Received 27/01/12

HERMITE-HADAMARD'S INEQUALITIES FOR DIFFERENT KINDS OF CONVEXITY VIA FRACTIONAL INTEGRALS

M. EMIN ÖZDEMIR$^{\blacktriangle}$ AND HAVVA KAVURMACI$^{\blacktriangle,\blacksquare}$

Abstract. In this paper, Hermite–Hadamard's inequalities for several kinds of convexity in the literature have been represented in fractional integral forms. So, we have established Hermite–Hadamard's inequalities for convex functions, m-convex fuctions and $g-$convex dominated functions via fractional integrals.

1. Introduction

The inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f:[a,b]\rightarrow\mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality.

The following theorem concerns derivatives of convex functions, in [14].

Theorem 1. *$f:(a,b)\rightarrow\mathbb{R}$ is convex if f there is at least one line of support for f at each $x_0\in(a,b)$, i.e.,*

$$f(x)\geq f(x_0)+\lambda(x-x_0)\quad,\quad\forall x\in(a,b)$$

where λ depends on x_0 and is given by $\lambda=f'(x_0)$ when f' exists and $\lambda\in\left[f'_-(x_0),f'_+(x_0)\right]$ when $f'_-(x_0)\neq f'_+(x_0)$.

In [18], Toader defined the concept of m-convexity as the following:

Definition 1. *The function $f:[0,b]\rightarrow\mathbb{R}$ is said to be $m-$convex, where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$ we have:*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0,b]$ for which $f(0)\leq0$.

Several papers have been written on $m-$convex functions and we refer the papers [1]-[5], [10], [19] and [20].

In [4], Dragomir proved the following theorem .

Date: December 14, 2012.

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.

Key words and phrases. Hermite–Hadamard's inequalities, Riemann–Liouville fractional integral, Integral inequalities, $g-$convex-dominated function, $m-$convex function.

Corresponding Author$^{\blacksquare}$.

Theorem 2. *Let $f:[0,\infty)\rightarrow\mathbb{R}$ be an $m-$convex function with $m\in(0,1]$ and $0\leq a<b$. If $f\in L_1[a,b]$, then one has the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \leq \frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right]. \tag{1.2}$$

In [6], Dragomir and Ionescu introduced the following class of functions.

Definition 2. *Let $g:I\rightarrow\mathbb{R}$ be a convex function on the interval I. The function $f:I\rightarrow\mathbb{R}$ is called $g-$convex dominated on I if the following condition is satisfied:*

$$\left|\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)\right| \leq \lambda g(x)+(1-\lambda)g(y)-g(\lambda x+(1-\lambda)y)$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

In [7], Dragomir et al. proved the following theorem for $g-$convex dominated functions related to (1.1).

Theorem 3. *Let $g:I\rightarrow\mathbb{R}$ be a convex fuction and $f:I\rightarrow\mathbb{R}$ be a $g-$convex dominated mapping. Then, for all $a,b\in I$ with $a<b$,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{b-a}\int_a^b g(x)\,dx-g\left(\frac{a+b}{2}\right) \tag{1.3}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)\,dx. \tag{1.4}$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-t}u^{\alpha-1}du$, here is $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral. Properties of this operator can be found in the references [8], [9] and [13].

In [15], Sarıkaya et al. represented Hadamard's inequality for convex functions in fractional integral forms as follows:

Theorem 4. *Let $f:[a,b]\rightarrow\mathbb{R}$, be positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractionals integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha>0$.

Several papers have been written on Hermite-Hadamard's inequalities for different kinds of convexity via fractional integrals and we refer the papers [11], [12], [15], [16] and [17].

The aim of this paper is to establish Hadamard's inequalities for convex, $m-$convex and g-convex dominated functions via Riemann-Liouville fractional integrals.

2. Fractional Integrals

Hermite–Hadamard's inequalities can be represented in fractional integral forms as follows.

Theorem 5. *Let $f:I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a convex function on I and $a,b\in I^{\circ}$, for all $x,t\in[a,b]$. Then we get the following inequality:*

$$\Gamma(\alpha)\left(J_{a^{+}}^{\alpha}f(x)+J_{b^{-}}^{\alpha}f(x)\right)\leq\frac{(x-a)^{\alpha}\left(\alpha f(a)+f(x)\right)+(b-x)^{\alpha}\left(\alpha f(b)+f(x)\right)}{\alpha(\alpha+1)} \tag{2.1}$$

for $f'(t)\in\left[f'_{-}(t),f'_{+}(t)\right]$ and $\alpha>0$.

