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SOME INEQUALITIES OF FURUTA'S TYPE FOR FUNCTIONS OF OPERATORS DEFINED BY POWER SERIES

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Abstract. Generalizations of Kato and Furuta inequalities for power series of bounded linear operators in Hilbert spaces are given. Applications for normal operators and some functions of interest such as the exponential, hyperbolic and trigonometric functions are provided as well.

1. Introduction

In the following we denote by $\mathcal{B}(H)$ the *Banach algebra* of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$.

If P is a *positive* selfadjoint operator on H, i.e. $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle, \tag{1.1}$$

for any $x,y\in H$.

The following inequality concerning the norm of a positive operator is of interest as well, see [13, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2\leq\|P\|\langle Px,x\rangle \tag{1.2}$$

for any $x\in H$.

The *"square root"* of a positive selfadjoint operator on H can be defined as follows, see for instance [13, p. 240]: *If the operator $A\in B(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B:=\sqrt{A}\in B(H)$ such that $B^2=A$. If A is invertible, then so is B.*

If $A\in\mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A|:=\sqrt{A^*A}$.

In 1952, Kato [14] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator T on H:

$$|\langle Tx,y\rangle|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2(1-\alpha)}y,y\right\rangle \tag{K}$$

for any $x,y\in H$ and $\alpha\in[0,1]$.

In order to generalize this result, in 1994 Furuta [12] obtained the following result:

$$\left|\left\langle T|T|^{\alpha+\beta-1}x,y\right\rangle\right|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2\beta}y,y\right\rangle \tag{F}$$

for any $x,y\in H$ and $\alpha,\beta\in[0,1]$ with $\alpha+\beta\geq 1$.

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If one analyses the proof from [12], that one realizes that the condition $\alpha, \beta \in [0,1]$ is taken only to fit with the result from the *Heinz-Kato inequality*

$$|\langle Tx, y\rangle| \leq \|A^{\alpha}x\| \left\|B^{1-\alpha}y\right\| \tag{HK}$$

for any $x, y \in H$ and $\alpha \in [0,1]$ where A and B are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^{*}y\| \leq \|By\|$ for all $x, y \in H$.

Therefore, one can state the more general result:

Theorem 1 (Furuta Inequality, 1994, [12])**.** *Let $T \in B(H)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$. Then for any $x, y \in H$ we have the inequality (F).*

If we take $\beta = \alpha$, then we get

$$\left|\left\langle T|T|^{2\alpha-1}x, y\right\rangle\right|^{2} \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T^{*}|^{2\alpha}y, y\right\rangle \tag{1.3}$$

for any $x, y \in H$ and $\alpha \geq \frac{1}{2}$. In particular, for $\alpha = 1$ we get

$$|\langle T|T|x, y\rangle|^{2} \leq \left\langle |T|^{2}x, x\right\rangle \left\langle |T^{*}|^{2}y, y\right\rangle \tag{1.4}$$

for any $x, y \in H$.

If we take $T = N$ a *normal operator*, i.e., we recall that $NN^{*} = N^{*}N$, then we get from (F) the following inequality for normal operators

$$\left|\left\langle N|N|^{\alpha+\beta-1}x, y\right\rangle\right|^{2} \leq \left\langle |N|^{2\alpha}x, x\right\rangle \left\langle |N|^{2\beta}y, y\right\rangle \tag{1.5}$$

for any $x, y \in H$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$.

This implies the inequalities

$$\left|\left\langle N|N|^{2\alpha-1}x, y\right\rangle\right|^{2} \leq \left\langle |N|^{2\alpha}x, x\right\rangle \left\langle |N|^{2\alpha}y, y\right\rangle \tag{1.6}$$

for any $x, y \in H$ and $\alpha \geq \frac{1}{2}$ and, in particular,

$$|\langle N|N|x, y\rangle|^{2} \leq \left\langle |N|^{2}x, x\right\rangle \left\langle |N|^{2}y, y\right\rangle \tag{1.7}$$

for any $x, y \in H$.

Making $y = x$ in (1.6) produces

$$\left|\left\langle N|N|^{2\alpha-1}x, x\right\rangle\right| \leq \left\langle |N|^{2\alpha}x, x\right\rangle$$

for any $x \in H$ and $\alpha \geq \frac{1}{2}$ and, in particular,

$$|\langle N|N|x, x\rangle| \leq \left\langle |N|^{2}x, x\right\rangle$$

for any $x \in H$.

