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SOME INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. If $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A on the Hilbert space H and $m=\min Sp(A)$ and $M=\max Sp(A)$, we show that for any continuous function $\varphi:[m,M]\to\mathbb{C}$ we have the inequality

$$\begin{aligned}|\langle\varphi(A)x,y\rangle|^2 &\leq \left(\int_{m-0}^{M}|\varphi(t)|\,d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right)\right)^2\\ &\leq \langle|\varphi(A)|x,x\rangle\langle|\varphi(A)|y,y\rangle\end{aligned}$$

for any vector x and y from H. Some related results and applications are also given.

1. Introduction

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, & \text{for } -\infty<s\leq\lambda,\\ \\ 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{1.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [13, p. 256]:

Theorem 1 (Spectral Representation Theorem)**.** *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda\leq E_{\lambda'}$ *for* $\lambda\leq\lambda'$;
b) $E_{m-0}=0, E_M=I$ *and* $E_{\lambda+0}=E_\lambda$ *for all* $\lambda\in\mathbb{R}$;
c) *We have the representation*

$$A=\int_{m-0}^{M}\lambda dE_\lambda. \tag{1.2}$$

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More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \leq \varepsilon \tag{1.3}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases} \tag{1.4}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{1.5}$$

where the integral is of Riemann-Stieltjes type.

Corollary 1. *With the assumptions of Theorem 1 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H \tag{1.6}$$

and

$$\langle \varphi(A)\, x, y \rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y \rangle \quad \text{for all } x, y \in H. \tag{1.7}$$

In particular,

$$\langle \varphi(A)\, x, x \rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x \rangle \quad \text{for all } x \in H. \tag{1.8}$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H. \tag{1.9}$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 1, see for instance [13, p. 258]:

Theorem 2. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min Sp(A)$ and $M = \max Sp(A)$. If $\{F_\lambda\}_{\lambda \in \mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 1, then $F_\lambda = E_\lambda$ for all $\lambda \in \mathbb{R}$ where E_λ is defined by (1.1).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A. The spectral family also reflects in a direct way the properties of the operator A as follows, see [13, p. 263-p.266]:

Theorem 3. *Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. If B is a bounded linear operator on H, then $AB = BA$ iff $E_\lambda B = BE_\lambda$ for all $\lambda \in \mathbb{R}$. In particular $E_\lambda A = AE_\lambda$ for all $\lambda \in \mathbb{R}$.*

Theorem 4. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and $\mu \in \mathbb{R}$. Then*

a) *μ is a regular value of A, i.e., $A-\mu I$ is invertible iff there exists a $\theta > 0$ such that $E_{\mu-\theta} = E_{\mu+\theta}$;*
b) *$\mu \in Sp(A)$ iff $E_{\mu-\theta} < E_{\mu+\theta}$ for all $\theta > 0$;*
c) *μ is an eigenvalue of A iff $E_{\mu-0} < E_\mu$.*

If P is a positive selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle, \tag{1.10}$$

for any $x, y \in H$.

The following inequality is of interest as well, see [13, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2 \leq \|P\| \langle Px, x\rangle \tag{1.11}$$

for any $x \in H$.

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [13, p. 240]:

Theorem 5. *If the operator $A \in \mathcal{B}(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$. If A is invertible, then so is B.*

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [14] proved the following generalization of Schwarz inequality:

$$|\langle Tx, y\rangle|^2 \leq \langle (T^*T)^\alpha x, x\rangle \left\langle (TT^*)^{1-\alpha} y, y\right\rangle, \tag{1.12}$$

for any $x, y \in H$, $\alpha \in [0,1]$ and T is a bounded linear operator on H.

Utilizing the modulus notation introduced before, we can write (1.12) as follows

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle. \tag{1.13}$$

If we choose in (1.13) $T = A$, a selfadjoint operator, and $\alpha = 1/2$ then we get from (1.13) the following inequality of interest:

$$|\langle Ax, y\rangle|^2 \leq \langle |A| x, x\rangle \langle |A| y, y\rangle \tag{1.14}$$

for any $x, y \in H$.

This result can be improved as follows:

$$|\langle Ax, y\rangle|^2 \leq \left(\int_{m-0}^{M} |t|\, d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x, y\rangle\right)\right)\right)^2 \leq \langle |A| x, x\rangle \langle |A| y, y\rangle \tag{1.15}$$

for any $x, y \in H$, where $m = \min Sp(A)$ and $M = \max Sp(A)$ while $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A. It can be obtained as a particular case of a more general result for complex-valued functions of selfadjoint operators provided in the next section.

For recent inequalities for functions of selfadjoint operators in Hilbert spaces, see the recent papers [1]-[10], the book [12] and the papers [15]-[18].