Proof. Since f is convex we can write

$$f(x)-f(t)\geq(x-t)f'(t). \tag{2.2}$$

Multiply both sides of 2.2 by $(x-t)^{\alpha-1}>0$ and then integrate the resulting inequality with respect to t over $[a,x]$. We get

$$\underbrace{\int_{a}^{x}(x-t)^{\alpha-1}\left(f(x)-f(t)\right)dt}_{I_1}\geq\underbrace{\int_{a}^{x}(x-t)^{\alpha-1}(x-t)f'(t)\,dt}_{I_2}. \tag{2.3}$$

$$\begin{aligned}I_1&=\int_{a}^{x}(x-t)^{\alpha-1}\left(f(x)-f(t)\right)dt\\&=f(x)\int_{a}^{x}(x-t)^{\alpha-1}dt-\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt\\&=f(x)\frac{(x-a)^{\alpha}}{\alpha}-\Gamma(\alpha)J_{a^{+}}^{\alpha}f(x)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}I_2&=\int_{a}^{x}(x-t)^{\alpha}f'(t)\,dt\\&=(x-t)^{\alpha}f(t)\big|_{a}^{x}-\alpha\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt\\&=-(x-a)^{\alpha}f(a)+\alpha\Gamma(\alpha)J_{a^{+}}^{\alpha}f(x).\end{aligned} \tag{2.5}$$

Then if we use 2.4 and 2.5 in 2.3, we get

$$\frac{(x-a)^{\alpha}\left(\alpha f(a)+f(x)\right)}{\alpha(\alpha+1)} \geq \Gamma(\alpha) J_{a+}^{\alpha} f(x). \tag{2.6}$$

Analogously we can write

$$f(t)-f(x) \leq (t-x) f'(t). \tag{2.7}$$

Before multiplying both sides of 2.7 by $(t-x)^{\alpha-1}>0$ and then integrate the resulting inequality with respect to the t over $[x,b]$. We get

$$\underbrace{\int_x^b (t-x)^{\alpha-1}\left(f(t)-f(x)\right) dt}_{I_3} \leq \underbrace{\int_x^b (t-x)^{\alpha} f'(t)\, dt}_{I_4}. \tag{2.8}$$

$$\begin{aligned} I_3 &= \int_x^b (t-x)^{\alpha-1}\left(f(t)-f(x)\right) dt \\ &= \int_x^b (t-x)^{\alpha-1} f(t)\, dt - f(x)\int_x^b (t-x)^{\alpha-1} dt \\ &= \Gamma(\alpha) J_{b-}^{\alpha} f(x) - f(x)\frac{(b-x)^{\alpha}}{\alpha} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} I_4 &= \int_x^b (t-x)^{\alpha} f'(t)\, dt \\ &= (t-x)^{\alpha} f(t)\Big|_x^b - \alpha\int_x^b (t-x)^{\alpha-1} f(t)\, dt \\ &= (b-x)^{\alpha} f(b) - \alpha\Gamma(\alpha) J_{b-}^{\alpha} f(x). \end{aligned} \tag{2.10}$$

Then if we use 2.9 and 2.10 in 2.8, we get

$$\frac{(b-x)^{\alpha}\left(\alpha f(b)+f(x)\right)}{\alpha(\alpha+1)} \geq \Gamma(\alpha) J_{b-}^{\alpha} f(x). \tag{2.11}$$

Then if we add inequalities in 2.6 and 2.11, we get the desired result in 2.1. □

Remark 1. *If we choose $\alpha=1$ and $x=\frac{a+b}{2}$ in Theorem 5, then we have*

$$\frac{1}{b-a}\int_a^b f(t)\, dt \leq \frac{f(a)+2f\left(\frac{a+b}{2}\right)+f(b)}{4}.$$

Remark 2. *If we choose $\alpha=1$ and $x=a$ or $x=b$ in Theorem 5, we get the right hand side of the inequality in 1.1.*