If we take $\beta = 1 - \alpha$ with $\alpha \in [0,1]$ in (1.5), then we get

$$|\langle Nx, y\rangle|^{2} \leq \left\langle |N|^{2\alpha}x, x\right\rangle \left\langle |N|^{2(1-\alpha)}y, y\right\rangle \tag{1.8}$$

for any $x, y \in H$.

We can state the following corollary of Furuta's inequality for the numerical radius w of an operator $V \in B(H)$, namely $w(V) = \sup_{\|x\|=1} |\langle Vx, x\rangle|$, which satisfies the following basic inequalities

$$\frac{1}{2}\|V\| \leq w(V) \leq \|V\|.$$

Corollary 1. *Let $T \in B(H)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$. Then we have*

$$w\left(T|T|^{\alpha+\beta-1}\right) \leq \frac{1}{2}\left\||T|^{2\alpha} + |T^*|^{2\beta}\right\|. \tag{1.9}$$

In particular, we also have

$$w\left(T|T|^{2\alpha-1}\right) \leq \frac{1}{2}\left\||T|^{2\alpha} + |T^*|^{2\alpha}\right\|, \tag{1.10}$$

for any $\alpha \geq \frac{1}{2}$ and, as a special case,

$$w(T|T|) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.11}$$

Proof. We have from (F) for any $x \in H$ that

$$\begin{aligned} \left|\left\langle T|T|^{\alpha+\beta-1}x, x\right\rangle\right| &\leq \left\langle |T|^{2\alpha}x, x\right\rangle^{1/2}\left\langle |T^*|^{2\beta}x, x\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left\langle\left[|T|^{2\alpha} + |T^*|^{2\beta}\right]x, x\right\rangle \end{aligned} \tag{1.12}$$

where $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$.

Utilising the inequality in (1.12) and taking the supremum over $x \in H, \|x\| = 1$ we get

$$\begin{aligned} w\left(T|T|^{\alpha+\beta-1}\right) &= \sup_{\|x\|=1}\left|\left\langle T|T|^{\alpha+\beta-1}x, x\right\rangle\right| \\ &\leq \frac{1}{2}\sup_{\|x\|=1}\left\langle\left[|T|^{2\alpha} + |T^*|^{2\beta}\right]x, x\right\rangle \\ &= \frac{1}{2}\left\||T|^{2\alpha} + |T^*|^{2\beta}\right\|. \end{aligned}$$

□

For various interesting generalizations, extension of Kato and Furuta inequalities, see the papers [3]-[12], [17]-[21] and [23].

Motivated by the above results, we establish in this paper some generalizations of Kato and Furuta inequalities for functions of operators that can be expresses as power series with real coefficients. Applications for some functions of interest such as the exponential, hyperbolic and trigonometric functions are provided as well.

2. Functional Inequalities

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_A = f$.

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and be $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions defined by power series with real coefficients and both of them convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If T is a bounded linear operator on the Hilbert space H and $z, u \in \mathbb{C}$ with the property that*

$$|z|^2, |u|^2, \|T\|^2 < R, \tag{2.1}$$

then we have the inequality

$$
\begin{aligned}
(2.2)\qquad & \left|\langle Tf(z|T|)\,g(u|T|)\,x,y\rangle\right|^2 \\
& \leq f_A\left(|z|^2\right) g_A\left(|u|^2\right)\left\langle f_A\left(|T|^2\right)x,x\right\rangle\left\langle |T^*|^2 g_A\left(|T^*|^2\right)y,y\right\rangle
\end{aligned}
$$

for any $x,y\in H$.

Proof. From Furuta's inequality (F) we have for any natural numbers $n\geq 0$ and $m\geq 1$ the following power inequality

$$
(2.3)\qquad \left|\left\langle T|T|^{n+m-1}x,y\right\rangle\right| \leq \left\langle |T|^{2n}x,x\right\rangle^{1/2}\left\langle |T^*|^{2m}y,y\right\rangle^{1/2},
$$

where $x,y\in H$.

If we multiply this inequality with the positive quantities $|a_n|\,|z|^n$ and $|b_{m-1}|\,|u|^{m-1}$, use the triangle inequality and the Cauchy-Bunyakowsky-Schwarz discrete inequality we have successively:

$$
\begin{aligned}
(2.4)\qquad & \left|\sum_{n=0}^{k}\sum_{m=1}^{l} a_n z^n b_{m-1}u^{m-1}\left\langle T|T|^{n+m-1}x,y\right\rangle\right| \\
& \leq \sum_{n=0}^{k}\sum_{m=1}^{l}|a_n|\,|z|^n\,|b_{m-1}|\,|u|^{m-1}\left|\left\langle T|T|^{n+m-1}x,y\right\rangle\right| \\
& \leq \sum_{n=0}^{k}|a_n|\,|z|^n\left\langle |T|^{2n}x,x\right\rangle^{1/2}\sum_{m=1}^{l}|b_{m-1}|\,|u|^{m-1}\left\langle |T^*|^{2m}y,y\right\rangle^{1/2} \\
& \leq \left(\sum_{n=0}^{k}|a_n|\,|z|^{2n}\right)^{1/2}\left\langle\sum_{n=0}^{k}|a_n|\,|T|^{2n}x,x\right\rangle^{1/2} \\
& \times\left(\sum_{m=1}^{l}|b_{m-1}|\,|u|^{2(m-1)}\right)^{1/2}\left\langle\sum_{m=1}^{l}|b_{m-1}|\,|T^*|^{2m}y,y\right\rangle^{1/2}
\end{aligned}
$$

for any $x,y\in H$ and $k\geq 0, l\geq 1$.

Observe also that

$$
\begin{aligned}
(2.5)\qquad & \sum_{n=0}^{k}\sum_{m=1}^{l} a_n z^n b_{m-1}u^{m-1}\left\langle T|T|^{n+m-1}x,y\right\rangle \\
& = \left\langle T\left(\sum_{n=0}^{k}a_n z^n|T|^n\right)\left(\sum_{m=1}^{l}b_{m-1}u^{m-1}|T|^{m-1}\right)x,y\right\rangle
\end{aligned}
$$

for any $x,y\in H$ and $k\geq 0, l\geq 1$.

Making use of (2.4) and (2.5) we get

$$\begin{aligned}(2.6)\qquad &\left|\left\langle T\left(\sum_{n=0}^{k} a_n z^n |T|^n\right)\left(\sum_{m=1}^{l} b_{m-1}u^{m-1}|T|^{m-1}\right)x,y\right\rangle\right| \\ &\leq \left(\sum_{n=0}^{k}|a_n||z|^{2n}\right)^{1/2}\left\langle \sum_{n=0}^{k}|a_n||T|^{2n}x,x\right\rangle^{1/2} \\ &\times\left(\sum_{m=1}^{l}|b_{m-1}||u|^{2(m-1)}\right)^{1/2}\left\langle |T^*|^2\sum_{m=1}^{l}|b_{m-1}||T^*|^{2(m-1)}y,y\right\rangle^{1/2}\end{aligned}$$

for any $x,y\in H$ and $k\geq 0, l\geq 1$.

Due to the assumption (2.1) in the theorem, we have that the series $\sum_{n=0}^{\infty} a_n z^n |T|^n$, $\sum_{m=0}^{\infty} b_m u^m |T|^m$, $\sum_{n=0}^{\infty}|a_n||T|^{2n}$ and $\sum_{m=0}^{\infty}|b_m||T^*|^{2m}$ are convergent in $B(H)$ and the series $\sum_{n=0}^{\infty}|a_n||z|^{2n}$ and $\sum_{m=0}^{\infty}|b_m||u|^{2m}$ are convergent in $\mathbb{R}$ and then, by taking the limit over $k\to\infty$ and $l\to\infty$ in (2.6), we deduce the desired result (2.2). □

Remark 1. *The above inequality (2.2) can provide various particular instances of interest.*

For instance, if we take $g=f$ in Theorem 2 then we get

$$\begin{aligned}(2.7)\qquad &\left|\left\langle Tf^2(z|T|)x,y\right\rangle\right| \\ &\leq f_A\left(|z|^2\right)\left\langle f_A\left(|T|^2\right)x,x\right\rangle^{1/2}\left\langle |T^*|^2 f_A\left(|T^*|^2\right)y,y\right\rangle^{1/2}\end{aligned}$$

for any $x,y\in H$.

Also if we take $g(z)=1$ in (2.2), then we get

$$(2.8)\qquad |\langle Tf(z|T|)x,y\rangle|^2\leq f_A\left(|z|^2\right)\left\langle f_A\left(|T|^2\right)x,x\right\rangle\left\langle |T^*|^2 y,y\right\rangle$$

for any $x,y\in H$.