2. The Results

We can prove the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y \rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$:

Theorem 6. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and let $m = \min Sp(A)$ and $M = \max Sp(A)$. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right]^2 \leq \langle (E_\beta - E_\alpha) x, x\rangle \langle (E_\beta - E_\alpha) y, y\rangle, \tag{2.1}$$

where $\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)$ *denotes the total variation of the function* $\left\langle E_{(\cdot)} x, y\right\rangle$ *on* $[\alpha, \beta]$.

Proof. Now, if $d : \alpha = t_0 < t_1 < ... < t_{n-1} < t_n = \beta$ is an arbitrary partition of the interval $[\alpha, \beta]$, then we have by Schwarz's inequality for positive operators (1.10) that

$$\begin{aligned} &\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \\ &= \sup_d \left\{\sum_{i=0}^{n-1} \left|\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y\right\rangle\right|\right\} \\ &\leq \sup_d \left\{\sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle^{1/2}\right]\right\} := B. \end{aligned} \tag{2.2}$$

By the Cauchy-Bunyakovsky-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned} &\sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle^{1/2}\right] \\ &\leq \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle\right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle\right]^{1/2} \\ &= \left[\langle (E_\beta - E_\alpha) x, x\rangle\right]^{1/2} \left[\langle (E_\beta - E_\alpha) y, y\rangle\right]^{1/2} \end{aligned} \tag{2.3}$$

for any $x, y \in H$. Taking the supremum over d in (2.3) we get

$$B \leq \left[\langle (E_\beta - E_\alpha) x, x\rangle\right]^{1/2} \left[\langle (E_\beta - E_\alpha) y, y\rangle\right]^{1/2}$$

for any $x, y \in H$ which together with (2.2) produce the desired result (2.1). □

Remark 1. *For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (2.1) the inequality*

$$\bigvee_{m-\varepsilon}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \leq \langle (1_H - E_{m-\varepsilon}) x, x\rangle^{1/2} \langle (1_H - E_{m-\varepsilon}) y, y\rangle^{1/2} \tag{2.4}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right) \leq \|x\| \|y\|, \tag{2.5}$$

where $\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ *denotes the limit* $\lim_{\varepsilon\to 0+}\left[\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$.

In order to prove our main results we need the following inequality for the Riemann-Stieltjes integral of continuous integrands and of bounded variation integrators. Since this result plays a key role in the proof of our main result we give here a short proof (see also [11]).

Lemma 1. *Let* $f,u:[a,b]\to\mathbb{C}$*. If* f *is continuous on* $[a,b]$ *and* u *is of bounded variation on* $[a,b]$*, then*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t(u)\right) \le \left[\bigvee_a^b(u)\right]^{\frac{1}{q}}\left\{\int_a^b |f(t)|^p\,d\left(\bigvee_a^t(u)\right)\right\}^{\frac{1}{p}} \le \max_{t\in[a,b]}|f(t)|\bigvee_a^b(u), \tag{2.6}$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then for any sequence of partitions $I_n^{(n)}: a=t_0^{(n)}<t_1^{(n)}<\cdots<t_{n-1}^{(n)}<t_n^{(n)}=b$ with the norm $v\left(I_n^{(n)}\right):=\max_{i\in\{0,\dots,n-1\}}\left(t_{i+1}^{(n)}-t_i^{(n)}\right)\to 0$ as $n\to\infty$, and for any intermediate points $\xi_i^{(n)}\in[t_i^{(n)},t_{i+1}^{(n)}]$, $i\in\{0,\dots,n-1\}$ we have:

$$\left|\int_a^b f(t)\,du(t)\right| = \left|\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[u\left(t_{i+1}^{(n)}\right)-u\left(t_i^{(n)}\right)\right]\right| \le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left|u\left(t_{i+1}^{(n)}\right)-u\left(t_i^{(n)}\right)\right|. \tag{2.7}$$

However,

$$\left|u\left(t_{i+1}^{(n)}\right)-u\left(t_i^{(n)}\right)\right| \le \bigvee_{t_i^{(n)}}^{t_{i+1}^{(n)}}(u) = \bigvee_a^{t_{i+1}^{(n)}}(u)-\bigvee_a^{t_i^{(n)}}(u),$$

for any $i\in\{0,\dots,n-1\}$, and by (2.7) we have

$$\left|\int_a^b f(t)\,du(t)\right| \le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[\bigvee_a^{t_{i+1}^{(n)}}(u)-\bigvee_a^{t_i^{(n)}}(u)\right] = \int_a^b |f(t)|\,d\left(\bigvee_a^t(u)\right),$$

and the last Riemann-Stieltjes integral exists since $|f|$ is continuous and $\bigvee_a^{\cdot}$ is monotonic nondecreasing on $[a,b]$.

The last part follows from the following well known Hölder type inequality for the Riemann-Stieltjes integral with monotonic integrator, namely:

$$\begin{aligned}(2.8)\qquad &\left|\int_a^b g(t)\,dv(t)\right| \\ &\leq [v(b)-v(a)]^{\frac{1}{q}}\left[\int_a^b |g(t)|^p\,dv(t)\right]^{\frac{1}{p}} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ &\leq \max_{t\in[a,b]}|g(t)|\,[v(b)-v(a)],\end{aligned}$$

holding for any g continuous on $[a,b]$ and v monotonic nondecreasing on $[a,b]$.

