langle \exp\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle \exp\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle \tag{3.9}$$

and

$$\begin{aligned}&\left|\left\langle T\sin\left(z|T|^{\alpha}\right)\cos\left(u|T|^{\beta}\right)|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2 \\ &\leq \sinh\left(|z|^2\right)\cosh\left(|u|^2\right)\left\langle \sinh\left(|T|^{2\alpha}\right)|T|^{2\alpha}x,x\right\rangle\left\langle \cosh\left(|T^*|^{2\beta}\right)|T^*|^{2\beta}y,y\right\rangle\end{aligned} \tag{3.10}$$

for $T \in \mathcal{B}(H)$, $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and for any $z, u \in \mathbb{C}$, $x, y \in H$.

By the same inequality (2.12) we also get

$$\left|\left\langle T\left(1_{H}\pm z\left|T\right|^{\alpha}\right)^{-1}\left(1_{H}\pm u\left|T\right|^{\beta}\right)^{-1}\left|T\right|^{\alpha+\beta-1}x,y\right\rangle\right|^{2} \leq \frac{\left\langle\left(1_{H}-\left|T\right|^{2\alpha}\right)^{-1}\left|T\right|^{2\alpha}x,x\right\rangle\left\langle\left(1_{H}-\left|T^{*}\right|^{2\beta}\right)^{-1}\left|T^{*}\right|^{2\beta}y,y\right\rangle}{\left(1-\left|z\right|^{2}\right)\left(1-\left|u\right|^{2}\right)} \tag{3.11}$$

for $T\in\mathcal{B}(H)$, $z,u\in\mathbb{C}$ with $\|T\|,|z|,|u|<1,\alpha,\beta\geq 0$ with $\alpha+\beta\geq 1$ and for any, $x,y\in H$.

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