Corollary 2. *With the assumptions of Theorem 2 we have the norm inequality*

$$\begin{aligned}(2.9)\qquad &\|Tf(z|T|)g(u|T|)\|^2 \\ &\leq f_A\left(|z|^2\right)g_A\left(|u|^2\right)\left\|f_A\left(|T|^2\right)\right\|\left\||T^*|^2 g_A\left(|T^*|^2\right)\right\|\end{aligned}$$

and the numerical radius inequality

$$\begin{aligned}(2.10)\qquad &w(Tf(z|T|)g(u|T|)) \\ &\leq \frac{1}{2}\left[f_A\left(|z|^2\right)g_A\left(|u|^2\right)\right]^{1/2}\left\|f_A\left(|T|^2\right)+|T^*|^2 g_A\left(|T^*|^2\right)\right\|.\end{aligned}$$

Proof. The inequality (2.9) follows from (2.2) by taking the supremum over $x,y\in H$ with $\|x\|=\|y\|=1$.

From (2.2) we also have the inequality

$$\begin{aligned}&|\langle Tf(z|T|)g(u|T|)x,x\rangle| \\ &\leq\left[f_A\left(|z|^2\right)g_A\left(|u|^2\right)\right]^{1/2}\left\langle f_A\left(|T|^2\right)x,x\right\rangle^{1/2}\left\langle |T^*|^2 g_A\left(|T^*|^2\right)x,x\right\rangle^{1/2} \\ &\leq\frac{1}{2}\left[f_A\left(|z|^2\right)g_A\left(|u|^2\right)\right]^{1/2}\left\langle\left[f_A\left(|T|^2\right)+|T^*|^2 g_A\left(|T^*|^2\right)\right]x,x\right\rangle^{1/2}\end{aligned}$$

for any $x \in H$, which, by taking the supremum over $\|x\| = 1$ produces the desired result (2.10). □

The following result also holds:

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If T is a bounded linear operator on the Hilbert space H with the property that $\|T\|^2 < R$, then we have the inequality*

$$\left|\left\langle T|T| f\left(|T|^2\right) x, y\right\rangle\right|^2 \leq \left\langle |T|^2 f_A\left(|T|^2\right) x, x\right\rangle \left\langle |T^*|^2 f_A\left(|T^*|^2\right) y, y\right\rangle \tag{2.11}$$

for any $x, y \in H$.

Proof. From Furuta's inequality (F) we have for any natural numbers $n \geq 1$ the power inequality

$$\left|\left\langle T|T|^{2n-1} x, y\right\rangle\right| \leq \left\langle |T|^{2n} x, x\right\rangle^{1/2} \left\langle |T^*|^{2n} y, y\right\rangle^{1/2} \tag{2.12}$$

where $x, y \in H$.

If we multiply this inequality with the positive quantities $|a_{n-1}|$, use the triangle inequality and the Cauchy-Bunyakowsky-Schwarz discrete inequality we have successively

$$\begin{aligned} &\left|\left\langle \sum_{n=1}^{k} a_{n-1} T|T|^{2n-1} x, y\right\rangle\right| \\ &\leq \sum_{n=1}^{k} |a_{n-1}| \left|\left\langle T|T|^{2n-1} x, y\right\rangle\right| \\ &\leq \sum_{n=1}^{k} |a_{n-1}| \left\langle |T|^{2n} x, x\right\rangle^{1/2} \left\langle |T^*|^{2n} y, y\right\rangle^{1/2} \\ &\leq \left\langle \sum_{n=1}^{k} |a_{n-1}| |T|^{2n} x, x\right\rangle^{1/2} \left\langle \sum_{n=1}^{k} |a_{n-1}| |T^*|^{2n} y, y\right\rangle^{1/2} \end{aligned} \tag{2.13}$$

for any $x, y \in H$ and $k \geq 1$.

Observe also that

$$\sum_{n=1}^{k} a_{n-1} T|T|^{2n-1} = T|T| \sum_{n=1}^{k} a_{n-1} |T|^{2(n-1)},$$

$$\sum_{n=1}^{k} |a_{n-1}| |T|^{2n} = |T|^2 \sum_{n=1}^{k} |a_{n-1}| |T|^{2(n-1)}$$

and

$$\sum_{n=1}^{k} |a_{n-1}| |T^*|^{2n} = |T^*|^2 \sum_{n=1}^{k} |a_{n-1}| |T^*|^{2(n-1)}$$

for any $k \geq 1$.

Therefore, by (2.13) we have the inequality

$$\left\langle T\left|T\right|\sum_{n=1}^{k}a_{n-1}\left|T\right|^{2(n-1)}x,y\right\rangle^{2} \tag{2.14}$$
$$\leq\left\langle \left|T\right|^{2}\sum_{n=1}^{k}\left|a_{n-1}\right|\left|T\right|^{2(n-1)}x,x\right\rangle\left\langle \left|T^{*}\right|^{2}\sum_{n=1}^{k}\left|a_{n-1}\right|\left|T^{*}\right|^{2(n-1)}y,y\right\rangle$$

for any $x,y\in H$ and $k\geq1$.