The details are omitted. □

Theorem 7. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and let $m=\min Sp(A)$ and $M=\max Sp(A)$. Then for any continuous function $\varphi:\mathbb{R}\to\mathbb{C}$ we have the inequality*

$$(2.9)\qquad |\langle\varphi(A)x,y\rangle|^2 \leq \left(\int_{m-0}^M |\varphi(t)|\,d\left(\bigvee_{m-0}^t \left(\langle E_{(\cdot)}x,y\rangle\right)\right)\right)^2$$

$$\leq \langle|\varphi(A)|\,x,x\rangle\,\langle|\varphi(A)|\,y,y\rangle$$

for any $x,y\in H$.

Proof. Fix $x,y\in H$. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. If we apply the first inequality in (2.6) on the interval $[m-\varepsilon,M]$ with $\varepsilon>0$, then we get

$$(2.10)\qquad \left|\int_{m-\varepsilon}^M \varphi(t)\,d\langle E_t x,y\rangle\right| \leq \int_{m-\varepsilon}^M |\varphi(t)|\,d\left(\bigvee_{m-\varepsilon}^t (u)\right)$$

where $u(\lambda):=\langle E_\lambda x,y\rangle$, $\lambda\in\mathbb{R}$.

Since $|\varphi|$ is continuous and u is monotonic nondecreasing then the Riemann-Stieltjes integral in the right hand side of (2.10) exists.

Taking the limit over $\varepsilon\to 0+$ and utilizing the representation (1.7) we get

$$(2.11)\qquad |\langle\varphi(A)x,y\rangle| = \left|\int_{m-0}^M \varphi(t)\,d\langle E_t x,y\rangle\right| \leq \int_{m-0}^M |\varphi(t)|\,d\left(\bigvee_{m-0}^t (u)\right).$$

This proves the first part of (2.9).

Moreover, since the Riemann-Stieltjes integral $\int_{m-0}^M |\varphi(t)|\,d\left(\bigvee_{m-0}^t (u)\right)$ exists, then for any sequence of partitions

$$I_n^{(n)}: t_0^{(n)} < m = t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} = M$$

with the norm $v\left(I_n^{(n)}\right) := \max_{i\in\{0,\dots,n-1\}}\left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$ as $n \to \infty$, and for any intermediate points $\xi_i^{(n)} \in [t_i^{(n)}, t_{i+1}^{(n)}]$, $i \in \{0,\dots,n-1\}$ we have:

$$
\begin{aligned}
(2.12)\qquad & \int_{m-0}^{M} |\varphi(t)|\, d\left(\bigvee_{m-0}^{t}(u)\right) \\
&= \lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} \left|\varphi\left(\xi_i^{(n)}\right)\right| \left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right] \\
&= \lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} \left|\varphi\left(\xi_i^{(n)}\right)\right| \bigvee_{t_i^{(n)}}^{t_{i+1}^{(n)}} \left(\langle E_{(\cdot)}x, y\rangle\right) \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} \left|\varphi\left(\xi_i^{(n)}\right)\right| \\
&\times \left[\left\langle \left(E_{t_{i+1}^{(n)}} - E_{t_i^{(n)}}\right)x, x\right\rangle \left\langle \left(E_{t_{i+1}^{(n)}} - E_{t_i^{(n)}}\right)y, y\right\rangle\right]^{1/2} \\
&:= T(x, y),
\end{aligned}
$$

where for the last inequality we have used (2.1) applied on the interval $[t_i^{(n)}, t_{i+1}^{(n)}]$, $i \in \{0,\dots,n-1\}$.

Now, we use the following Cauchy-Buniakowski-Schwarz's weighted discrete inequality

$$
\sum_{k=1}^{p} s_i a_i b_i \le \left(\sum_{k=1}^{p} s_i a_i^2\right)^{1/2} \left(\sum_{k=1}^{p} s_i b_i^2\right)^{1/2},
$$

where $s_i, a_i, b_i \ge 0$, to write that

$$
\begin{aligned}
(2.13)\qquad & \sum_{i=0}^{n-1} \left|\varphi\left(\xi_i^{(n)}\right)\right| \left[\left\langle \left(E_{t_{i+1}^{(n)}} - E_{t_i^{(n)}}\right)x, x\right\rangle \left\langle \left(E_{t_{i+1}^{(n)}} - E_{t_i^{(n)}}\right)y, y\right\rangle\right]^{1/2} \\
&\le \left[\sum_{i=0}^{n-1} \left|\varphi\left(\xi_i^{(n)}\right)\right| \left\langle \left(E_{t_{i+1}^{(n)}} - E_{t_i^{(n)}}\right)x, x\right\rangle\right]^{1/2} \\
&\times \left[\sum_{i=0}^{n-1} \left|\varphi\left(\xi_i^{(n)}\right)\right| \left\langle \left(E_{t_{i+1}^{(n)}} - E_{t_i^{(n)}}\right)x, x\right\rangle\right]^{1/2}.
\end{aligned}
$$