Theorem 6. *Let $f:[0,\infty)\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,\frac{b}{m}]$. If f is an $m-$convex function on $[0,\infty)$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a+}^{\alpha} f(b)+m^{\alpha+1} J_{\frac{b}{m}-}^{\alpha} f\left(\frac{a}{m}\right)}{2}\right] \\ &\leq \frac{1}{2}\left[\frac{\alpha f(a)+m f\left(\frac{a}{m}\right)}{\alpha+1}+m\frac{f\left(\frac{b}{m}\right)+m\alpha f\left(\frac{b}{m^2}\right)}{\alpha+1}\right]. \end{aligned} \tag{2.12}$$

with $\alpha > 0$ *and* $m \in (0,1]$

Proof. Since f is an $m-$convex function on $[0,\infty)$, we know that for any $(x,y) \in [0,\infty)$ and $\lambda = \frac{1}{2}$

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left[f(x) + mf\left(\frac{y}{m}\right)\right]. \tag{2.13}$$

If we choose $x = ta + (1-t)b$, $y = (1-t)a + tb$ and $t \in [0,1]$, $m \in (0,1]$, then we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f(ta + (1-t)b) + mf\left(\frac{(1-t)a + tb}{m}\right)\right]. \tag{2.14}$$

By multiplying both sides of (2.14) by $t^{\alpha-1}$, then by integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt \leq \frac{1}{2}\int_0^1 t^{\alpha-1}\left[f(ta + (1-t)b) + mf\left(\frac{(1-t)a + tb}{m}\right)\right]dt$$

i.e.

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a+}^{\alpha}f(b) + m^{\alpha+1}J_{\frac{b}{m}-}^{\alpha}f(\frac{a}{m})}{2}\right]. \tag{2.15}$$

It is easy to see that $\int_0^1 t^{\alpha-1}f(ta+(1-t)b)dt = \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J_{a+}^{\alpha}f(b)$ and $\int_0^1 t^{\alpha-1}f\left(\frac{(1-t)a+tb}{m}\right)dt = \frac{m^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}J_{\frac{b}{m}-}^{\alpha}f(\frac{a}{m})$. So the proof of the first inequality in (2.12) is completed.

For the proof of the second inequality in (2.12), we first note that if f is an $m-$convex on $[0,\infty)$, then $t \in [0,1]$, it yields

$$f(ta + (1-t)b) \leq tf(a) + m(1-t)f\left(\frac{b}{m}\right), \quad x = a, \quad y = b \tag{2.16}$$

and

$$f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) \leq (1-t)f\left(\frac{a}{m}\right) + mtf\left(\frac{b}{m^2}\right), \quad x = \frac{a}{m}, \quad y = \frac{b}{m}. \tag{2.17}$$

By multiplying the inequality in 2.16 by $\frac{1}{2}$ and mulltiplying the inequality in 2.17 by $\frac{m}{2}$ and then by adding the resulting inequalities we have

$$\begin{aligned} &\frac{1}{2}\left[f(ta + (1-t)b) + mf\left((1-t)\frac{a}{m} + t\frac{b}{m}\right)\right] \\ \leq\ &\frac{1}{2}\left[\left[tf(a) + m(1-t)f\left(\frac{a}{m}\right)\right] + \left[m(1-t)f\left(\frac{b}{m}\right) + m^2tf\left(\frac{b}{m^2}\right)\right]\right]. \end{aligned} \tag{2.18}$$

Then multiplying both sides of the inequality in **??** by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{J_{a^{+}}^{\alpha}f(b)+m^{\alpha+1}J_{\frac{b}{m}^{-}}^{\alpha}f(\frac{a}{m})}{2}\right] \\ \leq & \left\{\frac{f(a)\int_0^1 t^{\alpha}dt+mf\left(\frac{a}{m}\right)\int_0^1 t^{\alpha-1}(1-t)dt}{2}\right. \\ & \left.+m\frac{f\left(\frac{b}{m}\right)\int_0^1 t^{\alpha-1}(1-t)dt+mf\left(\frac{b}{m^2}\right)\int_0^1 t^{\alpha}dt}{2}\right\} \\ = & \frac{1}{2}\left[\frac{\alpha f(a)+mf\left(\frac{a}{m}\right)}{\alpha(\alpha+1)}+m\frac{f\left(\frac{b}{m}\right)+m\alpha f\left(\frac{b}{m^2}\right)}{\alpha(\alpha+1)}\right]. \end{aligned} \tag{2.19}$$