Due to the assumption $\left\|T\right\|^{2}<R$, we have that the series $\sum_{n=0}^{\infty}a_{n}\left|T\right|^{2n}$, $\sum_{n=0}^{\infty}\left|a_{n}\right|\left|T\right|^{2n}$ and $\sum_{n=0}^{\infty}\left|a_{n}\right|\left|T^{*}\right|^{2n}$ are convergent in $B\left(H\right)$ and taking the limit over $k\rightarrow\infty$ in (2.14) we deduce the desired result from (2.11). □

Corollary 3. *With the assumptions of Theorem 3 we have the norm inequality*

$$\left\|T\left|T\right|f\left(\left|T\right|^{2}\right)\right\|^{2}\leq\left\|\left|T\right|^{2}f_{A}\left(\left|T\right|^{2}\right)\right\|\left\|\left|T^{*}\right|^{2}f_{A}\left(\left|T^{*}\right|^{2}\right)\right\|$$

and the numerical radius inequality

$$w\left(T\left|T\right|f\left(\left|T\right|^{2}\right)\right)\leq\frac{1}{2}\left\|\left|T\right|^{2}f_{A}\left(\left|T\right|^{2}\right)+\left|T^{*}\right|^{2}f_{A}\left(\left|T^{*}\right|^{2}\right)\right\|.$$

The following result for functions of normal operators holds.

Theorem 4. *Let $f\left(z\right)=\sum_{n=0}^{\infty}a_{n}z^{n}$ be a function defined by power series with real coefficients and convergent on the open disk $D\left(0,R\right)\subset\mathbb{C}$, $R>0$. If N is a normal operator on the Hilbert space H and $\alpha,\beta\geq0$ with $\alpha+\beta\geq1$ with the property that $\left\|N\right\|^{2\alpha},\left\|N\right\|^{2\beta}<R$, then we have the inequality*

$$\left|\left\langle f\left(N\left|N\right|^{(\alpha+\beta-1)}\right)x,y\right\rangle\right|^{2}\leq\left\langle f_{A}\left(\left|N\right|^{2\alpha}\right)x,x\right\rangle\left\langle f_{A}\left(\left|N\right|^{2\beta}\right)y,y\right\rangle \tag{2.15}$$

for any $x,y\in H$.

Proof. Utilising Furuta's inequality written for N^{n} we have

$$\left|\left\langle N^{n}\left|N^{n}\right|^{\alpha+\beta-1}x,y\right\rangle\right|^{2}\leq\left\langle \left|N^{n}\right|^{2\alpha}x,x\right\rangle\left\langle \left|\left(N^{n}\right)^{*}\right|^{2\beta}y,y\right\rangle \tag{2.16}$$

for any $x,y\in H$.

Since N is normal, then

$$\begin{aligned}\left|N^{n}\right|^{2} &= \left(N^{n}\right)^{*}N^{n}=N^{*}...N^{*}N...N\\ &= N^{*}...NN^{*}...N=...\\ &= \left(N^{*}N\right)...\left(N^{*}N\right)=\left|N\right|^{2n}\end{aligned}$$

for any natural number n, and, similarly,

$$\left|\left(N^{n}\right)^{*}\right|^{2}=\left|\left(N^{*}\right)^{n}\right|^{2}=\left|N^{*}\right|^{2n}=\left|N\right|^{2n}$$

for any $n\in\mathbb{N}$.

These imply that $\left|N^{n}\right|^{2\alpha}=\left|N\right|^{2\alpha n}$, $\left|\left(N^{n}\right)^{*}\right|^{2\beta}=\left|N\right|^{2\beta n}$ and $\left|N^{n}\right|^{\alpha+\beta-1}=\left|N\right|^{(\alpha+\beta-1)n}$ for any $\alpha,\beta\geq0$ and for any $n\in\mathbb{N}$.

Utilising the spectral representation for Borel functions of normal operators on Hilbert spaces, see for instance [1, p. 67], we have for any $\alpha,\beta\geq0$ and for any

$n \in \mathbb{N}$ that

$$\begin{aligned} N^n |N|^{(\alpha+\beta-1)n} &= \int_{\sigma(N)} z^n |z|^{(\alpha+\beta-1)n} dP(z) \\ &= \int_{\sigma(N)} \left[z|z|^{(\alpha+\beta-1)}\right]^n dP(z) \\ &= \left[N|N|^{(\alpha+\beta-1)}\right]^n, \end{aligned}$$

where P is the spectral measure associated to the operator N and $\sigma(N)$ is its spectrum.