Taking the limit for $v\left(I_n^{(n)}\right) \to 0$ in (2.13) we get

$$T(x,y) \leq \left[\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|\varphi\left(\xi_i^{(n)}\right)\right|\left\langle\left(E_{t_{i+1}^{(n)}}-E_{t_i^{(n)}}\right)x,x\right\rangle\right]^{1/2} \tag{2.14}$$
$$\times\left[\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|\varphi\left(\xi_i^{(n)}\right)\right|\left\langle\left(E_{t_{i+1}^{(n)}}-E_{t_i^{(n)}}\right)x,x\right\rangle\right]^{1/2}$$
$$=\left[\int_{m-0}^{M}|\varphi(t)|\,d\langle E_t x,x\rangle\right]^{1/2}\left[\int_{m-0}^{M}|\varphi(t)|\,d\langle E_t y,y\rangle\right]^{1/2}$$
$$=\langle|\varphi(A)|\,x,x\rangle^{1/2}\langle|\varphi(A)|\,y,y\rangle^{1/2}.$$

On making use of (2.11), (2.12) and (2.14) we deduce the desired result (2.9). □

Remark 2. *If we take $\varphi(t)=t$ above, then we obtain the result announced in the introduction, namely the inequality (1.12).*

Corollary 2. *Let A be a selfadjoint operator on H. Then for any $x,y\in H$ we have the inequalities*

$$|\langle A^n x,y\rangle|^2 \leq \left(\int_{m-0}^{M}|t|^n\,d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right)\right)^2 \leq \langle|A|^n x,x\rangle\langle|A|^n y,y\rangle, \tag{2.15}$$

where $n\geq 1$ is a natural number, and

$$\left|\left\langle e^{iA}x,y\right\rangle\right| \leq \bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right) \leq \|x\|\,\|y\|. \tag{2.16}$$

If A is a positive definite operator on H then we have the logarithmic inequalities

$$|\langle \ln Ax,y\rangle|^2 \leq \left(\int_{m-0}^{M}|\ln t|\,d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right)\right)^2 \leq \langle|\ln A|\,x,x\rangle\langle|\ln A|\,y,y\rangle \tag{2.17}$$

and

$$|\langle A\ln Ax,y\rangle|^2 \leq \left(\int_{m-0}^{M}|t\ln t|\,d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right)\right)^2 \leq \langle|A\ln A|\,x,x\rangle\langle|A\ln A|\,y,y\rangle \tag{2.18}$$

for any $x,y\in H$.

Theorem 8. *With the assumptions in Theorem 7 we also have the inequality*

$$\begin{aligned}(2.19)\qquad & |\langle\varphi(A)x,y\rangle| \\ &\le\left[\bigvee_{m-0}^{M}(\langle E_{(\cdot)}x,y\rangle)\right]^{\frac{1}{q}}\left\{\int_{m-0}^{M}|\varphi(t)|^{p}d\left(\bigvee_{m-0}^{t}(\langle E_{(\cdot)}x,y\rangle)\right)\right\}^{\frac{1}{p}} \\ &\le\left[\bigvee_{m-0}^{M}(\langle E_{(\cdot)}x,y\rangle)\right]^{\frac{1}{q}}\left[\langle|\varphi(A)|^{p}x,x\rangle\right]^{\frac{1}{2p}}\left[\langle|\varphi(A)|^{p}y,y\rangle\right]^{\frac{1}{2p}} \\ &\le\left[\|x\|\|y\|\right]^{1/q}\left[\langle|\varphi(A)|^{p}x,x\rangle\right]^{\frac{1}{2p}}\left[\langle|\varphi(A)|^{p}y,y\rangle\right]^{\frac{1}{2p}}\end{aligned}$$

for any $x,y\in H$, where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Fix $x,y\in H$. If we apply the second inequality in (2.6) we get in a similar manner

$$\begin{aligned}(2.20)\qquad |\langle\varphi(A)x,y\rangle| &=\left|\int_{m-0}^{M}\varphi(t)\,d\langle E_{t}x,y\rangle\right| \\ &\le\left[\bigvee_{m-0}^{M}(u)\right]^{\frac{1}{q}}\left\{\int_{m-0}^{M}|\varphi(t)|^{p}d\left(\bigvee_{m-0}^{t}(u)\right)\right\}^{\frac{1}{p}}\end{aligned}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, $u(\lambda):=\langle E_{\lambda}x,y\rangle$, $\lambda\in\mathbb{R}$ and $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A.