If we combine the inequalities in 2.15 and 2.19, we get the second inequality in 2.12. □

Remark 3. *If we choose $\alpha=1$ in Theorem 6, then the inequalities 2.12 become the inequalities in 1.2 of Theorem 2.*

Theorem 7. *Let $f,g:[a,b]\to\mathbb{R}$ be positive functions with $0\leq a<b$ and $f,g\in L_1[a,b]$. If g is a convex function on $[a,b]$ and f is a g-convex dominated function, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & \left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \\ \leq & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}g(b)+J_{b^{-}}^{\alpha}g(a)\right]-g\left(\frac{a+b}{2}\right) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \\ \leq & \frac{g(a)+g(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}g(b)+J_{b^{-}}^{\alpha}g(a)\right] \end{aligned} \tag{2.21}$$

with $\alpha>0$.

Proof. By Definition 2 with $\lambda=\frac{1}{2}$, as the mapping f is $g-$convex dominated function, we have that

$$\left|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right|\leq\frac{g(x)+g(y)}{2}-g\left(\frac{x+y}{2}\right)$$

for all $x,y\in[a,b]$. If we choose $x=ta+(1-t)b$, $y=(1-t)a+tb$ and $t\in[0,1]$, then we get

$$\begin{aligned} & \left|\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}-f\left(\frac{a+b}{2}\right)\right| \\ \leq & \frac{g(ta+(1-t)b)+g((1-t)a+tb)}{2}-g\left(\frac{a+b}{2}\right). \end{aligned} \tag{2.22}$$

Multiplying both sides of 2.22 by $t^{\alpha-1}$, integrating the resulting inequality with respect to t over $[0,1]$, we deduce that

$$\left|\frac{\int_0^1 t^{\alpha-1}f(ta+(1-t)b)\,dt+\int_0^1 t^{\alpha-1}f((1-t)a+tb)\,dt}{2}-f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt\right|$$

$$\leq \frac{\int_0^1 t^{\alpha-1}g(ta+(1-t)b)\,dt+\int_0^1 t^{\alpha-1}g((1-t)a+tb)\,dt}{2}-g\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt$$

i.e.

$$\left|\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\frac{f\left(\frac{a+b}{2}\right)}{\alpha}\right|$$

$$\leq \frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\frac{g\left(\frac{a+b}{2}\right)}{\alpha}$$

and so the inequality in 2.20 is proved.

By using Definition 2, as the mapping f is $g-$convex dominated function,we have

$$|tf(a)+(1-t)f(b)-f(ta+(1-t)b)|$$

(2.23)

$$\leq tg(a)+(1-t)g(b)-g(ta+(1-t)b)$$

and

$$|(1-t)f(a)+tf(b)-f((1-t)a+tb)|$$

(2.24)

$$\leq (1-t)g(a)+tg(b)-g((1-t)a+tb).$$

By adding these inequalities in 2.23 and 2.24, we have

$$|[f(a)+f(b)]-[f(ta+(1-t)b)+f((1-t)a+tb)]|$$

(2.25)

$$\leq [g(a)+g(b)]-[g(ta+(1-t)b)+g((1-t)a+tb)].$$

Multiplying both sides of 2.25 by $t^{\alpha-1}$, integrating the resulting inequality with respect to t over $[0,1]$, we deduce that

$$\left|[f(a)+f(b)]\int_0^1 t^{\alpha-1}dt-\int_0^1 t^{\alpha-1}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]dt\right|$$

$$\leq [g(a)+g(b)]\int_0^1 t^{\alpha-1}dt-\int_0^1 t^{\alpha-1}\left[g(ta+(1-t)b)+g((1-t)a+tb)\right]dt$$

i.e.

$$\left|\frac{f(a)+f(b)}{\alpha}-\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|$$

$$\leq \frac{g(a)+g(b)}{\alpha}-\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]$$

and the inequality in 2.21 is proved. So, the proof is completed. □

Remark 4. *If we choose $\alpha=1$ in Theorem 7, then the inequalities in 2.20 and 2.21 become the inequalities in 1.3 and 1.4 of Theorem 3, respectively.*

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[▲]Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey

E-mail address: emos@atauni.edu.tr

E-mail address: hkavurmaci@atauni.edu.tr