Therefore, the inequality (2.16) can be written as

$$\left|\left\langle \left[N|N|^{(\alpha+\beta-1)}\right]^n x, y\right\rangle\right| \le \left\langle \left[|N|^{2\alpha}\right]^n x, x\right\rangle^{1/2} \left\langle \left[|N|^{2\beta}\right]^n y, y\right\rangle^{1/2} \tag{2.17}$$

for any $x, y \in H$ and for any $n \in \mathbb{N}$.

If we multiply the inequality (2.17) by $|a_n| \ge 0$, sum over n from 0 to $k \ge 1$ and utilize the Cauchy-Bunyakowsky-Schwarz discrete inequality, we have successively

$$\begin{aligned} &\left|\left\langle \sum_{n=0}^{k} a_n \left[N|N|^{(\alpha+\beta-1)}\right]^n x, y\right\rangle\right| \\ &\le \sum_{n=0}^{k} |a_n| \left|\left\langle \left[N|N|^{(\alpha+\beta-1)}\right]^n x, y\right\rangle\right| \\ &\le \sum_{n=0}^{k} |a_n| \left\langle \left[|N|^{2\alpha}\right]^n x, x\right\rangle^{1/2} \left\langle \left[|N|^{2\beta}\right]^n y, y\right\rangle^{1/2} \\ &\le \left\langle \sum_{n=0}^{k} |a_n| \left[|N|^{2\alpha}\right]^n x, x\right\rangle^{1/2} \left\langle \sum_{n=0}^{k} |a_n| \left[|N|^{2\beta}\right]^n y, y\right\rangle^{1/2} \end{aligned} \tag{2.18}$$

for any $x, y \in H$ and for any $k \ge 1$.

Since $\|N\|^{2\alpha}, \|N\|^{2\beta} < R$ then $\left\|N|N|^{(\alpha+\beta-1)}\right\| < R$ and the series

$$\sum_{n=0}^{\infty} |a_n| \left[|N|^{2\alpha}\right]^n, \sum_{n=0}^{\infty} |a_n| \left[|N|^{2\beta}\right]^n$$

and

$$\sum_{n=0}^{\infty} a_n \left[N|N|^{(\alpha+\beta-1)}\right]^n$$

are convergent in the Banach algebra $B(H)$.

Taking the limit over $k \to \infty$ in the inequality (2.18) we deduce the desired result from (2.15). □

Corollary 4. *With the assumptions of Theorem 4, we have the inequality*

$$\left\|f\left(N|N|^{(\alpha+\beta-1)}\right)\right\|^2 \le \left\|f_A\left(|N|^{2\alpha}\right)\right\| \left\|f_A\left(|N|^{2\beta}\right)\right\|. \tag{2.19}$$

Remark 2. *If we take $\beta = 1-\alpha$ with $\alpha \in [0,1]$ in (2.15), then we get the following generalization of Kato's inequality for normal operators (1.8)*

$$|\langle f(N) x, y\rangle|^2 \le \left\langle f_A\left(|N|^{2\alpha}\right) x, x\right\rangle \left\langle f_A\left(|N|^{2(1-\alpha)}\right) y, y\right\rangle \tag{2.20}$$

where $x, y \in H$ *and* $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R.$

3. Applications

As some natural examples that are useful for applications, we can point out that, if

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n = \ln \frac{1}{1+z}, \; z \in D(0,1); \tag{3.1}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \; z \in \mathbb{C};$$

$$h(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \; z \in \mathbb{C};$$

$$l(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \; z \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(z) = \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z}, \; z \in D(0,1); \tag{3.2}$$

$$g_A(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \; z \in \mathbb{C};$$

$$h_A(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \; z \in \mathbb{C};$$

$$l_A(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \; z \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad z \in \mathbb{C}, \tag{3.3}$$

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \qquad z \in D(0,1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0, \quad z \in D(0,1);$$

where Γ is the *Gamma function*.