Since $|\varphi|$ is continuous and u is monotonic nondecreasing then the Riemann-Stieltjes integral in the right hand side of (2.20) exists.

With a similar procedure as above we have

$$\begin{aligned}(2.21)\qquad & \int_{m-0}^{M}|\varphi(t)|^{p}d\left(\bigvee_{m-0}^{t}(u)\right) \\ &\le\left[\int_{m-0}^{M}|\varphi(t)|^{p}d\langle E_{t}x,x\rangle\right]^{1/2}\left[\int_{m-0}^{M}|\varphi(t)|^{p}d\langle E_{t}y,y\rangle\right]^{1/2} \\ &=\left[\langle|\varphi(A)|^{p}x,x\rangle\right]^{1/2}\left[\langle|\varphi(A)|^{p}y,y\rangle\right]^{1/2}\end{aligned}$$

which together with (2.20) produces the desired result (2.19). □

Remark 3. *If A is a selfadjoint operator on H, then we have the inequality*

$$\begin{aligned}(2.22)\qquad & |\langle Ax,y\rangle| \\ &\le\left[\bigvee_{m-0}^{M}(\langle E_{(\cdot)}x,y\rangle)\right]^{\frac{1}{q}}\left\{\int_{m-0}^{M}|t|^{p}d\left(\bigvee_{m-0}^{t}(\langle E_{(\cdot)}x,y\rangle)\right)\right\}^{\frac{1}{p}} \\ &\le\left[\bigvee_{m-0}^{M}(\langle E_{(\cdot)}x,y\rangle)\right]^{\frac{1}{q}}\left[\langle|A|^{p}x,x\rangle\right]^{\frac{1}{2p}}\left[\langle|A|^{p}y,y\rangle\right]^{\frac{1}{2p}} \\ &\le\left[\|x\|\|y\|\right]^{1/q}\left[\langle|A|^{p}x,x\rangle\right]^{\frac{1}{2p}}\left[\langle|A|^{p}y,y\rangle\right]^{\frac{1}{2p}}\end{aligned}$$

for any $x,y\in H$, where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

3. Some Applications

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} | \operatorname{Re}\left[(\Gamma - f(t)) \left(\overline{f(t)} - \overline{\gamma} \right) \right] \geq 0 \text{ for each } t \in [a,b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} | \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for each } t \in [a,b] \right\}.$$

The following representation result may be stated.

Lemma 2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2 - \left| z - \frac{\gamma+\Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma,\Gamma) = \{ f : [a,b] \to \mathbb{C} \mid & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for each } t \in [a,b] \}. \end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma,\Gamma) := \{ f : [a,b] \to \mathbb{C} \mid & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b] \}. \end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{3.4}$$

Proposition 1. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m =: \min Sp(A)$ and $M =: \max Sp(A)$. If the continuous function $\varphi : \mathbb{R} \to \mathbb{C}$*

belongs to the class $\bar{U}_{[m,M]}(\gamma,\Gamma)$ *for some* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *then we have the inequalities*

$$\left|\langle\varphi(A)x,y\rangle-\frac{\gamma+\Gamma}{2}\langle x,y\rangle\right| \le \int_{m-0}^{M}\left|\varphi(t)-\frac{\gamma+\Gamma}{2}\right|d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right) \le \left\langle\left|\varphi(A)-\frac{\gamma+\Gamma}{2}1_H\right|x,x\right\rangle^{1/2}\left\langle\left|\varphi(A)-\frac{\gamma+\Gamma}{2}1_H\right|y,y\right\rangle^{1/2} \le \frac{1}{2}|\Gamma-\gamma|\,\|x\|\,\|y\| \tag{3.5}$$

for any $x,y\in H$.

Alternatively, we have the inequalities

$$\left|\langle\varphi(A)x,y\rangle-\frac{\gamma+\Gamma}{2}\langle x,y\rangle\right| \le \int_{m-0}^{M}\left|\varphi(t)-\frac{\gamma+\Gamma}{2}\right|d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right) \le \frac{1}{2}|\Gamma-\gamma|\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\le\frac{1}{2}|\Gamma-\gamma|\,\|x\|\,\|y\| \tag{3.6}$$

for any $x,y\in H$.

The proof follows by Theorem 7 and by Lemma 2 and the details are omitted.

Now, if we choose for instance $\varphi(t)=t^p, t\ge0$ with $p>0$, then for any positive operator A with the property that $0\le m1_H\le A\le M1_H$ we get from (3.5) the following inequalities

$$\left|\langle A^p x,y\rangle-\frac{m^p+M^p}{2}\langle x,y\rangle\right| \le \int_{m-0}^{M}\left|t^p-\frac{m^p+M^p}{2}\right|d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right) \le \left\langle\left|A^p-\frac{m^p+M^p}{2}1_H\right|x,x\right\rangle^{1/2}\left\langle\left|A^p-\frac{m^p+M^p}{2}1_H\right|y,y\right\rangle^{1/2} \le \frac{1}{2}(M^p-m^p)\|x\|\,\|y\| \tag{3.7}$$

and

$$\left|\langle A^p x,y\rangle-\frac{m^p+M^p}{2}\langle x,y\rangle\right| \le \int_{m-0}^{M}\left|t^p-\frac{m^p+M^p}{2}\right|d\left(\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right) \le \frac{1}{2}(M^p-m^p)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\le\frac{1}{2}(M^p-m^p)\|x\|\,\|y\| \tag{3.8}$$

for any $x,y\in H$.

If $r > 0$ and $m > 0$ then we also have

$$\begin{aligned}
(3.9)\qquad & \left|\left\langle A^{-r}x, y\right\rangle - \frac{m^r + M^r}{2m^r M^r}\langle x, y\rangle\right| \\
& \leq \int_{m-0}^{M}\left|t^{-r} - \frac{m^r + M^r}{2m^r M^r}\right| d\left(\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right) \\
& \leq \left\langle\left|A^{-r} - \frac{m^r + M^r}{2m^r M^r}1_H\right| x, x\right\rangle^{1/2}\left\langle\left|A^{-r} - \frac{m^r + M^r}{2m^r M^r}1_H\right| y, y\right\rangle^{1/2} \\
& \leq \frac{1}{2}\frac{M^r - m^r}{M^r m^r}\|x\|\|y\|
\end{aligned}$$

and

$$\begin{aligned}
(3.10)\qquad & \left|\left\langle A^{-r}x, y\right\rangle - \frac{m^r + M^r}{2m^r M^r}\langle x, y\rangle\right| \\
& \leq \int_{m-0}^{M}\left|t^{-r} - \frac{m^r + M^r}{2m^r M^r}\right| d\left(\bigvee_{m-0}^{t}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right) \\
& \leq \frac{1}{2}\frac{M^r - m^r}{M^r m^r}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{1}{2}\frac{M^r - m^r}{M^r m^r}\|x\|\|y\|
\end{aligned}$$

for any $x, y \in H$.

The inequality in (2.9) can be useful to derive other upper bounds for the quantity $|\langle\varphi(A)x, y\rangle|$ when, for instance, a convexity property for the $|\varphi|$ is assumed:

Proposition 2. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m =: \min Sp(A)$ and $M =: \max Sp(A)$. If the continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ has the property that $|\varphi|$ is convex, then we have the inequality*

$$\begin{aligned}
(3.11)\qquad & |\langle\varphi(A)x, y\rangle| \\
& \leq \frac{1}{M - m}\Big[|\varphi(M)|\langle(A - 1_H m)x, x\rangle^{1/2}\langle(A - 1_H m)y, y\rangle^{1/2} \\
& + |\varphi(m)|\langle(1_H M - A)x, x\rangle^{1/2}\langle(1_H M - A)y, y\rangle^{1/2}\Big] \\
& \leq \left\langle\frac{|\varphi(M)|(A - 1_H m) + |\varphi(m)|(1_H M - A)}{M - m}x, x\right\rangle^{1/2} \\
& \times\left\langle\frac{|\varphi(M)|(A - 1_H m) + |\varphi(m)|(1_H M - A)}{M - m}y, y\right\rangle^{1/2}
\end{aligned}$$

for any $x, y \in H$.

Proof. Fix $x, y \in H$. Since $|\varphi|$ is continuous convex on $[m - \varepsilon, M]$, with $\varepsilon > 0$, then we have

$$|\varphi(t)| \leq \frac{(t - m + \varepsilon)|\varphi(M)| + (M - t)|\varphi(m - \varepsilon)|}{M - m + \varepsilon}$$

for any $t \in [m - \varepsilon, M]$.

Integrating on $[m-\varepsilon, M]$ we get for $\varepsilon > 0$ that

$$\begin{aligned}\int_{m-\varepsilon}^{M} |\varphi(t)|\, d\left(\bigvee_{m-\varepsilon}^{t}(u)\right) &\le \frac{1}{M-m+\varepsilon}\left[|\varphi(M)| \int_{m-\varepsilon}^{M} (t-m+\varepsilon)\, d\left(\bigvee_{m-\varepsilon}^{t}(u)\right)\right. \\ &\quad \left. + |\varphi(m-\varepsilon)| \int_{m-\varepsilon}^{M} (M-t)\, d\left(\bigvee_{m-\varepsilon}^{t}(u)\right)\right] \end{aligned} \tag{3.12}$$

where $u(\lambda) := \langle E_\lambda x, y\rangle$, $\lambda \in \mathbb{R}$ and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A.