Example 1. *Let $x, y \in H$.*

a) If we take $f(z) = \sin z$ and $g(z) = \cos z$ in (2.2), then we get

$$
\begin{aligned}
&\left|\left\langle T\sin(z|T|)\cos(u|T|)x, y\right\rangle\right|^2 \\
&\leq \sinh\left(|z|^2\right)\cosh\left(|u|^2\right) \\
&\times \left\langle \sinh\left(|T|^2\right)x, x\right\rangle \left\langle |T^*|^2\cosh\left(|T^*|^2\right)y, y\right\rangle
\end{aligned} \tag{3.4}
$$

for any $z \in \mathbb{C}$ and $T \in B(H)$.

b) If we take $f(z) = \ln\frac{1}{1+z}$ and $g(z) = \ln\frac{1}{1-z}$ in (2.2), then we get

$$
\begin{aligned}
&\left|\left\langle T\ln\left(1_H + z|T|\right)^{-1}\ln\left(1_H - z|T|\right)^{-1}x, y\right\rangle\right|^2 \\
&\leq \left(\ln\frac{1}{1-|z|^2}\right)^2 \\
&\times \left\langle \ln\left(1_H - |T|^2\right)^{-1}x, x\right\rangle \left\langle |T^*|^2\ln\left(1_H - |T^*|^2\right)^{-1}y, y\right\rangle
\end{aligned} \tag{3.5}
$$

for any $z \in \mathbb{C}$ and $T \in B(H)$ with $|z| < 1$ and $\|T\| < 1$.

c) If we take $f(z) = \exp(z)$ and $g(z) = \exp(z)$ in (2.2), then we get

$$
\begin{aligned}
&\left|\left\langle T\exp\left[(z+u)|T|\right]x, y\right\rangle\right|^2 \\
&\leq \exp\left(|z|^2\right)\exp\left(|u|^2\right) \\
&\times \left\langle \exp\left(|T|^2\right)x, x\right\rangle \left\langle |T^*|^2\exp\left(|T^*|^2\right)y, y\right\rangle
\end{aligned} \tag{3.6}
$$

for any $z, u \in \mathbb{C}$ and $T \in B(H)$.

d) By the inequality (2.8) we have

$$
\left|\left\langle T\sin^{-1}(z|T|)x, y\right\rangle\right|^2 \leq \sin^{-1}\left(|z|^2\right)\left\langle \sin^{-1}\left(|T|^2\right)x, x\right\rangle \left\langle |T^*|^2 y, y\right\rangle \tag{3.7}
$$

and

$$
\begin{aligned}
&\left|\left\langle T\tanh^{-1}(z|T|)x, y\right\rangle\right|^2 \\
&\leq \tanh^{-1}\left(|z|^2\right)\left\langle \tanh^{-1}\left(|T|^2\right)x, x\right\rangle \left\langle |T^*|^2 y, y\right\rangle
\end{aligned} \tag{3.8}
$$

for any $z \in \mathbb{C}$ and $T \in B(H)$ with $|z| < 1$ and $\|T\| < 1$.

Example 2. *Let $x, y \in H$.*

a) If we take $f(z) = \frac{1}{1\pm z}$ in (2.11), then we get

$$
\begin{aligned}
&\left|\left\langle T|T|\left(1_H \pm |T|^2\right)^{-1}x, y\right\rangle\right|^2 \\
&\leq \left\langle |T|^2\left(1_H - |T|^2\right)^{-1}x, x\right\rangle \left\langle |T^*|^2\left(1_H - |T^*|^2\right)^{-1}y, y\right\rangle
\end{aligned} \tag{3.9}
$$

for any $T \in B(H)$ with $\|T\| < 1$.

b) If we take $f(z) = \ln \frac{1}{1 \pm z}$ *in (2.11), then we get*

$$\left| \left\langle T|T| \ln \left(1_H \pm |T|^2 \right)^{-1} x, y \right\rangle \right|^2 \leq \left\langle |T|^2 \ln \left(1_H - |T|^2 \right)^{-1} x, x \right\rangle \left\langle |T^*|^2 \ln \left(1_H - |T^*|^2 \right)^{-1} y, y \right\rangle \tag{3.10}$$

for any $T \in B(H)$ *with* $\|T\| < 1$.

c) If we take $f(z) = \exp(z)$ *in (2.11), then we get*

$$\left| \left\langle T|T| \exp \left(|T|^2 \right) x, y \right\rangle \right|^2 \leq \left\langle |T|^2 \exp \left(|T|^2 \right) x, x \right\rangle \left\langle |T^*|^2 \exp \left(|T^*|^2 \right) y, y \right\rangle \tag{3.11}$$

for any $T \in B(H)$.

Example 3. *Let* N *be a normal operator on the Hilbert space* H, $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1$ *and* $x, y \in H$.

a) If we take $f(z) = \frac{1}{1 \pm z}$ *in (2.15), then we get*

$$\left| \left\langle \left(1_H \pm N|N|^{(\alpha+\beta-1)} \right)^{-1} x, y \right\rangle \right|^2 \leq \left\langle \left(1_H - |N|^{2\alpha} \right)^{-1} x, x \right\rangle \left\langle \left(1_H - |N|^{2\beta} \right)^{-1} y, y \right\rangle \tag{3.12}$$

provided $\|N\| < 1$.

In particular, we have

$$\left| \left\langle (1_H \pm N)^{-1} x, y \right\rangle \right|^2 \leq \left\langle \left(1_H - |N|^{2\alpha} \right)^{-1} x, x \right\rangle \left\langle \left(1_H - |N|^{2(1-\alpha)} \right)^{-1} y, y \right\rangle, \tag{3.13}$$

for $\alpha \in [0, 1]$.

b) If we take $f(z) = \exp(z)$ *in (2.15), then we get*

$$\left| \left\langle \exp \left(N|N|^{(\alpha+\beta-1)} \right) x, y \right\rangle \right|^2 \leq \left\langle \exp \left(|N|^{2\alpha} \right) x, x \right\rangle \left\langle \exp \left(|N|^{2\beta} \right) y, y \right\rangle. \tag{3.14}$$

As a special case, we have

$$|\langle \exp(N) x, y \rangle|^2 \leq \left\langle \exp \left(|N|^{2\alpha} \right) x, x \right\rangle \left\langle \exp \left(|N|^{2(1-\alpha)} \right) y, y \right\rangle, \tag{3.15}$$

for $\alpha \in [0, 1]$.

References

[1] W. Arveson, *A Short Course on Spectral Theory*, 2002, Springer-Verlag Inc., New York.

[2] S.S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications. *J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 52, 22 pp.

[3] M. Fujii, C.-S. Lin and R. Nakamoto, Alternative extensions of Heinz-Kato-Furuta inequality. *Sci. Math.* **2** (1999), no. 2, 215–221.

[4] M. Fujii and T. Furuta, Löwner-Heinz, Cordes and Heinz-Kato inequalities. *Math. Japon.* **38** (1993), no. 1, 73–78.

[5] M. Fujii, E. Kamei, C. Kotari and H. Yamada, Furuta's determinant type generalizations of Heinz-Kato inequality. *Math. Japon.* **40** (1994), no. 2, 259–267

[6] M. Fujii, Y.O. Kim, and Y. Seo, Further extensions of Wielandt type Heinz-Kato-Furuta inequalities via Furuta inequality. *Arch. Inequal. Appl.* **1** (2003), no. 2, 275–283
[7] M. Fujii, Y.O. Kim and M. Tominaga, Extensions of the Heinz-Kato-Furuta inequality by using operator monotone functions. *Far East J. Math. Sci.* (FJMS) **6** (2002), no. 3, 225–238
[8] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **128** (2000), no. 1, 223–228.
[9] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. II. *J. Inequal. Appl.* **3** (1999), no. 3, 293–302,
[10] T. Furuta, Equivalence relations among Reid, Löwner-Heinz and Heinz-Kato inequalities, and extensions of these inequalities. *Integral Equations Operator Theory* **29** (1997), no. 1, 1–9.
[11] T. Furuta, Determinant type generalizations of Heinz-Kato theorem via Furuta inequality. *Proc. Amer. Math. Soc.* **120** (1994), no. 1, 223–231.
[12] T. Furuta, An extension of the Heinz-Kato theorem. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 785–787.
[13] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.
[14] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208-212.
[15] F. Kittaneh, Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.
[16] F. Kittaneh, Norm inequalities for fractional powers of positive operators. *Lett. Math. Phys.* **27** (1993), no. 4, 279–285.
[17] C.-S. Lin, On Heinz-Kato-Furuta inequality with best bounds. J. *Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **15** (2008), no. 1, 93–101.
[18] C.-S. Lin, On chaotic order and generalized Heinz-Kato-Furuta-type inequality. *Int. Math. Forum* **2** (2007), no. 37-40, 1849–1858,
[19] C.-S. Lin, On inequalities of Heinz and Kato, and Furuta for linear operators. *Math. Japon.* **50** (1999), no. 3, 463–468.
[20] C.-S. Lin, On Heinz-Kato type characterizations of the Furuta inequality. II. *Math. Inequal. Appl.* **2** (1999), no. 2, 283–287.
[21] C.A. McCarthy, c_p. *Israel J. Math.*, **5**(1967), 249-271.
[22] G. Popescu, Unitary invariants in multivariable operator theory. *Mem. Amer. Math. Soc.* **200** (2009), no. 941, vi+91 pp.
[23] M. Uchiyama, Further extension of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **127** (1999), no. 10, 2899–2904.

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