Taking the limit over $\varepsilon \to 0+$ we get

$$\begin{aligned}\int_{m-0}^{M} |\varphi(t)|\, d\left(\bigvee_{m-0}^{t}(u)\right) &\le \frac{1}{M-m}\left[|\varphi(M)| \int_{m-0}^{M} (t-m)\, d\left(\bigvee_{m-0}^{t}(u)\right)\right. \\ &\quad \left. + |\varphi(m)| \int_{m-0}^{M} (M-t)\, d\left(\bigvee_{m-0}^{t}(u)\right)\right]. \end{aligned} \tag{3.13}$$

Now, observe that by the second inequality in (2.9) we have

$$\int_{m-0}^{M} (t-m)\, d\left(\bigvee_{m-0}^{t}(u)\right) \le \langle (A-1_H m)x, x\rangle^{1/2} \langle (A-1_H m)y, y\rangle^{1/2} \tag{3.14}$$

and

$$\int_{m-0}^{M} (M-t)\, d\left(\bigvee_{m-0}^{t}(u)\right) \le \langle (1_H M - A)x, x\rangle^{1/2} \langle (1_H M - A)y, y\rangle^{1/2} \tag{3.15}$$

which together with (3.13) produce

$$\begin{aligned}\int_{m-0}^{M} |\varphi(t)|\, d\left(\bigvee_{m-0}^{t}(u)\right) &\le \frac{1}{M-m}\left[|\varphi(M)| \langle (A-1_H m)x, x\rangle^{1/2} \langle (A-1_H m)y, y\rangle^{1/2}\right. \\ &\quad \left. + |\varphi(m)| \langle (1_H M - A)x, x\rangle^{1/2} \langle (1_H M - A)y, y\rangle^{1/2}\right]. \end{aligned} \tag{3.16}$$

Further, from the first inequality in (2.9) and the inequality (3.16) we deduce the first inequality in (3.11).

Now, if we employ the elementary Cauchy-Bunyakovsky-Schwarz's weighted discrete inequality

$$\sum_{k=1}^{2} s_i a_i b_i \le \left(\sum_{k=1}^{2} s_i a_i^2\right)^{1/2} \left(\sum_{k=1}^{2} s_i b_i^2\right)^{1/2}$$

where $s_i, a_i, b_i \geq 0, i = 1, 2$ we can write that

$$\begin{aligned}
&\left[|\varphi(M)| \langle (A - 1_H m) x, x\rangle^{1/2} \langle (A - 1_H m) y, y\rangle^{1/2}\right.\\
&\left.+ |\varphi(m)| \langle (1_H M - A) x, x\rangle^{1/2} \langle (1_H M - A) y, y\rangle^{1/2}\right]\\
&\leq [|\varphi(M)| \langle (A - 1_H m) x, x\rangle + |\varphi(m)| \langle (1_H M - A) x, x\rangle]^{1/2}\\
&\times [|\varphi(M)| \langle (A - 1_H m) y, y\rangle + |\varphi(m)| \langle (1_H M - A) y, y\rangle]^{1/2}\\
&= [\langle [|\varphi(M)| (A - 1_H m) + |\varphi(m)| (1_H M - A)] x, x\rangle]^{1/2}\\
&\times [\langle [|\varphi(M)| (A - 1_H m) + |\varphi(m)| (1_H M - A)] y, y\rangle]^{1/2}
\end{aligned}$$

which proves the second part of (3.11). □

Now, if we choose $\varphi(t) = t^n$, with n a natural number, then for any A a selfadjoint operator on H and for any $x, y \in H$ we have the inequalities

$$\begin{aligned}
&|\langle A^n x, y\rangle| \\
&\leq \frac{1}{M-m}\left[|M|^n \langle (A - 1_H m) x, x\rangle^{1/2} \langle (A - 1_H m) y, y\rangle^{1/2}\right.\\
&\left.+ |m|^n \langle (1_H M - A) x, x\rangle^{1/2} \langle (1_H M - A) y, y\rangle^{1/2}\right]
\end{aligned} \tag{3.17}$$

where $m =: \min Sp(A)$ and $M =: \max Sp(A)$.

Since the function $\varphi(t) = t^p$ is convex on $[0, \infty)$ for $p \geq 1$, then for a selfadjoint operator A with $0 \leq m1_H \leq A \leq M1_H$ we have the inequality

$$\begin{aligned}
&|\langle A^p x, y\rangle| \\
&\leq \frac{1}{M-m}\left[M^p \langle (A - 1_H m) x, x\rangle^{1/2} \langle (A - 1_H m) y, y\rangle^{1/2}\right.\\
&\left.+ m^p \langle (1_H M - A) x, x\rangle^{1/2} \langle (1_H M - A) y, y\rangle^{1/2}\right]
\end{aligned} \tag{3.18}$$

for any $x, y \in H$.

Now, if we use (3.18) for a positive operator $A = P$ and take $m = 0$ and $M = \|P\|$, then we get the inequality

$$|\langle P^p x, y\rangle| \leq \|P\|^{p-1} \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} \tag{3.19}$$

for any $x, y \in H$.

In particular we have

$$\left|\left\langle P^2 x, y\right\rangle\right| \leq \|P\| \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} \tag{3.20}$$

for any $x, y \in H$.

Since the function $\varphi(t) = \ln t$ is convex in absolute value for $t \in (0, 1]$, then for a selfadjoint operator A with $0 < m1_H \leq A \leq M1_H \leq 1_H$ we have the inequality

$$\begin{aligned}
&|\langle \ln A x, y\rangle| \\
&\leq \frac{1}{M-m}\left[|\ln(M)| \langle (A - 1_H m) x, x\rangle^{1/2} \langle (A - 1_H m) y, y\rangle^{1/2}\right.\\
&\left.+ |\ln(m)| \langle (1_H M - A) x, x\rangle^{1/2} \langle (1_H M - A) y, y\rangle^{1/2}\right]
\end{aligned} \tag{3.21}$$

for any $x, y \in H$.

The following lower bounds for the quantity $\langle|\varphi(A)|x,x\rangle^{1/2}\langle|\varphi(A)|y,y\rangle^{1/2}$ when a concavity condition for the $|\varphi|$ is assumed can be stated as well:

Proposition 3. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m =: \min Sp(A)$ and $M =: \max Sp(A)$. If the continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ has the property that $|\varphi|$ is concave, then we have the inequality*

$$\begin{aligned}\langle|\varphi(A)|x,x\rangle^{1/2}&\langle|\varphi(A)|y,y\rangle^{1/2} \\ &\geq \frac{1}{M-m}\left[|\varphi(M)|\,|\langle(A-m1_H)x,y\rangle| + |\varphi(m)|\,|\langle(M1_H-A)x,y\rangle|\right] \\ &\geq \left|\left\langle\left[\frac{\varphi(M)(A-m1_H)+\varphi(m)(M1_H-A)}{M-m}\right]x,y\right\rangle\right| \end{aligned} \tag{3.22}$$

for any $x, y \in H$.

Proof. Fix $x, y \in H$. By the concavity of $|\varphi|$ and utilizing a similar argument to the one from Proposition 2 we have

$$\begin{aligned}\int_{m-0}^{M}&|\varphi(t)|\,d\left(\bigvee_{m-0}^{t}(u)\right) \\ &\geq \frac{1}{M-m}\left[|\varphi(M)|\int_{m-0}^{M}(t-m)\,d\left(\bigvee_{m-0}^{t}(u)\right)\right. \\ &\left.+|\varphi(m)|\int_{m-0}^{M}(M-t)\,d\left(\bigvee_{m-0}^{t}(u)\right)\right],\end{aligned} \tag{3.23}$$

where $u(\lambda) := \langle E_\lambda x, y\rangle$, $\lambda \in \mathbb{R}$ and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A.

On making use of the first inequality in (2.9) we have

$$\int_{m-0}^{M}(t-m)\,d\left(\bigvee_{m-0}^{t}(u)\right) \geq |\langle(A-m1_H)x,y\rangle|$$

and

$$\int_{m-0}^{M}(M-t)\,d\left(\bigvee_{m-0}^{t}(u)\right) \geq |\langle(M1_H-A)x,y\rangle|$$

which together with (3.23) produce

$$\begin{aligned}\int_{m-0}^{M}&|\varphi(t)|\,d\left(\bigvee_{m-0}^{t}(u)\right) \\ &\geq \frac{1}{M-m}\left[|\varphi(M)|\,|\langle(A-m1_H)x,y\rangle| + |\varphi(m)|\,|\langle(M1_H-A)x,y\rangle|\right].\end{aligned} \tag{3.24}$$

Finally, by the second inequality in (2.9) and by (3.24) we deduce the first inequality in (3.22). The second inequality is obvious by the triangle inequality for modulus. □

Since the function $\varphi(t) = t^q$ with $q \in (0,1)$ is concave on the interval $[0,\infty)$, then for a selfadjoint operator A with $0 \le m1_H \le A \le M1_H$ we have the inequality

$$\langle A^q x, x\rangle^{1/2} \langle A^q y, y\rangle^{1/2} \ge \frac{M^q |\langle (A - m1_H) x, y\rangle| + m^q |\langle (M1_H - A) x, y\rangle|}{M - m} \tag{3.25}$$

for any $x, y \in H$.

Now, if we use (3.25) for a positive operator $A = P$ and take $m = 0$ and $M = \|P\|$, then we get the inequality

$$|\langle Px, y\rangle| \le \|P\|^{1-q} \langle P^q x, x\rangle^{1/2} \langle P^q y, y\rangle^{1/2} \tag{3.26}$$

for any $x, y \in H$, where $q \in [0,1]$.

In particular, we have

$$|\langle Px, y\rangle| \le \|P\|^{1/2} \left\langle P^{1/2} x, x\right\rangle^{1/2} \left\langle P^{1/2} y, y\right\rangle^{1/2} \tag{3.27}$$

for any $x, y \in H$.

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